Time-Varying Networks, I*

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Summary—This paper presents an analysis of some of the significant developments in time-varying network theory which have taken place during the past decade, with the emphasis placed on three topics: 1) characterization of time-varying networks, and in particular, transition from the impulsive response to the differential equation; 2) the problem of factorization, with emphasis on the contributions of Darlington, Batkov and Paul Levy; and 3) randomly-varying systems and, in particular, the question of stability of discrete-time systems of this type. The identification problem, the analysis of periodically-varying systems, the synthesis problem, and the filtering and prediction of nonstationary processes will be treated in Part II, to be published later.

This paper is primarily concerned with developments in time-varying-network theory which have taken place during the past decade. A survey paper by Bennett [1] provides a brief discussion of the developments prior to 1950.

The past ten years have witnessed a rapid growth of interest in time-varying networks as well as a marked shift in emphasis brought about by the advent of such new areas of research as missile guidance, detection of fluctuating targets, propagation through randomly-varying media, communication via satellites, parametric amplification, etc. Furthermore, the availability of machine computers has made, and is making, a profound impact on the whole process of constructing a model of a physical system and subjecting it to mathematical analysis, as well as on the reverse process of synthesizing a system from components having known characteristics. One consequence of the advent of machine computers is the growth in importance of the so-called identification problem, which, roughly speaking, is concerned with the determination of the characteristics of a given black box, based upon the observation of its external behavior. This stems from the fact that, with the aid of machine computers, the analysis of a system comprised of elements having known characteristics no longer presents essential difficulties. Thus the outstanding problem becomes that of identification, which in any case is a basic prerequisite to the analysis of a system by either machine computer or analytical means.

This paper is not purported to be an exhaustive survey. Rather, its limited aim is to present a connected and not too superficial account of some of the significant and lesser known contributions to time-varying-network theory made during the past ten years. Because of limitations on space, the paper is in two parts. Part I deals with the problem of transition from one mode of characterization to another, the so-called factorization problem, and with randomly-varying systems. The identification problem, the analysis of periodically-varying systems, the synthesis problem, the stability problem, the filtering and prediction of nonstationary processes, the resolution into elementary time functions and miscellaneous contributions will be treated in Part II, to be published later.

It should be noted, in the following exposition, that the amount of space devoted to a particular contribution is not necessarily a measure of its significance. As a general rule, the contributions which are well known or have appeared in widely available journals are referred to very briefly, while the results which appeared in unpublished reports or foreign publications are discussed in greater detail.

I. Characterization

It has long been known that the relation between the input and output of a time-varying system can be expressed in a variety of ways other than those based on the use of differential equations. Yet it was—and to some extent still is—standard practice to employ almost exclusively the differential-equation representation for the purpose of relating the input to the output, either directly or through the state variable.

It is largely during the past decade that a need for a wider variety of alternative representations became definitely established and their potentialities as well as limitations became more clearly understood. The reason for this development is twofold: first, it was found that many of the time-varying models of such systems as randomly-varying media, fluctuating targets, etc., cannot be characterized in terms of ordinary differential equations of finite order. Second, it became clear that in many of the problems of optimization, identification, detection, filtering, etc., it is much more convenient to use a representation tailored to the characteristics of a particular class of systems than to use an "all-purpose" representation involving a differential equation or a set of differential equations relating the input to the output.

In speaking of the modes of characterization other than those based on the use of differential equations, what we have in mind are the techniques centering on the resolution of input time functions into a family of "elementary signals" such as delta functions, exponentials, etc. If the responses of the system to such elementary signals can be determined, then the response to

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an arbitrary input can be determined by superposition. In this way, the problem of finding the response to a given input is reduced to a large number of simple problems, each involving the determination of the response of the system to a particular elementary signal.

Before proceeding to discuss some of the contributions to the characterization of time-varying systems, it would be helpful to summarize some of the well-known facts pertaining to the technique of resolving the input and output into elementary signals. (For a more detailed discussion, see Huggins [19], Lanning and Battin [16], Gerardi [12], and Zadeh [2].)

