A FUZZY-SET-THEORETIC APPROACH TO THE COMPOSITIONALITY OF MEANING: PROPOSITIONS, DISPOSITIONS AND CANONICAL FORMS

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Abstract

In its traditional interpretation, Frege's principle of compositionality is not sufficiently flexible to have a wide applicability to natural languages. In a fuzzy-set-theoretic setting which is outlined in this paper, Frege's principle is modified and broadened by allowing the meaning of a proposition, $p$, to be composed not from the meaning of the constituents of $p$, but, more generally, from the meaning of a collection of fuzzy relations which form a so-called explanatory database that is associated with $p$. More specifically, through the application of test-score semantics, the meaning of $p$ is represented as a procedure which tests, scores and aggregates the elastic constraints which are implicit in $p$. The employment of fuzzy sets in this semantics allows $p$ to contain fuzzy predicates such as tall, kind, much richer, etc.; fuzzy quantifiers such as most, several, few, usually etc.; modifiers such as very, more or less, quite, somewhat, etc.; and other types of semantic entities which cannot be dealt with within the framework of classical logic.

The approach described in the paper suggests a way of representing the meaning of dispositions, e.g., Overeating causes obesity, Icy roads are slippery. Young men like young women, etc. Specifically, by viewing a disposition, $d$, as a proposition with implicit fuzzy quantifiers, the problem of representing the meaning of $d$ may be decomposed into (a) restoring the suppressed fuzzy quantifiers and/or fuzzifying the nonfuzzy quantifiers in the body of $d$; and (b) representing the meaning of the resulting dispositional proposition through the use of test-score semantics.

To place in evidence the logical structure of $p$ and, at the same time, provide a high-level description of the composition process, $p$ may be expressed in the canonical form "$X$ is $F$" where $X = (X_1, ..., X_n)$ is an implicit $n$-ary variable which is constrained by $p$, and $F$ is a fuzzy $n$-ary relation which may be interpreted as an elastic constraint on $X$. This canonical form and the meaning-composition process for propositions and dispositions are illustrated by several examples among which is the proposition $p \Delta$ Over the past few years Naomi earned far more than most of her close friends.
1. Introduction

It is widely agreed at this juncture that Frege's principle of compositionality has a rather limited validity in application to natural languages (Hintikka (1982)). However, as is well known, its applicability may be extended, as it is done in Montague semantics (Partee (1976)), by the employment of higher-order type-theoretical constructs.

A different approach which is described in this paper is based on a broader interpretation of compositionality which allows the meaning of a proposition to be composed not from the meaning of its constituents, but, more generally, from the meaning of a collection of fuzzy relations in what is referred to as an explanatory database. With this interpretation of compositionality, Frege's principle regains much of its validity and, in its modified form, provides a basis for representing the meaning of complex propositions and other types of semantic entities. In particular, it may be used to represent the meaning of propositions containing fuzzy predicates exemplified by tall, kind, much younger, close friend, etc.; fuzzy quantifiers such as most, many, few, several, not very many, frequently, rarely, mostly, etc.; modifiers such as very, quite, more or less, somewhat, etc.; and qualifiers such as quite true, very unlikely, almost impossible, etc.

An especially important application of the approach described in this paper relates to the representation of the meaning of dispositions, that is, propositions with implicit fuzzy quantifiers. For example, the disposition Overeating causes obesity may be viewed as a result of suppressing the fuzzy quantifier most in the proposition Most of those who overeat are obese. Similarly, the disposition Young men like young women may be interpreted as an abbreviation of the proposition Most young men like mostly young women. On the other hand, the proposition Anne never tells a lie may be interpreted as the dispositional proposition Anne tells a lie very rarely, in which the fuzzy quantifier very rarely may be viewed as a fuzzified version of the nonfuzzy quantifier never. In general, a disposition may have a number of different interpretations and the restoration or explicitation of fuzzy quantifiers is an interpretation-dependent process.

2. Test-Score Semantics

The modified Frege's principle underlies a fuzzy-set-based meaning-representation system termed test-score semantics (Zadeh (1981)). In this system, a semantic entity such as a proposition, predicate, predicate-modifier, quantifier, qualifier, command, etc., is regarded as a system of elastic constraints whose domain is a collection of fuzzy relations in a database - a database which describes a state of affairs, a possible world, or more generally, a set of objects or derived objects in a universe of discourse. The meaning of a semantic entity, then, is represented as a test which when applied to the database
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yields a collection of partial test scores. Upon aggregation, these test scores lead to an overall vector test score, $\tau$, whose components are numbers in the unit interval, with $\tau$ serving as a measure of the compatibility of the semantic entity with the database. In this respect, test-score semantics subsumes both truth-conditional and possible-world semantics as limiting cases in which the partial and overall test scores are restricted to \{pass, fail\} or, equivalently, \{true, false\} or \{1, 0\}.

