FUZZY LOGIC AND COMPUTING WITH WORDS—A NEW LOOK

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WHAT IS FUZZY LOGIC?

- There are many misconceptions about fuzzy logic. To begin with, fuzzy logic is not fuzzy. Like traditional logical systems, fuzzy logic is precise. In large measure, fuzzy logic is designed to address an important class of problems which are not addressed by traditional logical systems—problems in which the central issues relate to imprecision, uncertainty, incompleteness of information, unreliability and partiality of truth.
WHAT IS FUZZY LOGIC?

- The importance of fuzzy logic derives from the fact that in much of the real world such problems are the norm rather than exception. Here are a few examples of simple problems which are not addressed by traditional logical systems.
CONTINUED

Most Swedes are tall

Most tall Swedes are blond

What fraction of Swedes are blond?

Most Swedes are tall

What is the average height of Swedes?

Most Swedes are tall

What is the truth value of "Many Swedes are not tall"?
X is the value of a real-valued variable. What is known about X is: (a) X is larger than approximately a; (b) X is smaller than approximately b. What is the probability that X is approximately c?

f is a function from reals to reals, Y=f(X). A linguistic summary of f is described as a collection of fuzzy if-then rules:

if X is small then Y is small
if X is medium then Y is large
if X is large then Y is small

What is the value of Y if X is larger than approximately a and smaller than approximately b?
f is a function from reals to reals which is described as a collection of fuzzy if-then rules:

- if X is small then usually (Y is small)
- if X is medium then usually (Y is large)
- if X is large then usually (Y is small)

What is the value of Y if usually (X is medium)?

- Computing with Words (CW or CWW or CWP) is a branch of fuzzy logic which is designed to address problems of this kind.
- The objects of computation/deduction in CW are propositions, predicates, questions, commands, etc which are described in natural language.
WHAT IS FUZZY LOGIC?--A NEW LOOK
FUZZY LOGIC—A NEW LOOK

- Fuzzy logic is a precise system of reasoning, deduction and computation in which the objects of discourse and analysis are associated with information which is, or is allowed to be, imprecise, uncertain, incomplete, unreliable, partially true or partially possible.

- The importance of fuzzy logic derives from the fact that in much of the real world such information is the norm rather than exception. Fuzzy logic is designed to deal with problems which are avoided by traditional logical systems.
Numbers are respected, words are not
THE PARADIGM SHIFT

nontraditional

(a) 

NL

unprecisiated words

progression

PNL

precisiated words

countertraditional

(b) 

numbers

progression summarization

PNL

precisiated words
Mechanization of linguistic summarization requires a paradigm shift
CLARIFICATION

- **Fuzzy logic, does not replace formalisms based on bivalent logic with formalisms based on fuzzy logic.**

- **Fuzzy logic adds to and generalizes formalisms based on bivalent logic.**
The new look is intended to clarify what fuzzy logic is and what it has to offer.

In the new look of fuzzy logic, the concept of cointensive precisiation plays a pivotal role.

One of the principal contributions of fuzzy logic is its high power of cointensive precisiation.
The point of departure in fuzzy logic—the center of fuzzy logic (FL)—is the concept of a fuzzy set.
(a): A set may be viewed as a special case of a fuzzy set. A fuzzy set is not a set;
(b) Basic attributes of a set/fuzzy set are boundary and measure (cardinality, count, volume)

- Fuzzy set theory is boundary-oriented
- Probability theory is measure-oriented
- Fuzzy logic is both boundary-oriented and measure-oriented
CONTINUED

- A set, A, in U is a class with a crisp boundary.
- A set is precisiated through association with a characteristic function \( c_A : U \rightarrow \{0,1\} \)
- A fuzzy set is precisiated through graduation, that is, through association with a membership function \( \mu_A : U \rightarrow [0,1] \), with \( \mu_A(u), u \in U \), representing the grade of membership of u in A.
Membership in a fuzzy set is a matter of degree.

In fuzzy logic everything is or is allowed to be a matter of degree.
THE CONCEPT OF GRADUATION

- Graduation of a fuzzy concept or a fuzzy set, A, serves as a means of precisiation of A.