**Resolution into Delta Functions**

In this case, the family of elementary signals has the form \( \{ \delta(t - \xi), -\infty < \xi < \infty, -\infty < t < \infty \} \) where \( \delta(t - \xi) \) denotes a unit impulse occurring at time \( \xi \), with \( \xi \) ranging over the interval \(( -\infty, \infty)\). Assuming, for simplicity, that the system has a single input \( u \) and a single output \( y \), and that it is initially at rest, the relation between the input and output can be written as

\[
y(t) = \int_{-\infty}^{\infty} h(t, \xi) u(\xi) d\xi
\]

where \( h(t, \xi) \) is called the impulse (or impulse) response of the system and is defined as the response of the system at rest to \( \delta(t - \xi) \). If a system is nonanticipative (i.e., if, for all \( t \), \( y(t) \) is independent of the values of the input for times greater than \( t \)), then \( h(t, \xi) \equiv 0 \) for \( t < \xi \) and the upper limit in (1) can be replaced by \( t \). (For a discussion of the relationship between the impulsive response and Green’s function, see Miller [3] and Zadeh [4].)

**Resolution into Exponential Functions**

Here, the elementary signals are of the form \( \{ e^{\omega t}, -\infty < \omega < \infty, -\infty < t < \infty \} \) or more generally \( \{ e^{\omega t}, \omega \in C \} \) where \( C \) is a Bromwich-Wagner contour in the \( s \) plane. In terms of such signals, a system \( B \) is characterized by its frequency-response function \( H(\omega, t) \), which is defined as

\[
H(\omega, t) = \frac{\text{response of } B \text{ to } e^{\omega t}}{e^{\omega t}}.
\]

The time-varying frequency-response function \( H(\omega, t) \) constitutes a natural generalization of \( H(\omega) \)—the frequency-response function of a time-invariant system. If the system is initially at rest and the Fourier transform of the input \( u \) is denoted by \( U(\omega) \), the expression for the output at time \( t \), \( y(t) \), in terms of \( H(\omega, t) \) and \( U(\omega) \) is

\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega, t) U(\omega) e^{\omega t} d\omega.
\]

**General Relations for Resolution into Elementary Time Functions**

If a family of time functions \( \{ k(t, \lambda) \} \), with \( \lambda \) taking values on some contour \( C \) in the \( \lambda \) plane and \( t \) ranging over a finite or infinite interval \( T \), form a basis for the input and output spaces of a system \( B \) [in the sense that the \( k(t, \lambda) \) and \( C \) are such that any signal in the input or the output space of \( B \) can be resolved into the \( k(t, \lambda) \)], then an input \( u \) can be expressed as

\[
u(t) = \int_{c} k(t, \lambda) U(\lambda) d\lambda, \quad t \in T,
\]

where \( U(\lambda) \) is an integral transform of \( u \) given formally by

\[
U(\lambda) = \int_{-\infty}^{\infty} k^{-1}(\lambda, t) u(t) dt.
\]

In this relation, the kernel \( k^{-1}(\lambda, t) \) is inverse to \( k(t, \lambda) \) in the sense that

\[
\int_{c} k(t, \lambda) k^{-1}(\lambda, \xi) d\lambda = \delta(t - \xi), \quad t, \xi \in T.
\]

\( B \) is characterized by its responses to the \( k(t, \lambda), \lambda \in C \). Thus, if the response of \( B \) to \( k(t, \lambda) \) is denoted by \( K(t, \lambda) \), then the response of \( B \) to \( u \) (with \( B \) initially at rest) can be written as:

\[
y(t) = \int_{c} K(t, \lambda) U(\lambda) d\lambda.
\]

Eqs. (1) and (3) are special cases of this relation corresponding to \( k(t, \lambda) = \delta(t - \lambda) \) and \( k(t, \lambda) = e^{\omega t}/2\pi j \), respectively.

**Resolution into Eigenfunctions**

A function \( k(t, \lambda), t \in T \), is an eigenfunction of \( B \) if the response of \( B \) to \( k(t, \lambda) \) (with \( B \) initially at rest) is of the form: constant (depending on \( \lambda \)) times \( k(t, \lambda) \). If \( B \) has a set of eigenfunctions \( \{ k(t, \lambda), \lambda \in C \} \) which can be used as a basis for the input and output spaces of \( B \), then the input-output relationship of \( B \) can be expressed in the simple form

\[
Y(\lambda) = R(\lambda) U(\lambda)
\]

where \( R(\lambda) \) is the response function of \( B \)

\[
R(\lambda) = \frac{\text{response of } B \text{ to } k(t, \lambda)}{k(t, \lambda)}
\]

and \( U(\lambda) \) and \( Y(\lambda) \) are the integral transforms of \( u \) and \( y \), respectively; i.e.,

\[
U(\lambda) = \int_{T} k^{-1}(\lambda, t) u(t) dt
\]

and

\[
Y(\lambda) = \int_{T} k^{-1}(\lambda, t) y(t) dt.
\]

It can be shown readily that if a system is time-invariant and homogeneous (in the sense that if \( u \) is any input in the input space of \( B \), then for all real constants \( k \), the response of \( B \) to \( ku \) is \( k \) times the response of \( B \) to \( u \)), then the functions \( \{ e^{\omega t}, -\infty < \omega < \infty \),
constitute an eigenfunction set for \( B \).
(Note that \( B \) need not be linear.) It is in this sense that
the exponential functions \( e^{int} \) constitute a “natural”
set of elementary time functions for linear time-invariant
systems.