In more specific terms, the process of meaning representation in test-score semantics involves three distinct phases. In Phase 1, an explanatory database frame or EDF, for short, is constructed. EDF consists of a collection of relational frames, i.e., names of relations, names of attributes and attribute domains whose meaning is assumed to be known. In consequence of this assumption, the choice of EDF is not unique and is strongly influenced by the knowledge profile of the addressee of the representation process as well as by the objective of explanatory effectiveness. For example, in the case of the proposition $p$ \cdot \text{Over the past few years Naomi earned far more than most of her close friends,}$ the EDF might consist of the following relations: INCOME [Name: Amount; Year], which lists the income of each individual identified by his/her name as a function of the variable Year; FRIEND [Name, $\mu$], where $\mu$ is the degree to which Name is a friend of Naomi; FEW [Number, $\mu$], where $\mu$ is the degree to which Number is compatible with the fuzzy number few: MOST [Proportion, $\mu$] in which $\mu$ is the degree to which Proportion is compatible with the fuzzy quantifier most; and FAR MORE [Income 1; Income 2; $\mu$], where $\mu$ is the degree to which Income 1 fits the fuzzy predicate far more in relation to Income 2. Each of these relations is interpreted as an elastic constraint on the variables which are associated with it.

In Phase 2, a test procedure is constructed which acts on the relations in the explanatory database and yields the test scores which represent the degree to which the elastic constraints induced by the constituents of the semantic entity are satisfied. For example, in the case of $p$, the test procedure would yield the test scores for the constraints induced by the relations FRIEND, FEW, MOST and FAR MORE.

In Phase 3, the partial test scores are aggregated into an overall test score, $\tau$, which, in general, is a vector which serves as a measure of the compatibility of the semantic entity with an instantiation of EDF. As was stated earlier, the components of this vector are numbers in the unit interval or, more generally, possibility/probability distributions over this interval. In particular, in the case of a proposition, $p$, for which the overall test score is a scalar, $\tau$ may be interpreted as the degree of truth of $p$ with respect to the explanatory database ED (i.e., an instantiation of EDF). It is in this sense that test-score semantics may be viewed as a generalization of truth-conditional and model-theoretic semantics.

In summary, the process described above may be regarded as a
test which assesses the compatibility of a given proposition, $p$, with
an explanatory database, $ED$. What is important to note is that the
meaning of $p$ is the test itself rather than the overall test score,
$t$, which it yields.

In effect, the test in question may be viewed as the process by
which the meaning of a proposition is composed from the meaning
of the constituent relations in the associated explanatory database.
As was stated earlier, the essential difference between this approach
to compositionality and that of Frege is that, in general, the meaning
of a proposition, $p$, is composed not from the meaning of the constitu-
ents of $p$ but from those of a database, $ED$, which is constructed
for the explicit purpose of explaining or representing the meaning
of $p$ in terms of fuzzy relations whose meaning is assumed to be
known to the addressee of the representation process.

In some instances, the names of constituent relations in the explana-
tory database may bear a close relation to the constituents of the
proposition. In general, however, the connection may be implicit
rather than explicit.

In testing the constituent relations in $ED$, it is helpful to have
a collection of standardized rules for computing the aggregated test
score of a combination of elastic constraints $C_1, ..., C_k$ from the knowl-
dge of the test scores of each constraint considered in isolation.
For the most part, such rules are default rules in the sense that they
are intended to be used in the absence of alternative rules supplied
by the user.

In test-score semantics, the elementary rules of this type are the
following:

Rules pertaining to unary modification

If the test-score for an elastic constraint $C$ in a specified context
is $t$, then in the same context the test score for
(a) not $C$ is $1 - t$ (negation).
(b) very $C$ is $t^2$ (intensification or concentration).
(c) more or less $C$ is $t^2$ (diffusion or dilation).

Rules pertaining to composition

If the test scores for elastic constraints $C_1$ and $C_2$ in a specified
context are $t_1$ and $t_2$, respectively, then in the same context the
test score for
(a) $C_1$ and $C_2$ is $t_1 \land t_2$, where $\land$ min (conjunction).
(b) $C_1$ or $C_2$ is $t_1 \lor t_2$, where $\lor$ max (disjunction).
(c) If $C_1$ then $C_2$ is $1 \land (1 - t_1 + t_2)$ (implication).

Rules pertaining to quantification

Let $Q$ be a fuzzy quantifier (i.e., a fuzzy number) which is character-
ized by its membership function $\mu_Q$.

Let $A$ and $B$ be fuzzy subsets of a universe of discourse $U = \{u_1, ..., u_n\}$. 

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with respective membership functions \( \mu_A \) and \( \mu_B \).