Examples

- Graduation of middle-age
- Graduation of the concept of earthquake via the Richter Scale
- Graduation of recession?
- Graduation of civil war?
- Graduation of mountain?
EXAMPLE—MIDDLE-AGE

- Imprecision of meaning = elasticity of meaning
- Elasticity of meaning = fuzziness of meaning
Control Rules:

1. If (speed is low) and (shift is high) then (-3)
2. If (speed is high) and (shift is low) then (+3)
3. If (throt is low) and (speed is high) then (+3)
4. If (throt is low) and (speed is low) then (+1)
5. If (throt is high) and (speed is high) then (-1)
6. If (throt is high) and (speed is low) then (-3)
The concept of FL-Generalization plays a pivotal role in fuzzy logic, its generation and its applications.

- $T$ denotes a bivalent-logic-based theory, formalism, algorithm or concept.
- Fuzzy $T$ is an FL-Generalized $T$. Examples: fuzzy set theory, fuzzy topology, fuzzy measure theory, fuzzy game theory, fuzzy control, etc.
CONTINUED

- FL-Generalization applied to T involves (a) adding to T concepts and techniques drawn from fuzzy logic; (b) employing these concepts and techniques to generalize T, resulting in fuzzy T; and (c) adding FL-relevant concepts and techniques drawn from fuzzy T to FL.

- FL is generated by applying FL-Generalization to various T’s in succession, starting with the concept of a fuzzy set and set theory.
Application of FL-Generalization to set theory is followed by application of FL-Generalization to logic, to relations and to theories related to knowledge representation, information, probability theory and possibility theory. FL-Generalization is a continuing process through which forms the basis for generation of fuzzy logic and its principal facets.
Starting with the concept of a fuzzy set, successive application of FL-Generalization to related theories leads to the principal facets of fuzzy logic.
BASIC STRUCTURE OF FL

fuzzy logic

applied fuzzy logic
  relational facet
  epistemic facet

theoretical fuzzy logic
  set-theoretic facet
  logical facet
A facet of FL consists of FL-generalization of a theory or FL-generalization of a collection of related theories.

The principal facets of FL are: logical, FLl; set theoretic, FLs; epistemic, FLe; and relational, FLr.
PRINCIPAL FACETS OF FL

- The logical facet of FL, FLI, is fuzzy logic in its narrow sense. FLI may be viewed as a generalization of multivalued logic. The agenda of FLI is similar in spirit to the agenda of classical logic.

- The set-theoretic facet, FLs, is focused on FL-generalization of set theory. Historically, the theory of fuzzy sets (Zadeh 1965) preceded fuzzy logic (Zadeh 1975c). The theory of fuzzy sets may be viewed as an entry to generalizations of various branches of mathematics, among them fuzzy topology, fuzzy measure theory, fuzzy graph theory and fuzzy algebra.
The epistemic facet of FL, FL_e, is concerned in the main with knowledge representation, semantics of natural languages, possibility theory, fuzzy probability theory, granular computing and the computational theory of perceptions.

The relational facet, FL_r, is focused on fuzzy relations and, more generally, on fuzzy dependencies. The concept of a linguistic variable—and the associated calculi of fuzzy if-then rules—play pivotal roles in almost all applications of fuzzy logic.
NOTE—SPECIALIZATION VS. GENERALIZATION

- Consider a concatenation of two words, MX, with the prefix, M, playing the role of a modifier of the suffix, X, e.g., small box.
- Usually M specializes X, as in convex set.
- Unusually, M generalizes X. The prefix fuzzy falls into this category. Thus, fuzzy set generalizes the concept of a set. The same applies to fuzzy topology, fuzzy measure theory, fuzzy control, etc. Many misconceptions about fuzzy logic are rooted in misinterpretation of fuzzy as a specializer rather than a generalizer.
The cornerstones of fuzzy logic are: graduation, granulation, precisiation and the concept of a generalized constraint.
THE CONCEPT OF GRANULATION

● The concept of granulation is unique to fuzzy logic and plays a pivotal role in its applications. The concept of granulation is inspired by the way in which humans deal with imprecision, uncertainty and complexity.

● Granulation serves as a means of imprecisiation.
GRADUATION / GRANULATION

A

graduation/precisiation

A*

grainulation/imprecisiation

*A

- graduation = precisiation
- granulation = imprecisiation
- Forced: singular values of variables are not known.