This concludes our brief review of the terminology
and basic relations pertaining to the characterization
of linear systems in terms of their responses to
elementary time functions.

II. TRANSITIONS BETWEEN DIFFERENT MODES
OF CHARACTERIZATION

In dealing with time-varying systems, one frequently
encounters a situation where a system \( B \) is characterized
in a given way \( W \), and it is desired—for one reason or
another—to characterize \( B \) in a particular way \( W^\star \)
which is different from \( W \). For example, \( B \) might be
initially characterized\(^1\) by a differential equation of the
form
\[
a_n(t) \frac{d^n y}{dt^n} + \cdots + a_0(t)y = b_m(t) \frac{d^m u}{dt^m} + \cdots + b_0(t)u, \tag{12}
\]
or more compactly
\[
L(p, t)y = M(p, t)u, \quad p = \frac{d}{dt}
\]
where
\[
L(p, t) = \sum_{i=1}^{n} a_i(t) p^i
\]
and the \( a_i(t) \) and \( b_i(t) \) are given functions of time.
The desired characterization of \( B \) is:
\[
y(t) = \int_{-\infty}^{\infty} h(t, \xi) u(\xi)d\xi \tag{13}
\]
where \( h(t, \xi) \) is the impulsive response of \( B \). Here, the
problem is that of solving the differential equation
\[
(a_n(t) p^n + \cdots + a_0(t)) h(t, \xi) = (b_m(t) p^m + \cdots + b_0(t)) s(t - \xi) \tag{14}
\]
subject to the initial condition
\[
\left. \frac{\partial^{n-1} h(t, \xi)}{\partial \xi^{n-1}} \right|_{t=\xi^-} = 0, \quad n = 0, 1, \ldots, n - 1.
\]
(For a more detailed discussion of this and related
problems, see Borsky \[6\], Solodov \[5\], and Zadeh \[4\].

\(^1\) It is generally unrecognized that a differential equation such as
(12) does not characterize a system uniquely unless it is tacitly under-
stood that the system must be nonanticipative. More specifically,
(12) defines \( n+1 \) nonequivalent systems of which one is nonantici-
pative, one is purely anticipative, and the rest operate on both the
past and the future of the input.

Usually, the problem is to pass from an implicit character-
ization employing differential operators to an ex-
plicit characterization involving such functions as the
impulsive response, frequency-response function, etc.
In some cases, however, one is faced with a converse
problem, namely, that of passing from, say, the im-
ulsive response to the differential equation. This may
happen, for example, when it is desired to simulate a
system on an analog computer which is designed to
handle differential equations.

An interesting special case of the latter type of problem
was considered by White \[7\]. Specifically, White consi-
ders the case where one is given the eigenfunction set
\( \{ k(t, \lambda) \} \) of an undetermined differential operator
\( L(p, t) \) which admits a representation of the form
\[
L(p, t) = \sum_{s=0}^{n} l_s \mu^s p^s \tag{15}
\]
where the \( l_s \) are unknown constants, and \( \mu \) and \( \nu \) range
over integers. Furthermore, it is assumed that \( k(t, \lambda) \)
can be expanded as a power series in \( \lambda \),
\[
k(t, \lambda) = \sum_{s=0}^{n} k_s(\lambda) \lambda^s, \tag{16}
\]
and that the response function \( R(\lambda) \) can likewise be ex-
panded as
\[
R(\lambda) = \sum_{s=0}^{n} R_s(\lambda) \lambda^s, \tag{17}
\]
where the coefficients \( R_s \) may be adjusted to yield a
simple solution to the problem if one exists.

Formally, the problem is solved by substituting (12)–
(14) in
\[
L(p, t) k(t, \lambda) = R(\lambda) k(t, \lambda), \tag{18}
\]
and equating the coefficients of the like powers of \( \lambda \) and
\( t \). This yields a set of equations
\[
\sum_{s=1}^{n} \frac{n!}{(n-s)!} h_s(\lambda-n) = h_{\lambda-s}
\]
which may be solved for the \( l_s \) (if a solution exists).