Define the sigma-count (i.e., the cardinality) of \( A \) as the real number

\[
\Sigma \text{ Count } (A) \triangleq \sum \mu_A(\mu_i)
\]

where \( \mu_A(\mu_i) \), \( i = 1, \ldots, n \), is the grade of membership of \( u_i \) in \( A \).

Define the relative sigma-count of \( B \) in \( A \) as the ratio

\[
\Sigma \text{ Count } (B/A) = \frac{\Sigma \text{ Count } (A \cap B)}{\Sigma \text{ Count } (A)} = \frac{\sum \mu_A(\mu_i) \land \mu_B(\mu_i)}{\sum \mu_A(\mu_i)}
\]

Then, the overall test score for the generic proposition

\[
p \triangleq Q \text{ A's are B's,}
\]

where \( A \)'s and \( B \)'s are generic names of the elements of \( A \) and \( B \), is given by

\[
\tau = \mu_Q(\Sigma \text{ Count } (B/A)).
\]

In effect, this expression indicates that the compatibility of \( p \) with the denotations of \( A \) and \( B \) is equal to the degree to which the proportion of \( B \)'s in \( A \) - or, more generally, the degree of containment of \( A \) in \( B \) - fits the denotation of \( Q \).

As an illustration of the use of some of these rules in test-score semantics, consider the proposition cited earlier, namely, \( p \triangleq \text{ Over the past few years Naomi earned far more than most of her close friends.} \)

In this case, we shall assume, as was done earlier, that the constituent relations in the explanatory database are:

\[
\text{EDF} \triangleq \text{INCOME} \ [\text{Name; Amount; Year}] + \\
\text{FRIEND} \ [\text{Name}; \mu] + \\
\text{FEW} \ [\text{Number; \mu}] + \\
\text{FAR MORE} \ [\text{Income 1; Income 2; \mu}] + \\
\text{MOST} \ [\text{Proportion; \mu}] .
\]

Note that some of these relations are explicit in \( p \); some are not; and that most of the constituent words in \( p \) do not appear in \( \text{EDF} \).

In what follows, we shall describe the process by which the meaning of \( p \) may be composed from the meaning of the constituent relations in \( \text{EDF} \). Basically, this process is a test procedure which tests, scores and aggregates the elastic constraints which are induced by \( p \).

1. Find Naomi's income, \( IN_i \), in \( \text{Year}_i \), \( i = 1, 2, 3, \ldots \), counting backward from present. In symbols,
IN_{1} \triangleq \text{Amount INCOME [Name=Naomi; Year=Year_{1}]}

which signifies that Name is bound to Naomi, Year to Year_{1}, and the resulting relation is projected on the domain of the attribute Amount, yielding the value of Amount corresponding to the values assigned to the attributes Name and Year.

2. Test the constraint induced by FEW:

\[ \mu_{\tilde{\gamma}} \triangleq \mu_{\text{FEW [Year=Year_{1}]}} \]

which signifies that the variable Year is bound to Year_{1} and the corresponding value of \mu is read by projecting on the domain of \mu.

3. Compute Naomi’s total income during the past few years:

\[ \text{TIN} = \sum_{\zeta} \mu_{\tilde{\gamma}} fN_{\tilde{\gamma}}, \]

in which the \mu_{\tilde{\gamma}} play the role of weighting coefficients. Thus, we are tacitly assuming that the total income earned by Naomi during a fuzzily specified interval of time is obtained by weighting Naomi’s income in year Year_{1} by the degree to which Year_{1} satisfies the constraint induced by FEW and summing up the weighted incomes.

4. Compute the total income of each Name_{1} (other than Naomi) during the past few years:

\[ \text{TINName}_{1} = \sum_{\zeta} \mu_{\tilde{\gamma}} f\text{Name}_{1} \]

where \text{TINName}_{1} is the income of Name_{1} in Year_{1}.

5. Find the fuzzy set of individuals in relation to whom Naomi earned far more. The grade of membership of Name_{1} in this set is given by

\[ \mu_{FM} (\text{Name}_{1}) = \mu_{\text{FAR MORE [Income}_{1}=\text{TIN}; \text{Income}_{2}=\text{TINName}_{1}].} \]

6. Find the fuzzy set of close friends of Naomi by intensifying (Zadeh (1978)) the relation FRIEND:

\[ CF \triangleq \text{CLOSE FRIEND} \triangleq 2 \text{FRIEND}. \]

which implies that

\[ \mu_{CF} (\text{Name}_{1}) = (\mu_{\text{FRIEND [Name=Name}_{1}])^2), \]

where the expression

\[ \mu_{\text{FRIEND [Name=Name}_{1}]} \]

represents \mu_{\tilde{\gamma}} (\text{Name}_{1}), that is, the grade of membership of Name_{1} in
the set of Naomi’s friends.

