Can Mathematics Deal with Computational Problems Which are Stated in a Natural Language?

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INTRODUCTION

CAN MATHEMATICS DEAL WITH COMPUTATIONAL PROBLEMS WHICH ARE STATED IN A NATURAL LANGUAGE?

- ***Here are a few very simple examples of computational problems which are stated in a natural language.***

- ***(a) Most Swedes are tall. What is the average height of Swedes?***

CONTINUED

- ***(b) Probably John is tall. What is the probability that John is short? What is the probability that John is very short? What is the probability that John is not very tall?***

CONTINUED

- *(c) Usually, most United flights from San Francisco leave on time. I am scheduled to take a United flight from San Francisco. What is the probability that my flight will be delayed?*

CONTINUED

- *(d) X is a real-valued random variable. Usually X is much larger than approximately a. Usually X is much smaller than approximately b. What is the probability that X is approximately c, where c is a number between a and b?*

CONTINUED

- *(e) A and B are boxes, each containing 20 balls of various sizes. Most of the balls in A are large, a few are medium and a few are small. Most of the balls in B are small, a few are medium and a few are large. The balls in A and B are put into a box, C. What is the number of balls in C which are neither large nor small?*

CONTINUED

- *(f) Usually Robert leaves his office at about 5 pm. Usually it takes Robert about an hour to get home from work. At what time does John get home?*

- *(g) A box contains about 20 balls of various sizes. There are many more large balls than small balls. What is the number of small balls?*

CONTINUED

- ***For convenience, such problems will be referred to as CNL (Computation with Natural Language) problems.***

PREAMBLE

- ***It is a long-standing tradition in mathematics to view computational problems which are stated in a natural language as being outside the purview of mathematics. Such problems are dismissed as ill-posed and not worthy of attention. In the instance of CNL problems, mathematics has nothing constructive to say.***

CONTINUED

- *In the following, this tradition is questioned and a system of computation is suggested which opens the door to construction of mathematical solutions of CNL problems. The system draws on the fuzzy-logic-based formalism of computing with words (CW). (Zadeh 2006)*

CONTINUED

- ***A concept which plays a pivotal role in CW is that of precisiation of meaning. More concretely, precisiation involves translation of natural language into a mathematical language in which the objects of computation are well-defined—though not conventional—mathematical constructs.***

THE MEANING POSTULATE

- ***The point of departure—and a key idea—in our approach to precisiation of natural language, is embodied in the meaning postulate. (Zadeh 2006) More concretely, let p be a proposition drawn from a natural language, and let X be a variable which is associated with p.***

CONTINUED

- *The meaning postulate asserts that the meaning of p may be represented as a restriction (generalized constraint), R(X), on the values which X can take, expressed as:*

$$p \longrightarrow R(X): X \text{ isr } R,$$

CONTINUED

where X is the restricted variable, R is the restricting relation and r is an indexical variable which defines the way in which R restricts X. X may be a function of another variable, Y, Y=f(X), in which case the restriction on Y is indirect. X isr R is referred to as the canonical form of p, with X being the focal variable.

CONTINUED

- *In effect, p is viewed as a carrier of information about X.*

- *The meaning postulate is the key to construction of mathematical solutions of CNL problems.*

NOTE

- ***Construction and use of mathematical solutions of CNL problems is an unexplored domain in mathematics. The importance of this domain derives from the fact that much of human knowledge, and particularly world knowledge, is described in natural language.***

SOLUTIONS OF PROBLEMS (a) and (b)

- *We begin our presentation by showing, through examples, that it is possible to construct mathematical solutions of computational problems—even if at first glance such problems make no sense. Use of Problems (a) and (b) as examples serves to facilitate an understanding of the basic ideas which underlie our approach.*

CONTINUED

Problem (a)

- ***Given information, p: Most Swedes are tall. Question, q: What is the average height of Swedes?***

- ***To begin with, what should be noted is that precisiation of meaning is a prerequisite to computation. Raw (unprecisiated) natural language cannot be computed with.***

CONTINUED

- ***The first step involves precisiation of imprecise terms—most and tall. We assume that most and tall are labels of fuzzy sets with specified membership functions, μ_a and μ_b, respectively.***
- ***The second step involves precisiation of the question, q: What is the average height of Swedes?***

CONTINUED

- ***For this purpose, let h be the height density function for a population of Swedes, defined as:***

h(u)du=proportion of Swedes

whose height lies in the interval [u, u+du]. If h_{min} and h_{max} are, respectively, the minimum and maximum heights in the population, we have:

$$\int_{h\ min}^{h\ max} h(u)du = 1$$

CONTINUED

- *In terms of the height density function the average height of Swedes, h_{ave}, may be expressed as:*

$$h_{ave} = \int_{h_{min}}^{h_{max}} u h(u) du$$

It follows that precisiation of the question, q, may be expressed as:*

What is the average height of Swedes?

precisiation →

$$?h_{ave} = \int_{h_{min}}^{h_{max}} u h(u) du$$

CONTINUED

- *The third step involves precisiation of the given information, p: Most Swedes are tall. A concept which is needed for precisiation of p is that of a possibility distribution. Let X be a variable taking values in U. Informally, the possibility distribution of X is a fuzzy set, A, of possible values of X, with the understanding that possibility is a matter of degree. This is written as:*

CONTINUED

X is A

The meaning of this expression may be represented as:

$$Poss(X=u)=\mu_A(u)$$

where μ_A is the membership function of A. As an illustration, if X is a real-valued variable, then

X is small

CONTINUED

implies that the possibility that X=u is $\mu_A(u)$. It should be noted that possibility is distinct from probability.

- *Returning to precisiation of p, p is rewritten as:*

Prop(tall.Swedes/Swedes) is most

where the fuzzy quantifier most is a fuzzy set with membership function μ_{most}. In terms of the height density function, h, the proportion of tall

CONTINUED

Swedes may be expressed as:

$$\int_{h\ min}^{h\ max} \mu_{tall}(u)h(u)du$$

- Consequently, precisiation of p, p*, may be represented as:

p* → $\int_{h\ min}^{h\ max} \mu_{tall}(u)h(u)du$ is most

or equivalently as:

$$\mu_{most}\left(\int_{h\ min}^{h\ max} \mu_{tall}(u)h(u)du\right)$$

CONTINUED

- The last expression is referred to as the generalized intension of p. The generalized intension of p is the mathematical meaning of p. Basically, the generalized intension is the possibility distribution of h, μ_{tall} and μ_{most}.
- At this point, the statement of Problem (a) in a natural language translates into a mathematical problem. More concretely, from precisiation of q, q*,

CONTINUED

$$q^* : ?\, h_{ave} = \int_{h\ min}^{h\ max} u h(u) du$$

and precisiation, p*, of the given information, p,

$$p^* : \int_{h\ min}^{h\ max} \mu_{tall}(u) h(u) du$$ is most,

we would like to compute h_{ave}.