- Deliberate: singular values of variables are known. There is a tolerance for imprecision. Precision carries a cost. Granular values are employed to reduce cost.
In fuzzy logic everything is or is allowed to be granulated. Granulation involves partitioning of an object, A, into granules. More generally, granulation involves an association with A of a system of granules. Informally, a granule is a clump of elements drawn together by indistinguishability, equivalence, similarity, proximity or functionality.

A granule, G, is precisiated through association with G of a generalized constraint.
EXAMPLE—GRANULATION

automobile engine
- Block
- Head
- Crank shaft
- Pistons
- Rings
- ...

human body
- Head
- Neck
- Chest
- Arms
- Hands
- ...

crisp boundaries

fuzzy boundaries
Drawing the boundary
1. ballpoint pen (bivalent logic)
2. chisel pen (intuitionistic logic)
3. spray pen with adjustable precisiated pattern (fuzzy logic)

Which is most realistic?
DIGRESSION—COINTENSIVE PRECISIATION

not hand ➔ hand

definitely not hand ➔ definitely hand

possibly

not hand ➔ hand

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A USEFUL ANALOGY

- In fuzzy logic the writing instrument is a spray pen with an adjustable and precisely known spray pattern. For purposes of binarization a colored line marks the center of gravity of the spray pattern.
Graduated granulation = fuzzy granulation
GRANULATION / SYSTEM OF GRANULES

- Graduated granulation = fuzzy granulation
EXAMPLE: GRANULATION OF AGE

Partition
- Age: young + middle-aged + old

System (Linguistic Variable)
- Age: young + middle-aged + old + very young + not very old + quite young + not very young and not very old + ...
**SINGULAR AND GRANULAR VALUES**

- Granular value of $X$:
  - Singularity
  - Universe of discourse

- Singular value of $X$:
  - 7.3% high
  - 0.8 high
  - 160/80 high

- Unemployment probability
- Blood pressure
GRANULAR PROBABILITIES

- A granular value is precisiated via a generalized constraint.
- A granular value of probability is a granular probability.

Examples:
Likely, not likely, unlikely, very likely, very unlikely, usually, low, high, etc.

- The concept of granular probability is more general than the concept of fuzzy probability.
Granulation may be applied to objects of arbitrarily complexity, in particular to variables, functions, relations, probability distributions, dynamical systems, etc.

Quantization is a special case of granulation.
Granulation may be viewed as a form of summarization/information compression.

Humans employ graduated granulation to deal with imprecision, uncertainty and complexity.

A linguistic variable is a granular variable with linguistic labels.
GRANULATION OF A FUNCTION
GRANULATION=SUMMARIZATION

if $X$ is small then $Y$ is small
if $X$ is medium then $Y$ is large
if $X$ is large then $Y$ is small

$f$: perception

$\star f$: summarization

if $X$ is small then $Y$ is small
if $X$ is medium then $Y$ is large
if $X$ is large then $Y$ is small
GRANULAR VS. GRANULE-VALUED DISTRIBUTIONS

\[ \text{distribution} \]

\[ P \]

\[ g(u): \text{probability density of } X \]

\[ 0 \]

\[ X \]

\[ A_1, A_2, A, A_n \]

\[ P_1, P_2, P, P_n \]

\[ \text{possibility distribution of probability distributions} \]

\[ \text{probability distribution of possibility distributions} \]

\[ p_1, \ldots, p_n \]

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**FUZZY LOGIC GAMBIT**

- Fuzzy Logic Gambit = deliberate granulation followed by graduation

- The Fuzzy Logic Gambit is employed in most of the applications of fuzzy logic in the realm of consumer products
COMPUTING WITH WORDS
Computing with Words relates to computation with information described in a natural language. More concretely, in CW the objects of computation are words, predicates or propositions drawn from a natural language. The importance of computing with words derives from the fact that much of human knowledge and especially world knowledge is described in natural language.
The point of departure in CW is the concept of an information set, \( I \), described in natural language (NL).

- \( I/\text{NL} \): information set
- \( q/\text{NL} \): question
- \( \text{ans}(q/I) \)

\( p_1, \ldots, p_n, p_{wk} \) are given propositions

\( p_{wk} \) is information drawn from world knowledge
EXAMPLE

- X is Vera’s age
  - $p_1$: Vera has a son in mid-twenties
  - $p_2$: Vera has a daughter in mid-thirties
  - $p_{wk}$: mother’s age at birth of her child is usually between approximately 20 and approximately 40

q: what is Vera’s age?
EXAMPLE

- **X is the value of a real-valued variable**
  - $p_1$: X is larger than approximately a
  - $p_2$: X is smaller than approximately b
  - q: What is the probability that X is approximately c?