7. Count the number of close friends of Naomi. On denoting the count in question by \( \Sigma \text{Count} \ (CF) \), we have:

\[
\Sigma \text{Count} \ (CF) = \Sigma \mu^\text{FRIEND} (Name_j)
\]

8. Find the intersection of \( FM \) with \( CF \). The grade of membership of \( Name_j \) in the intersection is given by

\[
\mu^\text{FM} \cap CF (Name_j) = \mu^\text{FM} (Name_j) \wedge \mu^\text{CF} (Name_j),
\]

where the min operator signifies that the intersection is defined as the conjunction of its operands.

9. Compute the sigma-count of \( FM \cap CF \):

\[
\Sigma \text{Count} \ (FM \cap CF) = \Sigma \mu^\text{FM} (Name_j) \wedge \mu^\text{CF} (Name_j).
\]

10. Compute the relative sigma-count of \( FM \) in \( CF \), i.e., the proportion of individuals in \( FM \cap CF \) who are in \( CF \):

\[
\rho = \frac{\Sigma \text{Count} \ (FM \cap CF)}{\Sigma \text{Count} \ (CF)}
\]

11. Test the constraint induced by \( \text{MOST} \):

\[
\tau = \mu \text{MOST} \ [\text{Proportion}=\rho],
\]

which expresses the overall test score and thus represents the compatibility of \( \rho \) with the explanatory database.

In general, the relations in \( EDF \) are context-dependent. As an illustration, consider the proposition

\[
p \triangleq \text{Both are tall},
\]

in which the standards of tallness are assumed to be class-dependent, e.g., depend on whether an individual is male or female. To reflect this, we may express the \( EDF \) for \( p \) in the following form:

\[
EDF \triangleq \text{POPULATION} \ [\text{Name}; \text{Height}; \text{Sex};], + \\
\text{Indexical} \rightarrow \text{Name}_a + \\
\text{Indexical} \rightarrow \text{Name}_b + \\
\text{TALL} \ [\text{Height}; \text{Sex}; \mu],
\]

in which the notation \( \text{Indexical} \rightarrow \text{Name}_a \) indicates that \( \text{Name}_a \) is an indexical object, i.e., is pointed to by the context. More specifically, we assume (a) that \( \text{Name}_a \) and \( \text{Name}_b \) are the names of two individuals in \( \text{POPULATION} \) who are pointed to by the context in which \( p \) is assert-
ed; and (b) that the relation TALL is sex-dependent, with \( \mu \) representing the degree to which an individual whose height is Height and whose sex is Sex is tall.

For the EDF in question, the steps in the test procedure which leads to the overall test score and thereby represents the meaning of \( p \) may be described as follows:

1. Find the height and sex of Name\(_a\) and Name\(_b\):

   \( \text{Height} (\text{Name}_a) \leq \text{Height} \text{POPULATION} [\text{Name}=\text{Name}_a] \)

   \( \text{Sex} (\text{Name}_a) \leq \text{Sex} \text{POPULATION} [\text{Name}=\text{Name}_a] \)

   \( \text{Height} (\text{Name}_b) \leq \text{Height} \text{POPULATION} [\text{Name}=\text{Name}_b] \)

   \( \text{Sex} (\text{Name}_b) \leq \text{Sex} \text{POPULATION} [\text{Name}=\text{Name}_b] \).

2. Find the degrees to which Name and Name are tall:

   \( \tau_a \leq \mu \text{TALL} [\text{Height}=\text{Height}(\text{Name}_a); \text{Sex}=\text{Sex}(\text{Name}_a)] \)

   \( \tau_b \leq \mu \text{TALL} [\text{Height}=\text{Height}(\text{Name}_b); \text{Sex}=\text{Sex}(\text{Name}_b)] \).

3. Aggregate the test scores found in \( \frac{1}{2} \):

   \( \tau = \tau_a \wedge \tau_b \)

   in which we use the min operator (\( \wedge \)) to combine the test scores \( \tau_a \) and \( \tau_b \) into the overall test score \( \tau \).

As an illustration of the compositionality of meaning in the case of dispositions, we shall consider, first, the following simple disposition:

\( d \triangleq \text{Claudine is a better tennis player than Michael.} \)

For concreteness, \( d \) will be assumed to have the interpretation expressed by the proposition

\( p \triangleq \text{When Claudine and Michael play tennis, Claudine usually wins.} \)

The EDF for \( p \) is assumed to consist of the relations

\( \text{EDF} \triangleq \text{PLAY TENNIS} [\text{Outcome}]^+ \)

\( \quad \text{USUALLY} [\mu]. \)

The relation PLAY TENNIS represents a tally of the outcomes of \( n \) plays between Claudine and Michael, with the variable Outcome ranging over the set \{Win, Lose\}, and with Win implying that Claudine won the game. The relation USUALLY is a temporal fuzzy quantifier with \( \mu \) representing the degree to which a numerical value of Proportion fits the intended meaning of USUALLY.