CONTINUED

- *What is needed for this purpose is the extension principle (Zadeh 1965, 1975a, b & c, 2011). Basically, the extension principle is a computational rule of inference which may be expressed as:*

$$
\frac{\begin{array}{l} ?Y=g(X) \\ f(X) \text{ is } A \end{array}}{\mu_Y(v)=\sup_u \mu_A(f(u))}
$$

subject to $v=g(u)$

CONTINUED

- *In this expression, f and g are known functions, A is a fuzzy set which plays the role of a restriction on X, and μ_Y is the membership function of Y. In essence, the extension principle is a rule for computation of the value of a function, g, when the argument, X, is a restriction on X rather than a value of X.*

CONTINUED

- *The structure of the extension principle is the same as the structure of the problem at hand. Consequently, we can write:*

q*: $? \; h_{ave} = \int_{h\,min}^{h\,max} uh(u)du$

p*: $\int_{h\,min}^{h\,max} \mu_{tall}(u)h(u)du$ is most

$$\mu_{h_{ave}}(v) = sup_h \, \mu_{most}\left(\int_{h\,min}^{h\,max} \mu_{tall}(u)h(u)du\right)$$

CONTINUED

subject to

$$v = \int_{h\ min}^{h\ max} u h(u) du$$

and

$$\int_{h\ min}^{h\ max} h(u) du = 1$$

CONTINUED

- *In this way, solution of Problem (a) is reduced to the solution of a variational problem in traditional mathematics. This reduction is viewed as the construction of a mathematical solution of Problem (a).*

IMPORTANT NOTE

- ***It is important to note that the solution is a fuzzy set which is a restriction on the values which h_{ave} can take. The fuzzy set may be viewed as the set of all values of h_{ave} which are consistent with the given information, p, with the understanding that consistency is a matter of degree.***

CONTINUED

Problem (b)

- Given information, p: Probably John is tall. Version 1. Question, q_1: What is the probability that John is short? Version 2. q_2: What is the probability that John is very short? Version 3. q_3: What is the probability that John is not very tall?

CONTINUED

- ***Solution of Problem (b).***

  Since the given information, p, is the same for Versions 1, 2 and 3. It is expedient to begin with precisiation of p. To this end, it is helpful to rewrite p as:

  Prob(Height(John) is tall) is probable

  where Height(John) is a random variable and Prob is the probability measure of the fuzzy set tall.

CONTINUED

- The imprecise term "probable" is a label of a fuzzy set which plays the role of the possibility distribution of Height(John). Let p_H be the probability density function of Height(John). In terms of p_H, the probability measure of the fuzzy set tall may be expressed as (Zadeh 1968):

$$\int_R \mu_{tall}(u)\, p_H(u)\, du$$

where R is the real line.

CONTINUED

Consequently, p may be precisiated as:

$$\int_R \mu_{tall}(u)p_H(u)du \text{ is probable}$$

with probable playing the role of a fuzzy restriction on the probability measure of tall. Equivalently, probable may be interpreted as the possibility distribution of the probability measure of tall.

- *What can be concluded at this point is that translation of p into a mathematical language may be expressed as:*

CONTINUED

Probably John is tall $\longrightarrow$

$$\int_R \mu_{tall}(u)p_H(u)du$$ *is probable*

or equivalently as:

$$\mu(p_H) = \mu_{probable}\left(\int_R \mu_{tall}(u)p_H(u)du\right)$$

NOTE

Note that for given $\mu_{probable}$, μ_{tall} and p_H, the right-hand side of the above expression evaluates to a number in the interval [0, 1] which may be interpreted as the possibility of p_H or, equivalently, the truth-value of p given $\mu_{probable}$, μ_{tall} and p_H. It is important to observe that the expression:

$$\mu(p_H) = \mu_{probable}\left(\int_R \mu_{tall}(u)p_H(u)du\right)$$

CONTINUED

plays the role of a semantic deep structure (generalized intension) of p. (Zadeh 2011) An abstracted version of the generalized intension—the protoform of p—reads:

$$\mu(p) = \mu_A\left(\int_R \mu_B(u)p(u)du\right)$$

CONTINUED

Step 2. Precisiation of questions

We assume that short, very short and not very tall, are fuzzy sets with specified membership functions μ_s, μ_{vs} and μ_{nvt}, respectively. Using the technique employed for precisiation of p, precisiations of questions may be expressed as follows.

CONTINUED

Version 1.

- **What is the probability that John is short?** $\longrightarrow$

$$?\, P_S = \int_R \mu_S(u)\, p_H(u)\, du$$

Version 2.

- **What is the probability that John is very short?** $\longrightarrow$

$$?\, P_{VS} = \int_R \mu_{VS}(u)\, p_H(u)\, du$$

CONTINUED

Version 3.

- What is the probability that John is not very tall? $\longrightarrow$

$$?P_{nvt} = \int_R \mu_{nvt}(u)p_H(u)du$$

where P_s, P_{vs} and P_{nvt}, are the probabilities that John is short, John is very short and John is not very tall, respectively.

CONTINUED

Step 3. Computation of answer(s) to question(s).

In the case of Version 1 and Version 2, there is a shortcut. For these versions, it is expedient to compute the probability that John is not tall. The membership function of not tall, μ_{nt}, is related to that of tall, μ_t, by:

$$\mu_{nt} = 1 - \mu_t$$

CONTINUED

- *Consequently, the probability of the event "John is not tall," may be expressed as:*

Prob(Height(John) is not tall)

$$= \int_R \mu_{nt}(u) p_H(u) du$$
$$= \int_R (1 - \mu_t) p_H(u) du$$
$$= 1 - \int_R \mu_t(u) p_H(u) du$$
$$= 1 - probable$$

CONTINUED

- ***The fuzzy set short is a subset of not tall. Consequently, the probability measure of short is less than or equal to the probability measure of not tall. From this it follows that:***

$$P_s \text{ is } \leq (1\text{-probable})$$

Note that 1-probable is a fuzzy set and that ≤ (1-probable) is the composition of ≤ and 1-probable. The expression for P_s is the answer to the question in Version 1.