- This simple example is intended to demonstrate that probability and possibility are distinct concepts.
Phase 1: Precisiation (prerequisite to computation)

$p_1 \rightarrow \cdots \rightarrow p_n \rightarrow p_{wk}$

$\pi^*$ is a generalized constraint

$p_1^* \rightarrow \cdots \rightarrow p_n^* \rightarrow p_{wk}^*$
Phase 2: Computation/Deduction

\[ p_1^*, \ldots, p_n^*, p_{wk}^* \rightarrow \text{generalized constraint propagation} \rightarrow \text{ans}(q/l) \]
PHASE 1
The Concepts of Precisiation and Cointensive Precisiation
In one form or another, precisiation of meaning has always played an important role in science. Mathematics is a quintessential example of what may be called a meaning precisiation language system.
Note: A language system differs from a language in that in addition to descriptive capability, a language system has a deductive capability. For example, probability theory may be viewed as a precisiation language system; so is Prolog. A natural language is a language rather than a language system.

PNL is a language system.
SEMANTIC IMPRECISION (EXPLICIT)

EXAMPLES

WORDS/CONCEPTS
- Recession
- Civil war
- Very slow
- Honesty
- Arthritis
- High blood pressure
- Cluster
- Hot

PROPOSITIONS
- It is likely to be warm tomorrow.
- It is very unlikely that there will be a significant decrease in the price of oil in the near future.
CONTINUED

EXAMPLES

COMMANDS

- Slow down
- Slow down if foggy
- Park the car
SEMANTIC IMPRECISION (IMPLICIT)

EXAMPLES

- Speed limit is 65 mph
- Checkout time is 1 pm
Can you explain to me the meaning of “Speed limit is 65 mph?”

No imprecise numbers and no probabilities are allowed.

Imprecise numbers are allowed. No probabilities are allowed.

Imprecise numbers are allowed. Precise probabilities are allowed.

Imprecise numbers are allowed. Imprecise probabilities are allowed.
NECESSITY OF IMPRECISION

- Can you precisiate the meaning of “arthritis”? 
- Can you precisiate the meaning of “recession”? 
- Can you precisiate the meaning of “beyond reasonable doubt”? 
- Can you precisiate the meaning of “causality”? 
- Can you precisiate the meaning of “near”? 

The concept of precision has a position of centrality in scientific theories. And yet, there are some important aspects of this concept which have not been adequately treated in the literature. One such aspect relates to the distinction between precision of value (v-precision) and precision of meaning (m-precision).
The same distinction applies to imprecision, precisiation and imprecisiation.
• \( p: X \) is in the interval \([a, b]\). \( a \) and \( b \) are precisely defined real numbers
• \( p \) is \( v \)-imprecise and \( m \)-precise

• \( p: X \) is a Gaussian random variable with mean \( m \) and variance \( \sigma^2 \). \( m \) and \( \sigma^2 \) are precisely defined real numbers
• \( p \) is \( v \)-imprecise and \( m \)-precise
PRECISIATION AND IMPRECISIATION

- A proposition, predicate, query or command may be precisiated or imprecisiated

Examples

- Data compression and summarization are instances of imprecisiation
MODALITIES OF m-PRECISIATION

- **m-precisiation**
  - **mh-precisiation**
    - human-oriented
  - **mm-precisiation**
    - machine-oriented
      - (mathematically well-defined)

*Example: bear market
mh-precisiation: declining stock market with expectation of further decline

mm-precisiation: 30 percent decline after 50 days, or a 13 percent decline after 145 days. (Robert Shuster)
BASIC CONCEPTS

precisiand = model of meaning
precisiation ≈ modelization
intension = attribute-based meaning
cointension = measure of proximity of meanings
= measure of proximity of the model and the object of modelization

precisiation = translation into a precisiation language system
Cointension: qualitative measure of the proximity of precisian to precisien.

