THE CONCEPT OF A Z-NUMBER—TOWARD A
HIGHER LEVEL OF GENERALITY IN UNCERTAIN
COMPUTATION

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INTRODUCTION
In large measure, science and engineering dwell in the word of measurements and numbers. In this world, a basic question which arises is: How reliable are the numbers which we deal with? This question plays a particularly important role in decision analysis, planning, economics, design and process analysis.
The concept of a Z-number is intended to provide a basis for computation with numbers which are not totally reliable. More concretely, a Z-number, $Z=(A,B)$, is an order pair of two fuzzy numbers. The first number, $A$, is a restriction on the values which a real-valued variable, $X$, can take.
The second number, B, is a restriction on the degree of certainty that X is A. Typically, A and B are described in a natural language.
EXAMPLES

- $X =$ anticipated budget deficit
- $A =$ approximately 2 million dollars
- $B =$ very likely

- $X =$ travel time by car from Berkeley to Palo Alto
- $A =$ approximately 1 hour
- $B =$ usually
(approximately 100, very sure)

(approximately 100, not very likely)

(low, sure)

(high, not sure)
COMPUTATION WITH Z-NUMBERS

- (approximately 1 hr, usually) + (approximately 45 min, usually)

- What is the square root of (approximately 100, very likely?)

- Computation with Z-numbers falls within the province of Computing with Words.
Z-NUMBERS AND COMPUTING WITH WORDS

Before proceeding further without discussion of computation with Z-numbers we will discuss briefly some of the pertinent features of Computing with Words. The methodology of Computing with Words is of interest in its own right.
COMPUTING WITH WORDS (CW OR CWW) PRINCIPAL CONCEPTS AND IDEAS
There are many misconceptions about what Computing with Words is and what it has to offer. A common misconception is that CW and NLP (Natural Language Processing) are closely related. In fact, this is not the case. CW and NLP have different agendas and address different problems. A very simple example of a problem in CW is the following.
EXAMPLE

CN
- Dana is 25 years old
- Tandy is 3 years older than Dana
- Tandy is (25+3) years old

CW
- Dana is young
- Tandy is a few years older than Dana
- Tandy is (young + few) years old

In CW, young and few are interpreted as labels of fuzzy numbers. Fuzzy arithmetic is used to find the sum of young and few.
The point of departure in CW is a question, $q$, of the form: What is the value of a variable, $Y$? The answer to this question is expected to be derived from a collection of propositions, $I$, $I = (p_1, \ldots, p_n)$, which is referred to as the information set. In essence, $I$ is a collection of question-relevant propositions.
The terminus consists of an answer of the form: Y is ans(q/l). Generally, ans(q/l) is not a value of Y but a restriction (generalized constraint) on the values which Y is allowed to take. (Zadeh 2006)
BASIC STRUCTURE OF CW

\[ \text{ans}(q/l) \]

\[ q, p, +p, \text{NL} \rightarrow \text{CW engine} \rightarrow \text{NL} \]
In essence, CW is a system of computation in which the objects of computation are words, phrases and propositions drawn from a natural language. The carriers of information are propositions. It should be noted that CW is the only system of computation which offers a capability to compute with information described in a natural language.
A prerequisite to computation is precisiation of meaning. Raw (unprecisiated) natural language cannot be computed with. A key idea in CW, is that of precisiating the meaning of a proposition, $p$, as a restriction.
SUMMARY

- CW = precisiation of objects of computation followed by computation with precisiated objects.
I am asked: What is the value of a real-valued variable $X$? My answer is: I do not know the value precisely but I have a perception which I can express as a restriction (generalized constraint) on the values which $X$ can take.
EXAMPLES

- $8 \leq X \leq 10$

- $X$ is small

- $X$ is normally distributed with mean 9 and variance 2.