CONTINUED

- *In Version 2, very short, like short, is a subset of not tall. Consequently, as in Version 1:*

$$P_{vs} \text{ is } \leq (1\text{-probable})$$

It may appear to be counterintuitive that P_s and P_{vs} are identical. The reason for identity is that P_s and P_{vs} are not the values but fuzzy restrictions on the values of the probability measures of short and very short. In fact, the answer, ≤ (1-probable), is the same for all subsets of not tall.

CONTINUED

Version 3.

- ***The shortcut does not work in the case of Version 3 since not very tall may not be a subset of not tall. What has to be employed in Version 3 is a version of the extension principle.***

- ***What we have at this point are precisiations of the question, q_3, and the given information, p.***

CONTINUED

More concretely, we have

$$? P_{nvt} = \int_R \mu_{nvt}(u) p_H(u) du$$

and

$$\int_R \mu_t(u) p_H(u) du$$ is probable,

respectively. Applying the extension principle, the answer to the question may be expressed as a fuzzy probability which is the solution to the variational problem:

CONTINUED

$$\mu_{P_{nvt}}(v) = \sup_{p_H} \mu_{probable}\left(\int_R \mu_t(u)p_H(u)du\right)$$

subject to

$$v = \int_R \mu_{nvt}(u)p_H(u)du$$

$$\int_R p_H(u)du = 1$$

CONCLUDING REMARK

- *In conclusion, as in the case of Problem (a), solution of Version 3 of Problem (b) reduces to the solution of a variational problem. Problems (a) and (b) are simple to state but not easy to solve. This is typical of mathematical solutions of CNL problems.*

NOTE

- *It should be observed that solutions of Problems (a) and (b) involve concepts and techniques which are not a part of mainstream mathematics. This applies, in particular, to the formalism of precisiation of meaning.*

CONTINUED

- *It is of interest to note that the problem of precisiation—in the sense employed in my lecture—is not addressed in the literature of semantics of natural languages. In a general setting, precisiation of propositions drawn from a natural language is discussed in the following section.*

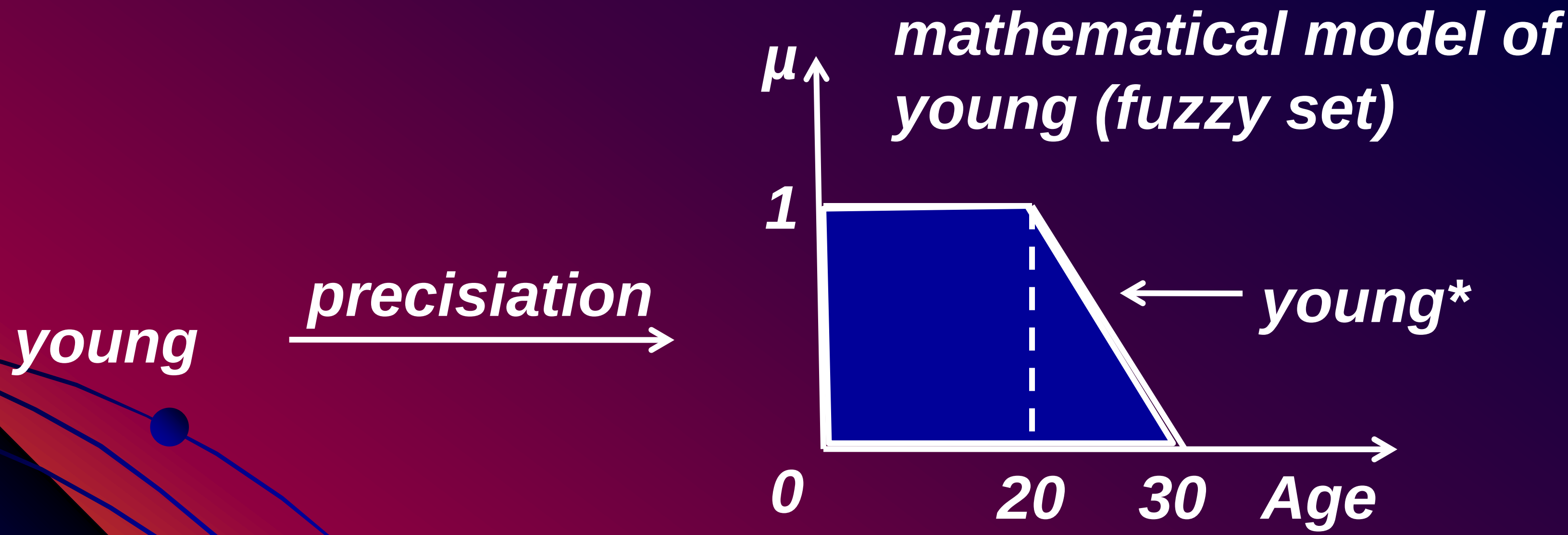
PHASE 1—PRECISIATION

Restriction-based Semantics (RS)

PRECISIATION OF MEANING

- Basically, precisiation involves construction of mathematical models of words, phrases, propositions, questions and other types of semantic entities.
- Precisiation of meaning goes beyond representation of meaning.
- Understanding of meaning is a prerequisite to precisiation of meaning.