For example,
\[
k(t, \lambda) = J_\nu(\lambda t) \quad (\text{Bessel function}) \tag{20}
\]
\[
\sum_{n=0}^{\infty} (-1)^{n/2} \frac{(\lambda t)^n}{2^n [(n+1)/2]^n} = 0, \quad n = 0, 2, 4, \ldots. \tag{21}
\]

In this case, it is expedient to set \( h_2 = -1, h_0 = h_1 = h_3 = \ldots = 0 \). Then
\[
\sum_{s=1}^{n} \frac{n!}{(n-s)!} = -1 [-4(n/2)^2], \tag{22}
\]
or
\[
+l_{h-2} + l_{h-2} u + l_{h-2} u(n-1) + \cdots = n^2 \tag{23}
\]
from which it follows that
\[
l_{h-2} = 1, \quad l_{h-2} = 1, \quad \text{all other } l_{sh} = 0.
\]
Thus, the desired operator is
\[ L(p, t) = p^2 + \frac{1}{t} p. \]  
(24)

The relation given by (19) can readily be extended to the case where \( k(t, \lambda) \) admits a more general representation of the form
\[ k(t, \lambda) = \sum_n \sum_m k_{nm} \lambda^n t^m. \]  
(25)

Then (19) becomes
\[ \sum_n \sum_m k_{nm} \lambda^{n-m} \frac{m!}{(m - \nu)!} = \sum_n h_{n-m} k_{nu}, \]  
(26)

which, for the special case of (16), reduces to (19).

If the eigenfunction set is identified with the functions \( \{ t^\nu, \ 0 < t < \infty \} \), with \( \lambda \) taking values on a Bromwich-Wagner contour in the \( \lambda \) plane, then the differential operators relating \( y \) to \( u \) assume the form
\[ (a_n p^n + a_{n-1} p^{n-1} + \cdots + a_0) y = (b_n p^n + \cdots + b_0) u, \]  
(27)

where the \( a \)'s and \( b \)'s are constants. Networks described by equations of this form have been studied by Aseltine [8], Aseltine and Trautman [9], Davis [10], Ho and Davis [11], Gerardi [12], and others, and are referred to in the literature as the Euler-Cauchy networks. In effect, such networks can be obtained from time-invariant networks by the transformation of timescale \( t \rightarrow t \log t \). Based on this observation, it can readily be shown that the impulsive response of a non-anticipative Euler-Cauchy network is of the form
\[ k(t, \xi) = 1(t - \xi) \frac{1}{t} \xi \left(\frac{t}{\xi}\right) \]  
(28)

where \( g(t-\xi) \) is the impulsive response of the time-invariant network from which the Euler-Cauchy network is derived.

If \( k(t, \lambda) \) is identified with \( 1/(2\pi j) (t^\lambda) \), then the inverse kernel is given by
\[ k^{-1}(\lambda, l) = l^{-\lambda}, \quad 0 \leq t < \infty. \]  
(29)

Thus, the resolution of a signal \( u(t \geq 0) \) into the \( k(t, \lambda) \) assumes the form
\[ u(t) = \frac{1}{2\pi j} \int_c \sum_n a_n \lambda^n U(\lambda) d\lambda \]  
(30)

and
\[ U(\lambda) = \int_0^\infty u(t) t^{\lambda-1} d\lambda \]  
(31)

where \( U(\lambda) \) is the Mellin transform of \( u \). In some cases, it is more convenient to employ a modified Mellin transform of \( u \) which is related to (31) by the translation \( t \rightarrow t+a \), where \( a \) is a constant. Such transforms have been used for illustrative purposes by Davis [10]. More recently, Peschon [13] has found a useful application for modified Mellin transforms in final value control problems.

An interesting idea which was motivated by the work of Aseltine and Trautman is contained in the paper by Davis [10] cited above. Specifically, let
\[ L = a_1(t) p + a_0(t) \]  
(32)

be a first-order differential operator and let \( k(t, \lambda) \) and \( k^{-1}(\lambda, t) \) be, respectively, the solutions of
\[ L y = \lambda y, \quad y(0) = 1 \]  
(33)

and
\[ L^* y = \lambda y, \quad y(0) = 1 \]  
(34)

where
\[ L^* = - p a_1(t) + a_0(t) \]  
(35)

is the adjoint of \( L \). Then the kernels \( k(t, \lambda) \) and \( k^{-1}(\lambda, t) \) are inverse to one another in the sense of (6), and, under mildly restrictive conditions, one can write
\[ U(\lambda) = \int_0^\infty k^{-1}(\lambda, t) u(t) dt \]  
(36)

and
\[ u(t) = \frac{1}{2\pi j} \int_c k(t, \lambda) U(\lambda) d\lambda. \]  
(37)