Cointensive precisiation: cointension of precisian is high.
Cointensive precisiation (fuzzy precisian, fuzzy precisian)

Achievement of cointension precisiation necessitates that if the precisian is fuzzy so must be the precisian.

Crisp definitions of fuzzy concepts is the norm in science. What is widely unrecognized is that crisp definitions of fuzzy concepts are generally not cointensive.

In fuzzy logic one writes with a spray pen which has an adjustable precisiated spray pattern.
Precisiation is a form of modelization.

- $mh$-precisianand $\approx h$-model

- $mm$-precisianand $\approx m$-model

- Nondeterminism of natural languages implies that a semantic entity, e.g., a proposition or a predicate has a multiplicity of models
MM-PRECISIATION OF “approximately a,” *a
(MODELS OF MEANING OF *a)

Bivalent Logic

number

interval

probability

It is a common practice to ignore imprecision, treating what is imprecise as if it were precise. LAZ 11/21/08
Fuzzy Logic: Bivalent Logic + ...

Fuzzy interval

Fuzzy interval type 2

Fuzzy probability

Fuzzy logic has a much higher expressive power than bivalent logic.
GOODNESS OF MODEL OF MEANING

goodness of model = (cointension, computational complexity)

*a: approximately a

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v-IMPRECISIATION

- Imperative (forced)
- Intentional (deliberate)

imperative: value is not known precisely
intentional: value need not be known precisely

- data compression and summarization are instances of v-imprecisiation
THE CONCEPT OF COINTENSIVE PRECISIATION

- m-precisiation of a concept or proposition, p, is cointensive if p* is cointensive with p.

Example: bear market

We classify a bear market as a 30 percent decline after 50 days, or a 13 percent decline after 145 days. (Robert Shuster)

This definition is clearly not cointensive
mm-PRECISIATION

Basic question

- Given a proposition, \( p \), how can \( p \) be cointesively mm-precisiated?

Key idea

- In generalized-constraint-based semantics, mm-precisiation is carried out through the use of the concept of a generalized constraint.
- What is a generalized constraint?
THE CONCEPT OF A GENERALIZED CONSTRAINT

A BRIEF INTRODUCTION
PREAMBLE

- The concept of a generalized constraint is the centerpiece of generalized-constraint-based semantics.

- In scientific theories, representation of constraints is generally oversimplified. Oversimplification of constraints is a necessity because bivalent-logic-based constraint definition languages have a very limited expressive power.
The concept of a generalized constraint is intended to provide a basis for construction of a maximally expressive meaning precisiation language for natural languages.

Generalized constraints have elasticity.

Elasticity of generalized constraints is a reflection of elasticity of meaning of words in a natural language.
GENERALIZED CONSTRAINT (Zadeh 1986)

- **Bivalent constraint (hard, inelastic, categorical):**
  \[ X \in C \]
  constraining bivalent relation

- **Generalized constraint on X: GC(X) (elastic)**

  - \( GC(X): X \text{ isr } R \)
  - constraining non-bivalent (fuzzy) relation
  - index of modality (defines semantics)
  - constrained variable

- \( r: \in | = | \leq | \geq | \subset | ... | \text{blank} | p | v | u | rs | fg | ps | ... \)
  - bivalent
  - primary

- **open GC(X): X is free (GC(X) is a predicate)**
- **closed GC(X): X is instantiated (GC(X) is a proposition)**
GENERALIZED CONSTRAINT—MODALITY r

\[ X \text{ is}_r R \]

- \( r: = \) equality constraint: \( X = R \) is abbreviation of \( X \text{ is}_= R \)
- \( r: \leq \) inequality constraint: \( X \leq R \)
- \( r: \subset \) subsethood constraint: \( X \subset R \)
- \( r: \text{ blank} \) possibilistic constraint; \( X \text{ is}_R R \); \( R \) is the possibility distribution of \( X \)
- \( r: v \) veristic constraint; \( X \text{ is}_v R \); \( R \) is the verity distribution of \( X \)
- \( r: p \) probabilistic constraint; \( X \text{ is}_p R \); \( R \) is the probability distribution of \( X \)

Standard constraints: bivalent possibilistic, bivalent veristic and probabilistic
PRIMARY GENERALIZED CONSTRAINTS

- Possibilistic: X is R
- Probabilistic: X isp R
- Veristic: X isv R

- Primary constraints are formalizations of three basic perceptions: (a) perception of possibility; (b) perception of likelihood; and (c) perception of truth