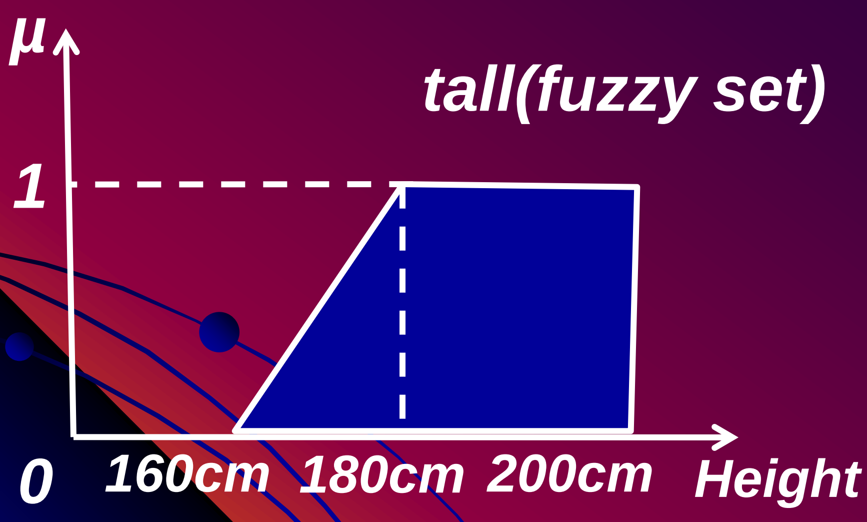
SIMPLE EXAMPLES OF PRECISIATION OF WORDS

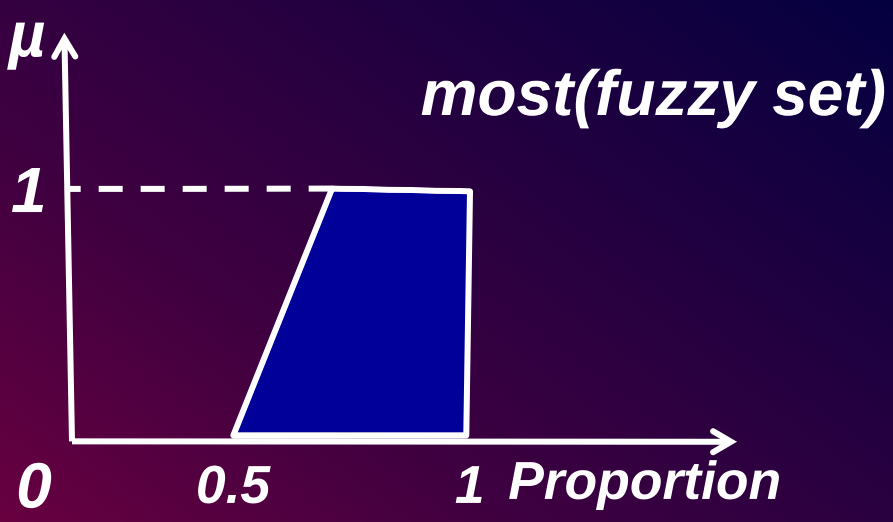


Note: Parameters of the trapezoidal membership function are context-dependent

CONTINUED

- *Most Swedes are tall* *precisiation* → *Proportion(tall.Swedes/Swedes) is most*





- *Precisiation of words is followed by composition.*

PRECISIATION OF PROPOSITIONS—A KEY IDEA

- ***As was stated earlier, a key idea in CW involves representation of the meaning of p as a restriction, R(X), of the form:***

$$p \longrightarrow R(X): \; X \text{ isr } R$$

R(X) restricts the values which X can take.

RESTRICTIONS

- *There are many different kinds of restrictions. The principal restrictions are: possibilistic (r=blank); probabilistic (r=p) and combinations of possibilistic and probabilistic restrictions. A special case of a combination of possibilistic and probabilistic restrictions is a Z-restriction (r=z).*

EXAMPLES OF RESTRICTIONS

- Possibilistic (fuzzy) restriction (r=blank):

$$R(X): \quad X \text{ is } A$$

where A is a fuzzy set in U with the membership function μ_A. A plays the role of the possibility distribution of X,

$$Poss(X=u) = \mu_A(u)$$

PROBABILISTIC RESTRICTION

- ***Probabilistic restriction (r=p):***

R(X): X isp P

where P plays the role of the probability distribution of X,

$$Prob(u \leq X \leq u+du) = p(u)du$$

where p is the probability density function of X.

Z-RESTRICTION

- *Z-restriction (r=z):*

  R(X): X isz Z

  where Z is a combination of possibilistic and probabilistic restrictions defined by:

  Prob(X is A) is B

  in which A and B are fuzzy sets. Usually, A and B are labels drawn from a natural language. The ordered pair, (A,B) is referred to as a Z-number.

EXAMPLES

- Usually temperature is low $\longrightarrow$ temperature isz (low, usually)
- Probably John is tall $\longrightarrow$ Height(John) isz (tall, probable).
- Important note

Usually X is A

where A is a fuzzy set, is a Z-restriction

DIRECT AND INDIRECT RESTRICTIONS

- A restriction is direct if it is of the form:

  R(X): X isr R

- A restriction is indirect if it is of the form:

  R(X): f(X) isr R

where f is a specified function or functional.

EXAMPLE OF INDIRECT RESTRICTION

$$R(X): \int_R \mu(u)p(u)du$$ is likely

Is an indirect restriction on p.

- Note: The term "restriction" is sometimes applied to R.

REPRESENTATIONS OF PROPOSITIONS AS POSSIBILISTIC RESTRICTIONS—SIMPLE EXAMPLES

- *p: Robert is young ⟶ Age(Robert) is young*

  X = Age(Robert), *R* = young

- *p: Most Swedes are tall ⟶*

  Proportion(tall Swedes/Swedes) is most

  X = Proportion(tall Swedes/Swedes), *R* = most

- *p: Probably John is tall ⟶*

  (Prob(Height(John)) is tall) is probable

  X = Height(John), *R* = probable

NOTE

- *X need not be a scalar variable.*

Example:

- *p: Robert gave a ring to Anne, X may be represented as the 3-tuple (Giver, Recipient, Object), with the corresponding values of R being (Robert, Anne, Ring).*

NOTE

- *A semantic network may be viewed as a canonical form of p, with X as an n-ary variable.*

RESTRICTION-BASED SEMANTICS

- *In CW, precisiation of propositions is carried out through the use of what is referred to as **restriction-based semantics**, RS.*
- *The point of departure in restriction-based semantics is the equality:*

Proposition = Restriction

- *More concretely,*

$$p \xrightarrow{\text{precisiation}} X \text{ isr } R$$

KEY IDEA

- ***Given p, three basic questions arise:***

1. ***What is the variable, X, which is restricted?***
2. ***What is the restricting relation, R?***
3. ***How does R restrict X?***

- ***The answers to these questions define the meaning of p.***

KEY POINTS

- Typically, in the case of propositions drawn from a natural language, X and R are implicit (hidden).
- Generally, X and R are identified by inspection. Choice of X is influenced by world knowledge. R depends on X.