The steps in the test procedure are as follows.

1. Find the proportion of plays won by Claudine:

   \( p = \frac{\text{Count} \text{(PLAY TENNIS [Outcome=Win])}}{n} \).

2. Test the constraint induced by USUALLY:

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\[ \tau = \mu_{USUALLY} [Proportion=q]. \]

This expression for \( \tau \) represents the overall test score for \( d \).

We can make use of the above result to represent the meaning of a more complex disposition, namely,

\[ d \triangleq \text{Men are better tennis players than women}, \]

which will be assumed to be interpreted as the proposition

\[ p \triangleq \text{Most men are better tennis players than most women}, \]

with the associated EDF consisting of the relations

\[ EDF \triangleq \text{POPULATION} [M. Name; F. Name; \mu_j] \]

\[ \quad \text{MOST} \ [Proportion; \mu]. \]

For simplicity, we assume that there are \( n \) men and \( n \) women in \( \text{POPULATION} \), with \( \mu \) representing the degree - computed as in the above example - to which \( M. \text{Name}_i \) is a better tennis player than \( F. \text{Name}_j \).

(More specifically, \( \mu_{ij} \) is the degree to which \( M. \text{Name}_i \) is a better tennis player than \( F. \text{Name}_j \), \( i,j=1,...,n \).)

The steps in the test procedure are as follows:

1. For each \( M. \text{Name}_i \), find the proportion (i.e. the relative sigma-count) of women tennis players in relation to whom \( M. \text{Name}_i \) is a better tennis player:

\[ \Omega_i \triangleq \frac{1}{n} \sum_j \mu_{ij}. \]

2. For each \( M. \text{Name}_i \), find the degree to which \( M. \text{Name}_i \) is a better tennis player than most women:

\[ \tau_i \triangleq \mu_{MOST} [Proportion=q]. \]

3. Compute the proportion of men who are better tennis players than most women:

\[ \rho = \frac{1}{n} \sum_i \tau_i. \]

4. Compute the test score for the constraint induced by \( \text{MOST} \):

\[ \tau = \mu_{MOST} [Proportion=q]. \]

This \( \tau \) represents the overall test score for \( d \).

As an additional illustration, consider the disposition

\[ d \triangleq \text{Young men like young women} \]

which, as stated earlier, may be interpreted as the proposition

\[ p \triangleq \text{Most young men like mostly young women}. \]

The candidate EDF for \( p \) is assumed to consist of the following relations:

\[ EDF \triangleq \text{POPULATION} [\text{Name}; \text{Sex}; \text{Age}] + \]

\[ \text{LIKE} [\text{Name 1}; \text{Name 2}; \mu] + \]
MOST [Proportion: $\mu$],
in which $\mu$ in LIKE is the degree to which Name 1 likes Name 2.

To represent the meaning of $p$, it is expedient to replace $p$ with the semantically equivalent proposition

$q \triangleq \text{Most young men are } P,$

where $P$ is the fuzzy dispositional predicate

$P \triangleq \text{likes mostly young women}.$

In this way, the representation of the meaning of $p$ is decomposed into two simpler problems, namely, the representation of the meaning of $P$, and the representation of the meaning of $q$ knowing the meaning of $P$.

The meaning of $P$ is represented by the following test procedure.

1. Divide POPULATION into the population of males, $M. \text{ POPULATION}$, and population of females, $F. \text{ POPULATION}$:

$M. \text{ POPULATION} \triangleq \text{Sex=Male } \text{ POPULATION}$

$F. \text{ POPULATION} \triangleq \text{Sex=Female } \text{ POPULATION},$

where $\text{ POPULATION}$ denotes the projection of POPULATION on the attributes Name and Age.

2. For each Name$_j$, $j=1, \ldots, n$, in $F. \text{ POPULATION}$, find the age of Name$_j$:

$A_j \triangleq \text{Age } \text{ POPULATION} (\text{Name}=\text{Name}_j).$

3. For each Name$_i$, find the degree to which Name$_i$ is young:

$\alpha_i \triangleq \mu \text{ YOUNG } (\text{Age}=A_i),$

where $\alpha_i$ may be interpreted as the grade of membership of Name$_i$ in the fuzzy set, YW, of young women.

4. For each Name$_i$, $i=1, \ldots, k$, in $M. \text{ POPULATION}$, find the age of Name$_i$:

$B_i \triangleq \text{Age } \text{ POPULATION} (\text{Name}=\text{Name}_i).$

5. For each Name$_i$, find the degree to which Name$_i$ is young:

$\delta_i \triangleq \mu \text{ YOUNG } (\text{Age}=B_i),$

where $\delta_i$ may be interpreted as the grade of membership of Name$_i$ in the fuzzy set, YM, of young men.