- In this perspective, probability may be viewed as an attribute of perception of likelihood
STANDARD CONSTRAINTS

- **Bivalent possibilistic**: $X \in C$ (crisp set)
- **Bivalent veristic**: $\text{Ver}(p)$ is true or false
- **Probabilistic**: $X$ is $p$ $R$
- **Standard constraints are instances of generalized constraints which underlie methods based on bivalent logic and probability theory**
GENERALIZED CONSTRAINT LANGUAGE (GCL)

- GCL is an abstract language
- GCL is generated by combination, qualification, propagation and counterpropagation of generalized constraints
- Examples of elements of GCL
  - X/Age(Monika) is R/young (annotated element)
  - (X isp R) and (X,Y) is S)
  - (X isr R) is unlikely) and (X iss S) is likely
  - If X is A then Y is B
- The language of fuzzy if-then rules is a sublanguage of GCL

- Deduction = generalized constraint propagation and counterpropagation
Meaning postulate

Equivalently, mm-precisiation of $p$ may be realized through translation of $p$ into GCL.
EXAMPLES: POSSIBILISTIC

annotation

- *Lily is young* → *Age (Lily) is young*

- *most Swedes are tall* → *Count (tall.Swedes/Swedes) is most*
PHASE 2
Computation
with
Precisiated Information
Representing the meaning of a proposition as a generalized constraint reduces the problem of computation with information described in natural language to the problem of computation with generalized constraints. In large measure, computation with generalized constraints involves the use of rules which govern propagation and counterpropagation of generalized constraints. Among such rules, the principal rule is the extension principle (Zadeh 1965, 1975).
EXTENSION PRINCIPLE (POSSIBILISTIC)

- $X$ is a variable which takes values in $U$, and $f$ is a function from $U$ to $V$. The point of departure is a possibilistic constraint on $f(X)$ expressed as $f(X)$ is $A$ where $A$ is a fuzzy relation in $V$ which is defined by its membership function $\mu_A(v), \quad v \in V$.

- $g$ is a function from $U$ to $W$. The possibilistic constraint on $f(X)$ induces a possibilistic constraint on $g(X)$ which may be expressed as $g(X)$ is $B$ where $B$ is a fuzzy relation. The question is: What is $B$?
CONTINUED

\[
\begin{align*}
\text{If } f(X) \text{ is } A & \\
g(X) \text{ is } ?B \\
\mu_B(w) &= \sup_u \mu_A(f(u)) \\
w &= g(u)
\end{align*}
\]

subject to

\[\mu_A \text{ and } \mu_B \text{ are the membership functions of } A \text{ and } B, \text{ respectively.}\]
STRUCTURE OF THE EXTENSION PRINCIPLE

$g(f^{-1}(A))$
I: p: most Swedes are tall
p*: \( \sum \text{Count(tall.Swedes/Swedes) is most} \)

q: How many are short?

Further precisiation

\( X(h): \text{height density function (not known)} \)

\( X(h)dh: \text{fraction of Swedes whose height is in } [h, h+dh], a \leq h \leq b \)

\[ \int_{a}^{b} X(h)dh = 1 \]
CONTINUED

- fraction of tall Swedes: \[ \int_a^b X(h) \mu_{tall}(h) \, dh \]

- constraint on \( X(h) \)

\[ \int_a^b X(h) \mu_{tall}(h) \, dh \] is most granular value

\[ \pi(X) = \mu_{most}(\int_a^b X(h) \mu_{tall}(h) \, dh) \]
CONTINUED

deduction:

\[ \int_{a}^{b} X(h) \mu_{\text{tall}}(h) dh \text{ is most } \quad \text{given} \]

\[ \int_{a}^{b} X(h) \mu_{\text{short}}(h) dh \text{ is } ? \text{ Q } \quad \text{needed} \]

solution:

\[ \mu_{Q}(v) = \sup_{X}(\mu_{\text{most}}(\int_{a}^{b} X(h) \mu_{\text{tall}}(h) dh)) \]

subject to

\[ v = \int_{a}^{b} X(h) \mu_{\text{short}}(h) dh \]

\[ \int_{a}^{b} X(h) dh = 1 \]
In a general setting, computation/deduction is governed by the Deduction Principle.