NOTES

- *Restriction-based semantics, RS, is rooted in test-score semantics (Zadeh 1981)*

- *Restriction-based semantics may be viewed as a generalization of traditional approaches to semantics of natural languages—mainly possible-world semantics and truth-conditional semantics.*

NOTES

- *Restriction-based semantics has a far greater precisiation capability than possible-world and truth-conditional semantics.*
- *Equating a proposition drawn from a natural language to a restriction bridges the divide between linguistics and mathematics.*
- *The basics of restriction-based semantics are described in the following.*

CANONICAL FORM of p: CF(p)

- When the meaning of p is represented as a restriction, the restriction is referred to as the canonical form of p, CF(p). In the case of direct restrictions,

CF(p): X isr R

- The concept of a canonical form of p has a position of centrality in precisiation of meaning of p.

NOTE

- *It is important to note that representing a proposition as a restriction is greatly facilitated by the fact that restrictions in a natural language are predominantly possibilistic. Possibilistic restrictions are easiest to compute with.*

THE CONCEPT OF AN EXPLANATORY DATABASE

- Constructing the canonical form of p is merely a first step in precisiation of p, since X and R are expressed in a natural language and hence require precisiation. What is needed for precisiation of X and R is the concept of an explanatory database.

THE CONCEPT OF AN EXPLANATORY DATABASE (ED)

- *In restriction-based semantics, the concept of an explanatory database, ED, serves as a basis for precisiation of meaning of p. (Zadeh 1984) More concretely, ED is a collection of relations, with the names of relations drawn, but not exclusively, from the constituents of p. Instantiated ED is denoted as ED^+.*

CONTINUED

- ***Basically, ED may be viewed as the information which is needed to define X and R. Alternatively, ED may be viewed as the information which is needed to assess the truth-value of p.***

  Example. For the proposition, p: Most Swedes are tall, ED may be represented as:

CONTINUED

ED=POPULATION.SWEDES[Name; Height]+TALL[Height;μ]+ MOST[Proportion;μ],

where + plays the role of the comma.

- ***Note: For Problem (a) the explanatory database may be represented more simply by surpressing the arguments:***

$$ED=h+\mu_{tall}+\mu_{most}$$

CONTINUED

- ***Similarly, for Problem (b), the explanatory database may be represented more simply as:***

$$ED=p_H+\mu_{tall}+\mu_{probable}$$

CONTINUED

- *It is important to underscore that precisiation of X and R is needed because typically X and R are described in a natural language.*
- *In relation to possible-world semantics, ED^+ may be viewed as the description of a possible world.*

THE CONCEPT OF A PRECISIATED CANONICAL FORM, CF(p)*

- ***After X and R have been identified and the explanatory database, ED, has been constructed, X and R may be defined as functions of ED. As was noted earlier, definition of X and R may be viewed as precisiation of X and R. Precisiated X and R are denoted as X* and R*, respectively.***

CONTINUED

- *A canonical form, CF*(p), with precisiated values of X and R, X* and R*, will be referred to as a precisiated canonical form.*
- *In RS, the precisiated canonical form of p is equated to the meaning of p.*
- *In the following, construction of the precisiated canonical form of p is discussed in greater detail.*

FROM p TO CF*(p): X* isr R*

MEANING = PRECISIATED CANONICAL FORM

- ***The concepts discussed so far provide a basis for a relatively straightforward procedure for constructing the precisiated canonical form of a given proposition, p. The precisiated canonical form may be viewed as a mathematical model of p. Effectively, the precisiated canonical form may be interpreted as a representation of precisiated meaning of p.***

- ***A summary of the procedure for computing the precisiated canonical form of p is presented in the following.***

A PRELIMINARY STEP—CLARIFICATION OF MEANING

- ***A preliminary step is that of clarification, if needed, of the meaning of the given proposition. This step requires world knowledge.***

Examples:

- ***Overeating causes obesity*** ***clarification*** → ***Most of those who overeat are obese.***

- ***Obesity is caused by overeating*** ***clarification*** → ***Most of those who are obese, overeat.***

CONTINUED

- *Young men like young women* —clarification→ *Most young men like mostly young women.*

- *Swedes are much taller than Italians*

  —clarification1→ *Most Swedes are much taller than most Italians.*

  —clarification2→ *The average height of Swedes is much greater than the average height of Italians.*

CONTINUED

- ***Clarification (disambiguation) of a predicate.***

Most tall Swedes

clarification1 → ***Mostly tall Swedes***

clarification2 → ***Most of tall Swedes***

SUMMARY OF PRECISIATION PROCEDURE

- ***Step 1. Identification (explicitation) of X and R.***

  Identify the restricted (focal) variable, X, and the corresponding restricting relation, R. R depends on X. Generally, X and R are identified by inspection, based on world knowledge.

CONSTRUCTION OF ED

- ***Step 2. Construction of ED.***

  What information is needed to precisiate (define) X and R? An answer to this question identifies the explanatory database, ED. Equivalently—as was noted earlier—ED may be viewed as an answer to the question: What information is needed to compute the truth-value of p?

PRECISIATION OF X AND R

- ***Step 3. Precisiation of X and R.***

  How can the information in ED be used to precisiate the values of X and R? This step leads to precisiated values of X and R, X* and R*, and thus results in the precisiated canonical form, CF*(p). X* and R* are constructed by inspection.

- ***Precisiated X* and R* are expressed as functions of ED.***

- ***More concretely, X* and R* may be expressed as:***

$$X^* = f(ED),\ R^* = g(ED)$$

CONTINUED

- ***The precisiated canonical form, CF*(p), may be expressed as:***

$$CF^*(p) = X^* \text{ isr } R^*$$

- ***The precisiated canonical form may be viewed as the precisiated meaning of p.***
- ***The precisiated canonical form is a restriction on ED.***
- ***The precisiated canonical form is the generalized intension of p.***

A HELPFUL RETROSPECTIVE SUMMARY OF PROBLEMS (a) AND (b)

- *Problem (a)*

  Given information, I, p: Most Swedes are tall

  Question, q: What is the average height of Swedes?