6. For each Name$_i$, find the degree to which Name$_i$ likes Name$_j$:

$\beta_{i,j} \triangleq \mu \text{ LIKE } (\text{Name }=\text{Name}_i; \text{Name }=\text{Name}_j),$

with the understanding that $\beta_{i,j}$ may be interpreted as the grade of membership of Name$_i$ in the fuzzy set, WL$_i$, of women whom Name$_j$ likes.

7. For each Name$_i$ find the degree to which Name$_i$ likes Name$_j$ and Name$_j$ is young.
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\[ \gamma_{i,j} \triangleq a_{i,j} \land b_{i,j} \]

Note: As in previous examples, we employ the aggregation operator min \((\land)\) to represent the effect of conjunction. In effect, \(\gamma_{i,j}\) is the grade of membership of Name in the intersection of the fuzzy sets \(WL_i\) and \(YW\).

3. Compute the relative sigma-count of young women among the women whom Name\(_i\) likes:
   \[ \sigma_i \triangleq \sum \text{Count} (YW/WL_i) \]
   \[ = \frac{\sum \text{Count} (YW \cap WL_i)}{\sum \text{Count} (WL_i)} \]
   \[ = \frac{\sum_{j=1}^{n} \gamma_{i,j}}{\sum_{j=1}^{n} b_{i,j}} = \frac{\sum_{j=1}^{n} a_{i,j} \land b_{i,j}}{\sum_{j=1}^{n} b_{i,j}} \]

9. Test the constraint induced by MOST:
   \[ \tau_i \triangleq \mu_{\text{MOST}} [\text{Proportion}=\sigma_i] \]
   This test score, then, represents the degree to which Name\(_i\) has the property expressed by the predicate \(P \triangleq \text{likes mostly young women}\).

Continuing the test procedure, we have:

10. Compute the relative sigma-count of men who have property \(P\) among young men:
   \[ \rho \triangleq \sum \text{Count} (P/YM) \]
   \[ = \frac{\sum \text{Count} (P \cap YM)}{\sum \text{Count} (YM)} \]
   \[ = \frac{\sum_{i=1}^{n} \tau_i \land \delta_i}{\sum_{i=1}^{n} \delta_i} \]

11. Test the constraint induced by MOST:
    \[ \tau = \mu_{\text{MOST}} [\text{Proportion}=\rho] \]
    This test score represents the overall test score for the disposition Young men like young women.

3. Canonical Form

The test procedures described in the preceding section provide, in effect, a characterization of the process by which the meaning of a proposition, \(p\), may be composed from the meaning of the constituent relations in the EDF which is associated with \(p\). However, the
details of the test procedure tend to obscure the higher-level features of the process of composition and thus make it difficult to discern its underlying modularity and hierarchical structure.

The concept of a canonical form of $p$, which plays an important role in PRUF (Zadeh (1973)), provides a way of displaying the logical structure of $p$ and thereby helps to place in a clearer perspective the role of the consecutive steps in the test procedure in the representation of meaning of $p$. Specifically, as was stated earlier, a proposition, $p$, may be viewed as a system of elastic constraints whose domain is the collection of fuzzy relations in the explanatory database. In more concrete terms, this implies that $p$ may be represented in the canonical form

$$ p \rightarrow X \text{ is } F, $$

where $X=(X_1,...,X_n)$ is an $n$-ary base variable whose components $X_1,...,X_n$ are the variables which are constrained by $p$; and $F$ - which is a fuzzy subset of the universe of discourse $U=U_1 \times ... \times U_n$, where $U_i, i=1,...,n$, denotes the domain of $X_i$ - plays the role of an elastic constraint on $X$.

In general, both the base variable and $F$ are implicit rather than explicit in $p$.

As a simple illustration, consider the proposition

$$ p \triangleq \text{Virginia is slim}. $$

In this case, the base variables are $X_1 \triangleq \text{Height (Virginia)}, X_2 \triangleq \text{Weight (Virginia)}$; the constraint set is SLIM; and hence the canonical form of $p$ may be expressed as

$$ (\text{Height (Virginia)}, \text{Weight (Virginia)}) \text{ is SLIM}, $$

where SLIM is a fuzzy subset of the rectangle $U_1 \times U_2$, with $U_1=[0,200cm]$ and $U_2=[0,100\text{ kg}]$.

If the assertion "$X$ is $F$" is interpreted as an elastic constraint on the possible values of $X$, then the canonical form of $p$ may be expressed as the possibility assignment equation (Zadeh (1973))

$$ \Pi(X_1,...,X_n)=F, $$

in which $\Pi(X_1,...,X_n)$ denotes the joint possibility distribution of $X_1,...,X_n$. In more concrete terms, this equation implies that the possibility that the variables $X_1,...,X_n$ may take the values $u_1,...,u_n$, respectively, is equal to the grade of membership of the $n$-tuple $(u_1,...,u_n)$ in $F$, that is,

$$ \text{Poss} \{X_1=u_1,...,X_n=u_n\} = \mu_F(u_1,...,u_n), $$

where $\mu_F$ denotes the membership function of $F$.