- It is likely that $X$ is between 8 and 10.
A restriction (generalized constraint), $R(X)$, may be represented as:

$$R(X): \ X \ isr \ R$$

where $X$ is the restricted (constrained) variable, $R$ is the restricting (constraining) relation and $r$ is an indexical variable which defines how $R$ restricts $X$. 
**EXAMPLE**

- **Possibilistic restriction** \((r=\text{blank})\):

\[
R(X): \ X \text{ is } A
\]

where \(A\) is a fuzzy set in \(U\) with the membership function \(\mu_A\). \(A\) plays the role of the possibility distribution of \(X\)

\[
\text{Poss}(X=u) = \mu_A(u)
\]
PRINCIPAL LEVELS OF GENERALITY OF RESTRICTIONS

- **Z-numbers** (Level 3)
- **Fuzzy numbers** (Level 2)
- **Random numbers** (Level 2)
- **Intervals** (Level 1)
- **Real numbers** (ground level)

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A Z-mouse is an electronic implementation of a spray pen. The cursor is a round fuzzy mark called an f-mark. The color of the mark is a matter of choice. A dot identifies the centroid of the mark. The cross-section of a f-mark is a trapezoidal fuzzy set with adjustable parameters.
I believe that Robert is very honest
MORE ON Z-MOUSE

- If I am not sure what the degree is, and I am allowed to use a Z-mouse, I will put a fuzzy f-mark on the scale.

- A fuzzy f-mark reflects imprecision of my perception.

- A Z-mouse reads my f-mark and represents it internally as a trapezoidal fuzzy set—a fuzzy set which serves as an object of computation for the machinery of Computing with Words.
I am scheduled to fly from San Francisco to Los Angeles. My flight is scheduled to leave at 5pm. I have to be at the airport about an hour before departure. Usually it takes about forty five minutes to get to the airport from my home. I would like to be pretty sure that I arrive at the airport in time. At what time should I leave my home?
FROM COMPUTING WITH WORDS TO COMPUTATION WITH Z-NUMBERS
MORE ON Z-NUMBERS

- Z-numbers and computation with Z-numbers open the door to computation with information in which reliability of information is an important issue. Among the primary fields of applications of Z-numbers are economics, decision analysis, risk assessment, design and process analysis, prediction, planning, biomedicine and rule-based manipulation of imprecise functions and relations.
The ordered triple \((X,A,B)\) is referred to as a Z-valuation. A Z-valuation is equivalent to an assignment statement, \(X\) is \((A,B)\). \(X\) is an uncertain variable if \(A\) is not a singleton. In a related way, uncertain computation is a system of computation in which the objects of computation are not values of variables but restrictions on values of variables.
In the following, unless stated to the contrary, X is assumed to be a random variable. Simple examples of Z-valuations are:

- (anticipated budget deficit, close to 2 million dollars, very likely)
- (population of Spain, about 45 million, quite sure)
- (degree of Robert's honesty, very high, absolutely)
(degree of Robert's honesty, high, not sure)

(travel time by car from Berkeley to San Francisco, about 30 minutes, usually)

(price of oil in the near future, significantly over 100 dollars/barrel, very likely)
(Height(John), tall, probable)
It is important to note that, in large measure, computation with probabilities and events described in a natural language, the information set, $I$, consists of a collection of Z-valuations.

Z-information = collection of Z-valuations.
If $X$ is a random variable, then $X$ is a fuzzy event in $R$, the real line. The probability measure of this event, $p$, may be expressed as:

$$p = \int_{R} \mu_A(u)p_X(u)du,$$

where $p_X$ is the underlying (hidden) probability density of $X$. A Z-number may be viewed as a summary of $p_X$. 

CONTINUED
More compactly, $P$ may be represented as the scalar product of $\mu_A$ and $p_X$,

$$P = \mu_A \cdot p_X$$

In effect, a Z-valuation $(X,A,B)$ may be viewed as a restriction (generalized constraint) on $X$ defined by:

$$\text{Prob}(X \text{ is } A) \text{ is } B$$
MEMBERSHIP FUNCTION OF A AND PROBABILITY DENSITY FUNCTION OF X
A Z-rule is an if-then rule in which the antecedent and/or the consequent are Z-valuations. Examples:

- if (anticipated budget deficit, about two million dollars, very likely) then (reduction in staff, about ten percent, very likely)
- if (degree of Robert’s honesty, high, not sure) then (offer a position, not, sure)
- if (X, small) then (Y, large, usually.)
What follows is a brief discussion of the basics of computation with Z-numbers.