- *Precisiation of question:* $q \xrightarrow{\text{precisiation}}$

$$? h_{ave} = \int_{h\ min}^{h\ max} u h(u) du$$

CONTINUED

- *Precisiation of given information, p.*
- *X = Prop(tall.Swedes/Swedes)*

  R= most
- *Canonical form, CF(p): Prop(tall.Swedes/Swedes) is most*
- *Explanatory Database, ED:* $h+\mu_{tall}+\mu_{most}$
- *Precisiation of X and R*

$$\bullet\ X^* = \int_{h\ min}^{h\ max} \mu_{tall}(u)h(u)du$$

CONTINUED

$R^* = most(\mu_{most})$

- Precisiated canonical form, CF*(p):

$$\int_{h\,min}^{h\,max} \mu_{tall}(u)h(u)du \text{ is most}$$

- Generalized intension, μ(p):

$$\mu_p = \mu_{most}\left(\int_{h\,min}^{h\,max} \mu_{tall}(u)h(u)du\right)$$

CONTINUED

- ***Problem (b), Version 1***

  Given information, I, p: Probably John is tall

  Question, q: What is the probability that John is short?

- ***Precisiation of question: q*** $\xrightarrow{\textit{precisiation}}$

$$?P_s = \int_R \mu_{tall}(u)p_H(u)du$$

where P_s is the probability that John is short.

CONTINUED

- *Precisiation of given information, p:*
- *X= Prob(Height(John) is tall)*

  R= probable
- *Canonical form, CF(p):*
  Prob(Height(John) is tall) is probable
- *Explanatory Database, ED:*
  $p_H + \mu_{tall} + \mu_{probable}$
- *Precisiation of X and R*

$$X^* = \int_R \mu_{tall}(u) p_H(u) du$$

CONTINUED

$R^* = probable(\mu_{probable})$

- Precisiated canonical form, CF*(p):

$$\int_R \mu_{tall}(u)p_H(u)du \text{ is probable}$$

- Generalized intension, μ(p):

$$\mu(p) = \mu_{probable}\left(\int_R \mu_{tall}(u)p_H(u)du\right)$$

IMPORTANT POINTS

- The canonical form X* isr R* induces a restriction on ED which plays the role of the generalized intension of p, μ(p).
- The generalized intension of p may be viewed as a representation of a deep semantic structure of p.

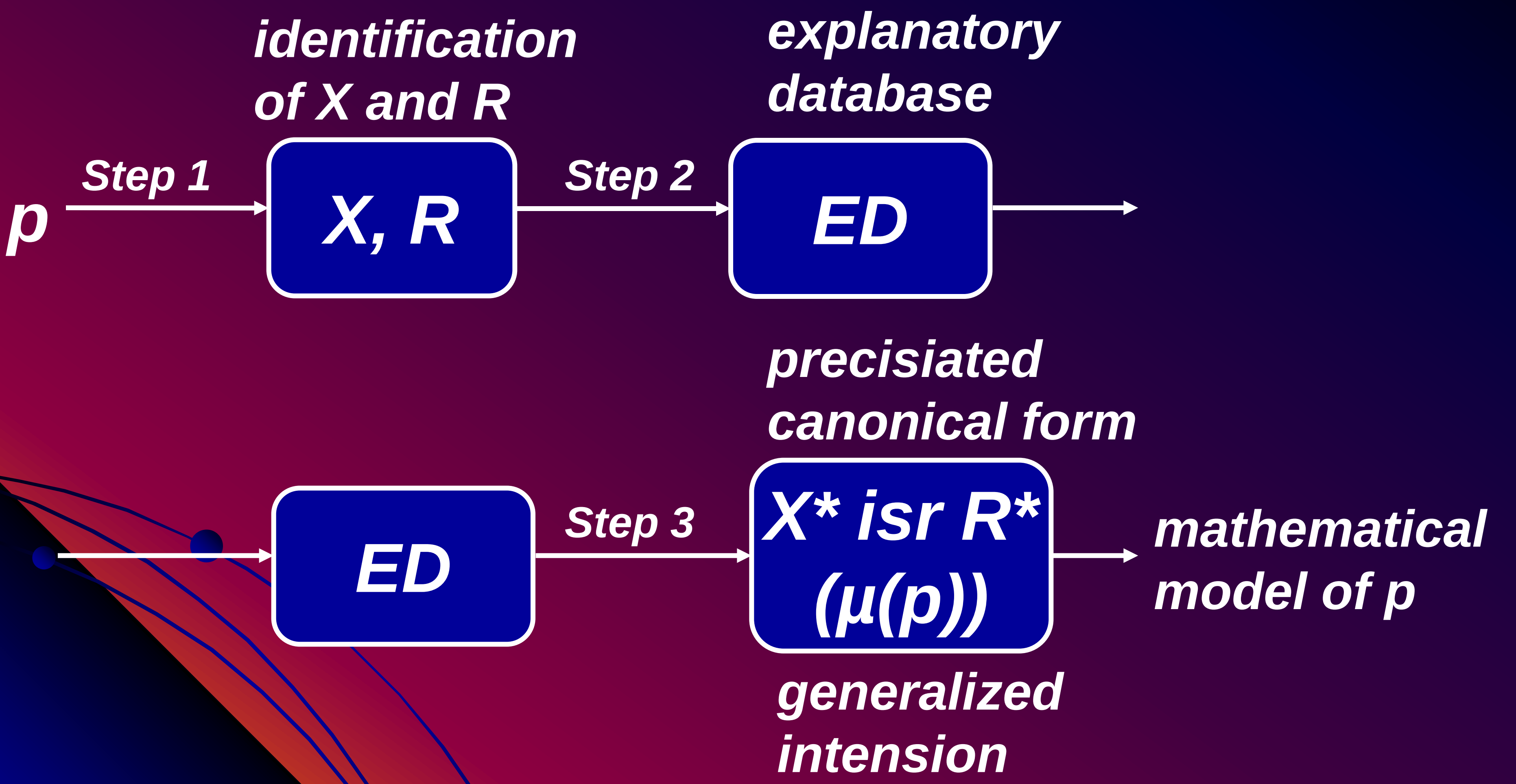
IMPORTANT POINTS

- *μ(p) may be interpreted as the possibility distribution of ED given p.*

- *Equivalently, μ(p) may be interpreted as a function from ED to truth-values of p. More concretely, we can write:*

$$Poss(ED^{+}|p)=Tr(p|ED^{+})$$

SUMMARY OF PROCEDURE



BASIC STRUCTURE OF CNL PROBLEMS

- Given information, I/NL $\xrightarrow[\text{RS}]{\text{precisiation}}$ f(X) isr R
- Question, q/NL $\xrightarrow[\text{RS}]{\text{precisiation}}$ g(X) iss ?S

$$\left.\begin{array}{l} f(X) \text{ isr } R \\ g(X) \text{ iss } ?S \end{array}\right\} \text{extension principle} \xrightarrow[\text{RS}]{\text{computation}} S$$