As an illustration, consider the disposition

$$ d \triangleq \text{Fat men are kind}, $$

which may be interpreted as an abbreviation of the proposition

$$ p \triangleq \text{Most fat men are kind}. $$

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Let \textit{FAT} and \textit{KIND} denote the fuzzy sets of \textit{fat men} and \textit{kind men}, respectively, in \( U \). Now, the fuzzy quantifier \textit{most} in \( p \) may be interpreted as a fuzzy characterization of the relative sigma-count of \textit{kind men} in \textit{fat men}. From this, it follows that the canonical form of \( p \) may be expressed as

\[
\Sigma \text{Count (KIND/FAT)} \text{ is MOST}
\]

or, equivalently, as the possibility assignment equation

\[
\Pi_X = \text{MOST}
\]

where

\[
X = \Sigma \text{Count (KIND/FAT)},
\]

and \textit{MOST} is a fuzzy subset of the unit interval \([0,1]\).

Along the same lines, consider the proposition

\( p \models \text{Most big men are not very agile.} \)

As in the previous example, \textit{BIG} will be assumed to be a fuzzy subset of the rectangle \([0,2000 \text{ cm}] \times [0,100 \text{ kg}]\). As for the fuzzy predicate \textit{not very agile}, its denotation may be expressed as

\[
\text{not very agile} \rightarrow (2\text{AGILE})'
\]

where \( 2\text{AGILE} \) represents the denotation of \textit{very agile} and ' denotes the complement. More concretely, the membership function of \( 2\text{AGILE} \) is given by

\[
\mu_{2\text{AGILE}} = (\mu_{\text{AGILE}})^2
\]

and thus

\[
\mu_{(2\text{AGILE})'} = 1 - (\mu_{\text{AGILE}})^2.
\]

By relating the denotation of \textit{not very agile} to that of \textit{agile}, the canonical form of \( p \) may be expressed compactly as

\[
p \rightarrow \Sigma \text{Count ((2AGILE)'/BIG)} \text{ is MOST}.
\]

As expected, this canonical form places in evidence the manner in which the meaning of \( p \) may be composed from the meaning of the fuzzy relations \textit{AGILE}, \textit{BIG} and \textit{MOST}.

As a further example, consider the proposition

\( p \models \text{Peggy lives in a small city near San Francisco,} \)

with which we associate the \textit{EDF}

\[
\text{EDF} \models \text{RESIDENCE [Name; City]}+
\]

\text{SMALL CITY [City; \mu]+}

\text{NEAR [City 1; City 2; \mu].}

In \textit{RESIDENCE}, \textit{City} is the city in which \textit{Name} lives; in \textit{SMALL CITY}, \( \mu \) is the degree to which \textit{City} is small; and in \textit{NEAR}, \( \mu \) is the degree to which \textit{City 1} is near \textit{City 2}.
The fuzzy set of cities which are near San Francisco may be expressed as

$$CNSF \triangleq \text{city, NEAR [City 2=San Francisco]},$$

and hence the fuzzy set of small cities which are near San Francisco is given by the intersection

$$SCNSF \triangleq \text{SMALL, CITY \cap CNSF},$$

which is, in effect, the fuzzy constraint set $F$ in the canonical form "$X$ is $F$". In terms of this set, then, the canonical form of $p$ may be expressed as

$$p \rightarrow \text{Location (Residence (Peggy)) is}$$

$$\text{SMALL CITY \cap \text{city, NEAR [City 2=San Francisco]}.}$$

To illustrate a different aspect of canonical forms, consider the proposition

$$p \triangleq \text{Mia had high fever.}$$

In this case, we have to assume that the base variable

$$X(t) \triangleq \text{Temperature (Mia, t)}$$

$$\triangleq \text{Temperature of Mia at time t}$$

is time-dependent. Furthermore, the verb "had" induces a fuzzy or, equivalently, elastic constraint on time which may be expressed as

$$\text{had} \Rightarrow t \text{ is PAST}$$

with the understanding that $\text{PAST}$ is a fuzzy subset of the interval $(-\infty, \text{present time})$ which is indexical in the sense that it is characterized more specifically by the context in which $p$ is ascerted. Using this interpretation of $\text{PAST}$, the canonical form of $p$ may be written as

$$p \rightarrow \text{Temperature (Mia, t is PAST) is HIGH} .$$

To conclude our examples, we shall construct canonical forms for two of the propositions considered in Section 2. We begin with the proposition

$$p \triangleq \text{Most young men like mostly young women.}$$

As before, we represent $p$ as the proposition

$$p \triangleq \text{Most young men are P},$$

where $P$ is the dispositional predicate $\text{likes mostly young women}$. In this way, the canonical form of $p$ may be expressed as