Point of departure: question, q

Information set: \( I = (X_1/u_1, \ldots, X_n/u_n) \)

\( u_i \) is a generic value of \( X_i \)

\( \text{ans}(q/I): \) answer to q/I
If we knew the values of the $X_i$, $u_1$, ..., $u_n$, we could express $\text{ans}(q/l)$ as a function of the $u_i$

$$\text{ans}(q/l) = g(u_1, ..., u_n) \quad u = (u_1, ..., u_n)$$

Our information about the $u_i$, $I(u_1, ..., u_n)$ is a generalized constraint on the $u_i$. The constraint is defined by its test-score function

$$f(u) = f(u_1, ..., u_n)$$
Use the extension principle

\[ \mu_{\text{Ans}(q)}(v) = \sup_u (ts(u)) \]

subject to

\[ v = g(u) \]
EXAMPLE

I: p: Most Swedes are much taller than most Italians
q: What is the difference in the average height of Swedes and Italians?

Solution

Step 1. precisiation: translation of p into GCL

$S = \{S_1, \ldots, S_n\}$: population of Swedes
$l = \{l_1, \ldots, l_n\}$: population of Italians
$g_i = \text{height of } S_i$ , $g = (g_1, \ldots, g_n)$
$h_j = \text{height of } l_j$ , $h = (h_1, \ldots, h_n)$
$\mu_{ij} = \mu_{\text{much.taller}}(g_i, h_j)$ = degree to which $S_i$ is much taller than $l_j$
CONTINUED

\[ r_i = \frac{1}{n} \sum_{j} \mu_{ij} \]

Relative \( \Sigma \) Count of Italians in relation to whom \( S_i \) is much taller

\[ t_i = \mu_{\text{most}}(r_i) = \text{degree to which } S_i \text{ is much taller than most Italians} \]

\[ v = \frac{1}{m} \sum t_i \]

Relative \( \Sigma \) Count of Swedes who are much taller than most Italians

\[ t_s(g, h) = \mu_{\text{most}}(v) \]

\[ p \rightarrow \text{generalized constraint on } S \text{ and } I \]

\[ q: d = \frac{1}{m} \sum g_i - \frac{1}{n} \sum h_j \]
Step 2. Deduction via extension principle

\[ \mu_q(d) = \sup_{g,h} ts(g,h) \]

subject to

\[ d = \frac{1}{m} \sum_i g_i - \frac{1}{n} \sum_j h_j \]
SUMMATION

- In large measure, existing scientific theories are based on bivalent logic—a logic in which everything is black or white, with no shades of gray allowed.
- What is not recognized, to the extent that it should, is that bivalent logic is in fundamental conflict with reality.
- Fuzzy logic is not in conflict with bivalent logic—it is a generalization of bivalent logic in which everything is, or is allowed to be, a matter of degree.
- Fuzzy logic, does not replace formalisms based on bivalent logic with formalisms based on fuzzy logic.
- Fuzzy logic adds to and generalizes formalisms based on bivalent logic.
Factual Information About the Impact of Fuzzy Logic

PATENTS

- Number of fuzzy-logic-related patents applied for in Japan: 17,740
- Number of fuzzy-logic-related patents issued in Japan: 4,801
- Number of fuzzy-logic-related patents issued in the US: around 1,700
Count of papers containing the word “fuzzy” in title, as cited in INSPEC and MATH.SCI.NET databases. Compiled on July 17, 2008.

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JOURNALS  (“fuzzy” in title)
1. Fuzzy in title
2. Fuzzy Sets and Systems
3. IEEE Transactions on Fuzzy Systems
4. Fuzzy Optimization and Decision Making
5. Journal of Intelligent & Fuzzy Systems
6. Fuzzy Economic Review
10. International Review of Fuzzy Mathematics
11. Fuzzy Systems and Soft Computing
12. Fuzzy Information Engineering
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From computing with numbers to computing with words --from manipulation of measurements to manipulation of perceptions, IEEE Transactions on Circuits and Systems 45, 105-119, 1999.


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- Fuzzy probabilities and their role in decision analysis, Proc. MIT/ONR Workshop on C\u3\d, MIT, Cambridge, MA., 1981.

- Fuzzy sets vs. probability, (correspondence item), Proc. IEEE 68, 421, 1980.