CONCLUDING REMARK

- *In RS, the precisiated meaning of a proposition, p, is equated to its precisiated canonical form, X* isr R*, with the understanding that X* isr R* is described as a procedure.*
- *The precisiated canonical form translates into a restriction on the explanatory database, ED. The generalized intension, μ(p), plays the role of a mathematical model of p.*

CONTINUED

- *Representation of p as a restriction is the centerpiece of restriction-based semantics, RS. Through precisiation of meaning, RS opens the door to a wide-ranging enlargement of the role of natural languages in scientific theories and engineering systems. Importantly, RS plays a pivotal role in empowering mathematics to construct mathematical solutions to computational problems which are stated in a natural language.*

PHASE 2— COMPUTATION

COMPUTATION WITH RESTRICTIONS

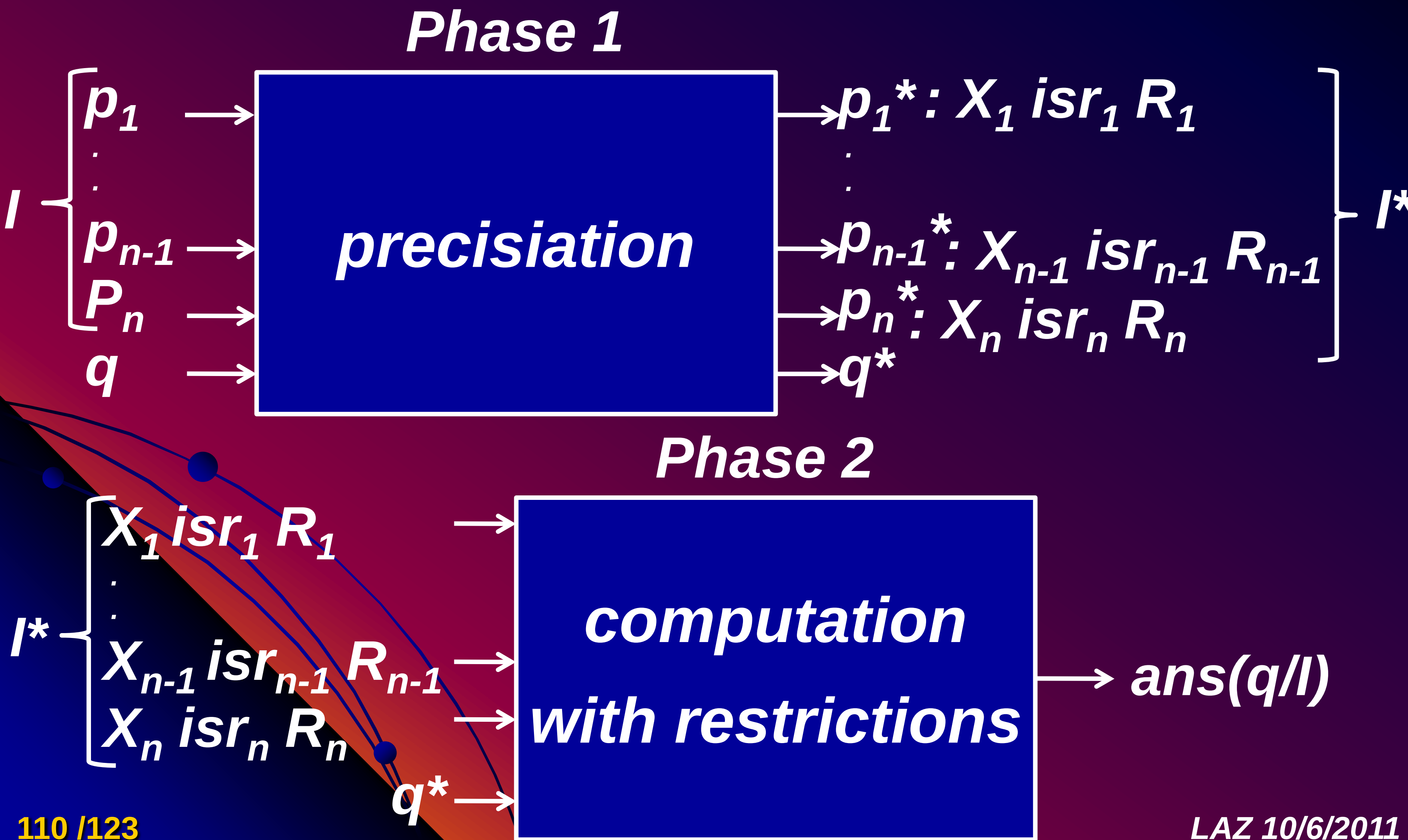
- *In CW, through representation of the meaning of a proposition as a restriction, the problem of computation with information described in natural language reduces to the problem of computation with restrictions. As was noted earlier, in the realm of natural languages, restrictions are for the most part possibilistic.*

CONTINUED

For this reason, in the following attention is focused on computation with possibilistic restrictions.

FROM PRECISIATION TO COMPUTATION

(I=GIVEN INFORMATION=COLLECTION OF PROPOSITIONS)



CONTINUED

- *In large measure, computation with restrictions (generalized constraints) involves the use of rules which govern propagation and counterpropagation of restrictions (Calculus of fuzzy restrictions, Zadeh 1974). Among such rules, the principal rule is the extension principle (Zadeh 1965, 1975 a, b and c, 2011).*

CONTINUED

- ***There are many versions of the extension principle. Let Y be a function of $X_1, \ldots, X_n$, $Y=f(X_1, \ldots, X_n)$. Basically, an extension principle is a rule which governs computation of the restriction on Y, R(Y) given restrictions on $X_1, \ldots, X_n$.***

EXTENSION PRINCIPLE (POSSIBILISTIC)

- X is a variable which takes values in U, and g is a function from U to V, Y=g(X). The point of departure is a possibilistic restriction on f(X) expressed as f(X) is A, where A is a fuzzy set in V which is defined by its membership function $\mu_A(v)$, $v \in V$.