$$\Sigma \text{Count (P/YM) is MOST;}$$

where $P$ is the fuzzy set which represents the denotation of $\text{likes mostly young women}$ in $M. \text{POPULATION}$, and $YM$ is the fuzzy subset of young men in $M. \text{POPULATION}$.
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To complete the construction of the canonical form, we must show how to construct $p$. To this end, we shall express in the canonical form the proposition

$$p_i \triangleq \text{Name}_i \text{ is } P,$$

where $\text{Name}$ is the name of $i$th man in $M. \text{POPULATION}$. As before, let $WL_i$ and $YW$ denote, respectively, the fuzzy set of women whom $\text{Name}_i$ likes and the fuzzy set of young women in $F. \text{POPULATION}$. Then, the canonical form of $p_i$ may be represented as

$$\text{Name}_i \text{ is } P \rightarrow \Sigma \text{Count} (YW/WL_i) \text{ is } \text{MOST}.$$

In the above analysis, we have employed a two-stage process to represent the meaning of $p$ through the construction of two canonical forms. Alternatively, we can subsume the second form in the first, as follows.

First, we note that, for each $\text{Name}_i$, the relative sigma-count $\Sigma \text{Count} (YW/WL_i)$ is a number in the interval $[0,1]$. Let $R$ denote a fuzzy subset of $M. \text{POPULATION}$ such that

$$\mu_R(\text{Name}_i) = \Sigma \text{Count} (YW/WL_i).$$

Then, the fuzzy set of men who like mostly young women may be represented as

$$P \triangleq \text{MOST} (R),$$

with the understanding that $\text{MOST} (R)$ should be evaluated through the use of the extension principle (Zadeh (1978)). This implies that the grade of membership of $\text{Name}_i$ in $P$ is related to the grade of membership of $\text{Name}_i$ in $R$ through the composition

$$\mu_P(\text{Name}_i) = \mu_{\text{most}}(\mu_R(\text{Name}_i)), \quad i = 1, \ldots, k.$$

Using this representation of $P$, the canonical form of $p$ may be expressed more compactly as

$$p \rightarrow \Sigma \text{Count} (\text{MOST} (R)/YM) \text{ is } \text{MOST}.$$

Using the same approach, the canonical form of the proposition

$$p \triangleq \text{Over the past few years Naomi earned far more than most of her close friends}$$

may be constructed as follows.

First, we construct the canonical form

$$p \rightarrow \Sigma \text{Count} (FM/2F) \text{ is } \text{MOST},$$

where

$$CF \triangleq \text{fuzzy set of close friends of Naomi}$$

and

$$FM \triangleq \text{fuzzy set of individuals in relation to whom Naomi earned}$$
far more during the past few years.

Second, we construct the canonical form for the proposition which defines FM. Thus,

\[ \text{Name}_j \text{ is } FM \Rightarrow (\text{TIN}, \text{TIName}_j) \text{ is FAR MORE}, \]

in which the base variables are defined by

\[ \text{TIN} \triangleq \text{total income of Naomi during the past few years.} \]
\[ = \sum_i \mu_{\text{IN}}(i)\text{IN} \]

and

\[ \text{TIName}_j \triangleq \text{total income of Name}_j \text{ during the past few years.} \]
\[ = \sum_i \mu_{\text{IName}}(i)\text{IName}_{ji} \]

where \( \text{IN}_j \) is Naomi's income in year \( \text{Year}_i \), \( i=1,2,3,... \), and \( \text{IName}_{ji} \) is Name\(_j\)'s income in Year\(_i\).

It is possible, as in the previous example, to absorb the second canonical form in the first form. The complexity of the resulting form, however, would make it more difficult to perceive the modularity of the meaning-representation process.

Concluding Remark

The fuzzy-set-theoretic approach outlined in the preceding sections is intended to provide a framework for representing the meaning of propositions and dispositions which do not lend themselves to semantic analysis by conventional techniques. The principal components of this framework are (a) the explanatory database which consists of a collection of fuzzy relations; (b) the procedure which tests, scores and aggregates the elastic constraints, and thereby characterizes the process by which the meaning of a proposition is composed from the meaning of the constituent relations in the explanatory database; and (c) the canonical form which represents a proposition as a collection of elastic constraints on a set of base variables which are implicit in the proposition.

Notes

* To Walter and Sally Sedelow.

Research supported in part by the NSF Grants ECS-8209679 and IST-8018196.

1. A more detailed discussion of the rules in question may be found in Zadeh (1978).
2. The concept of cardinality is treated in greater detail in Zadeh (1982 b).

3. To obtain the projection in question, all columns other than Name and Age in the relation POPULATION [Sex=Female] should be deleted.

References and related publications


L.A. ZADEH


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