Computation with Z-numbers is discussed in greater detail in “A note on Z-numbers,” Information Sciences, 2011.
Computation with Z-numbers involves, in the main, computation with functions whose arguments are Z-numbers. Example: \( f \) is a function whose arguments are real numbers, \( Z = f(X,Y) \). Assume that what we know are not the values of \( X \) and \( Y \) but restrictions, \( R(X) \) and \( R(Y) \), respectively. Restrictions on \( X \) and \( Y \) induce a restriction on \( Z \), \( R(Z) \). The problem is that of computing \( R(Z) \) given \( f \), \( R(X) \) and \( R(Y) \).
For example: if $f$ is the sum of $X$ and $Y$ then the problem is that of computing $R(X+Y)$, given $R(X)$ and $R(Y)$.

In computation with Z-numbers the principal tool is the Extension Principle. (Zadeh 1965, 1975\textsuperscript{a, b, c, 2011})

A basic version of the Extension Principle is stated in the following.
\[ f(X) \text{ is } A \]
\[ g(X) \text{ is } B \]

The answer to this question is the solution of a mathematical program expressed as:

\[
\mu_B(w) = \sup_u \mu_A(f(u))
\]

subject to

\[ w = g(u) \]

where \( \mu_A \) and \( \mu_B \) are the membership functions of \( A \) and \( B \), respectively.
Let $X=(A_X, B_X)$ and $Y=(A_Y, B_Y)$. The sum of $X$ and $Y$ is a Z-number, $Z=(A_Z, B_Z)$. The sum of $(A_X, B_X)$ and $(A_Y, B_Y)$ is defined as:

$$(A_X, B_X) + (A_Y, B_Y) = (A_X + A_Y, B_Z)$$

where $A_X + A_Y$ is the sum of fuzzy numbers $A_X$ and $A_Y$ computed through the use of fuzzy arithmetic. The main problem is computation of $B_Z$. 
Let $p_X$ and $p_Y$ be the underlying probability density functions in the Z-valuations $(X, A_X, B_X)$ and $(Y, A_Y, B_Y)$, respectively. If $p_X$ and $p_Y$ were known, the underlying probability density function in $Z$ would be the convolution of $p_X$ and $p_Y$, $p_Z = p_X \circ p_Y$, expressed as:

$$p_{X+Y}(v) = \int_R p_X(u)p_Y(v-u)du$$

where $R$ is the real line.
What we know are not $p_X$ and $p_Y$ but restrictions on $p_X$ and $p_Y$ which are expressed as:

$$
\int_{R} \mu_{A_X}(u)p_X(u)du \quad \text{is} \quad B_X
$$

$$
\int_{R} \mu_{A_Y}(u)p_Y(u)du \quad \text{is} \quad B_Y
$$
Using the Extension Principle we can compute the restriction on $p_Z$. It reads:

\[ \mu_{p_Z}(p_Z) = \sup_{p_X, p_Y} (\mu_{B_X} (\int_R \mu_{A_X} (u) p_X(u) du) \wedge \mu_{B_Y} (\int_R \mu_{A_Y} (u) p_Y(u) du)) \]

subject to

\[ p_Z = p_X \circ p_Y \]

\[ \int_R p_X(u) du = 1 \]

\[ \int_R p_Y(u) du = 1 \]
If \( p_Z \) were known, \( B_Z \) would be given by:

\[
B_Z = \int_R \mu_{A_Z}(u) p_Z(u) \, du,
\]

where

\[
\mu_{A_Z}(u) = \sup_v (\mu_{A_X}(v) \land \mu_{A_Y}(u-v))
\]
Since what we know is not $p_z$ but a restriction on $p_z$ which is defined by $\mu_{p_Z}(p_z)$, which was computed earlier, the Extension Principle is needed to compute $B_Z$. This step completes computation of the sum of $X$ and $Y$. 
Computation with Z-numbers is a move into uncharted territory. A variety of issues remain to be explored. One such issue is that of informativeness of results of computations. To enhance informativeness and reduce complexity of computations it may be expedient to make simplifying assumptions about the underlying probability distributions. For details see “A Note on Z-Numbers.”