- f is a function from U to W. The possibilistic restriction on f(X) induces a possibilistic restriction on g(X) which may be expressed as g(X) is ?B, where B is a fuzzy set in W. The question is: What is B? In symbols,

CONTINUED

$$\frac{f(X) \text{ is } A}{g(X) \text{ is } ?B}$$

The answer to this question is the solution of a variational problem expressed as:

$$\mu_B(w) = \sup_u \mu_A(f(u))$$

subject to

$$w = g(u)$$

where μ_A and μ_B are the membership functions of A and B, respectively.

CONTINUED

- ***Equivalently, as noted earlier (Problem (a)), the possibilistic extension principle may be expressed as:***

?Y=g(X)
f(X) is A

$$\mu_Y(v) = sup_u \, \mu_A(f(u))$$

subject to

$$v = g(u)$$

NOTE

- *In more general settings, the extension principle is concerned with propagation of many different kinds of restrictions. Schematically,:*

$$Y = g(X)$$

- *It is of interest to note that in many practical applications of fuzzy logic what is restricted is g.*

CONTINUED

- *A simple example of a restriction on g is expressed as:*

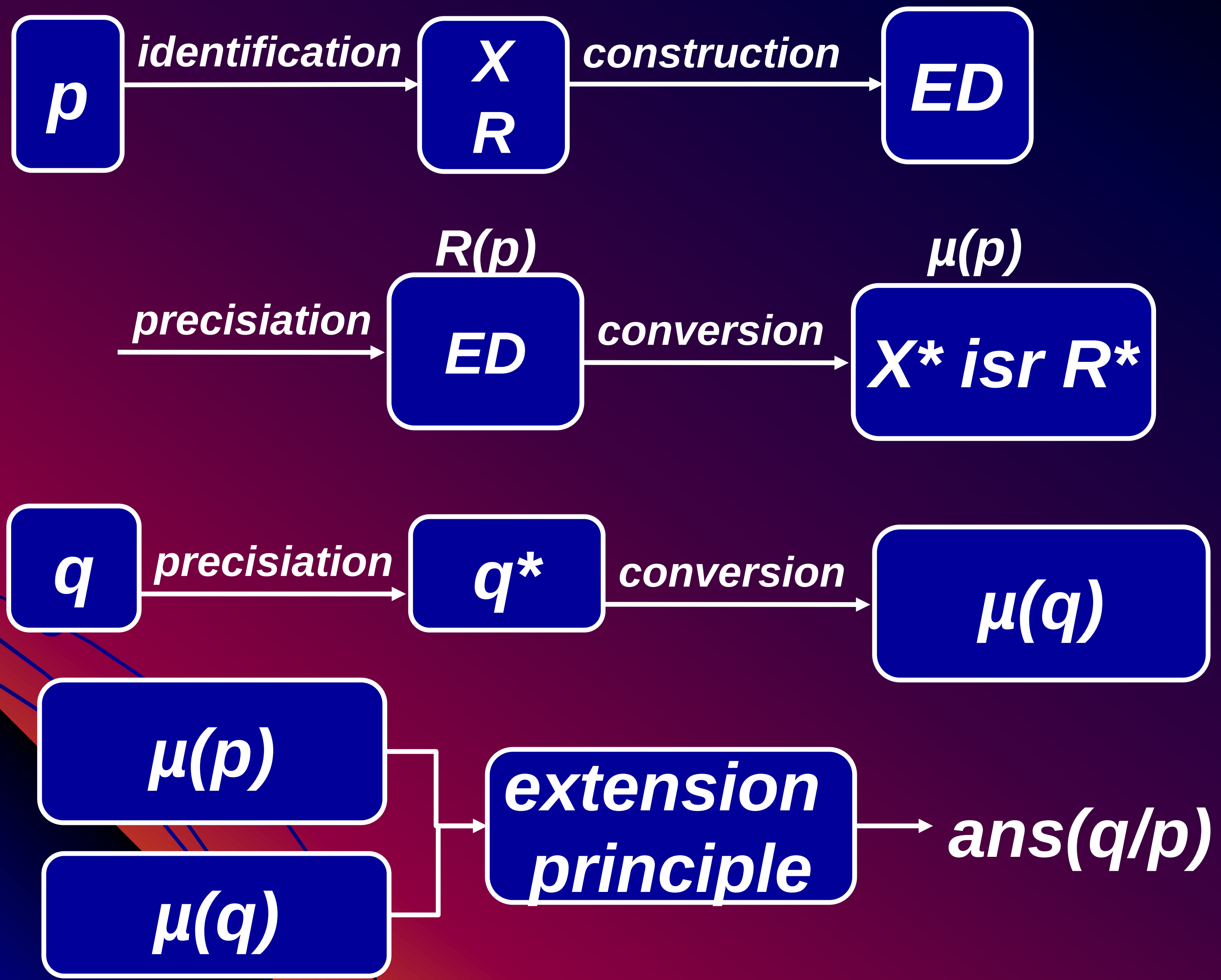
R(g): if X is small then Y is small
if X is medium then Y is large
if X is large then Y is small

- *Rationale: Precision carries a cost*

A SUMMARY OF COMPUTATION OF ans(q/p)

- ***For convenience, in the following the given information, is represented as a composite proposition, $p=(p_1, \ldots, p_n)$.***

CW—BASIC COMPUTATIONAL PROCESS



CONCLUDING REMARK

- ***Traditionally, computational problems which are stated in a natural language (CNL problems) are viewed as problems which do not lie within the province of mathematics.***
- ***The approach which is outlined in this lecture challenges this tradition by opening the door to construction of mathematical solutions of CNL problems.***

CONTINUED

- *The approach employs unconventional concepts and techniques drawn from the fuzzy-logic-based formalism of computing with words (CW).*
- *The importance of the capability to construct mathematical solutions of CNL problems derives from the fact that much of human knowledge, and particularly world knowledge, is described in natural language.*

SUMMATION

- *In large measure, traditional mathematics is based on bivalent logic.*
- *Bivalent-logic-based formalisms cannot deal with CNL problems.*
- *Moving from bivalent logic to multi-valued logic is a step in the right direction, but it is not sufficient.*
- *To deal with CNL problems what are needed are formalisms based on fuzzy logic.*

CONTINUED

- *The fuzzy-logic-based formalism which is described in this lecture opens the door to construction of mathematical solutions of CNL problems.*
- *Construction of mathematical solutions of CNL problems will grow in visibility and importance as we move further into the age of automation of everyday reasoning and decision-making.*