Now, given a time-invariant network \( N \), one can transform it into a time-varying network \( N^* \) by replacing each unit inductor (which corresponds to the operator \( p \)) with a network \( l \) corresponding to operator \( L \), and replacing each unit capacitor with a network \( l' \) corresponding to the operator \( L^{-1} \) (inverse of \( L \)), leaving all resistances unchanged. The networks \( l \) and \( l' \) corresponding to \( L \) and \( L^{-1} \), respectively, are described by the voltage-current relations \( v = Li \) and \( i = Lv \). Thus, \( l' \) can be obtained from \( l \) by dualization. Clearly, if a network function associated with time-invariant \( N \) is a function \( H(s) \) of complex-frequency \( s \), then the corresponding response function for time-varying \( N^* \) will be \( H(\lambda) \), with \( \lambda \) being the parameter entering into \( k(t, \lambda) \). In this way, one can derive from a prototype time-invariant network \( N \) a wide variety of time-varying networks \( N^* \) which are described by the same equations.

The adjoint of a differential operator
\[ L = \sum_n a_n(t) \lambda^n \]  
(38)

is defined as
\[ L^* = \sum_n (-1)^n p a_n(t), \]  
(39)

where \( p a_n(t) \) signifies that \( p \) operates on the product of \( a_n(t) \) and the operand; e.g., \( (p a_1(t)+a_0(t))u(t) \) means \( d/dt(a_1(t)u(t)) + a_0(t)u(t) \). For a more detailed discussion see, for instance, Ince [14], Friedman [15], and Lanning and Battin [16]. It should be noted, alternatively, that the adjoint of a system \( B \) with impulsive response \( k(t, \xi) \) can be defined as a system \( B^* \) whose impulsive response \( k^*(t, \xi) \) is \( k(\xi, t) \), i.e., \( k^*(t, \xi) = k(\xi, t) \). With this as the starting point, it is a simple matter to show that if \( B \) is characterized by the differential equation
\[ Ly = u, \]  
then \( B^* \) is characterized by the differential equation
\[ L^* y = u, \]  
where \( L^* \) is the adjoint of \( L \) in the sense defined above. However, if \( B \) is characterized by \( Ly = M u \), then, except in special cases, the characterizing equation for \( B \) is not \( L^* y = M u \).
in terms of \( \lambda \) as \( N \) is in terms of \( s \). In effect, Davis' method amounts to an extension of the familiar frequency-transformation technique (see Laurent [17], Bode [17a], and Zadeh [17b]), to time-varying networks. More recently, somewhat overlapping results have been obtained independently by Wattenburg [18].

There are several other contributions, notably by Huggins [19], Lai [20], Gerlach [21], Narendra [22], and others, which are concerned with various techniques of resolution of time functions into elementary signals. These contributions will be discussed in Part II of the paper.

**Transition from Impulsive Response to the Differential Equation**

The problem of finding the differential equation of a system which is characterized by its impulsive response arises in situations where it is simpler to simulate or synthesize a system in terms of its differential equations rather than its impulsive response. It arises also in connection with the so-called factorization problem which is discussed in Section III of this paper.

This problem was first formulated and partially solved by Batkov [23]. Batkov's approach is based on the well-known fact (see, for instance, Coddington and Levinson [24], and Miller [3]) that the impulsive response of a system characterized by the differential equation

\[
Ly = (a_n(t)p^n + \cdots + a_0(t))y(t) = u(t)
\]

has the form

\[
h(t, \xi) = 1(t - \xi) \sum_{i=1}^{n} \psi_i(\xi)\psi_i^*(\xi)
\]

(39)

where the \( \psi_i(\xi) \) are any \( n \) linearly independent solutions of the homogeneous equation \( Ly = 0 \) and the \( \psi_i^*(\xi) \) are \( n \) linearly independent solutions of the adjoint equation

\[
L^*y = ((-1)^p a_p(t) + \cdots + a_0(t))y(t) = 0.
\]

(40)

The \( \psi_i^*(\xi) \) can be expressed in terms of the \( \psi_i(t) \) by solving the system of \( n \) linear equations in the \( \psi_i^*(\xi) \)

\[
\frac{\partial^k h(t, \xi)}{\partial t^k} \bigg|_{t=\xi} = 0, \quad k = 0, 1, \ldots, n - 2
\]

(41)

\[
\frac{\partial^{n-1} h(t, \xi)}{\partial t^{n-1}} \bigg|_{t=\xi} = \frac{1}{a_n(\xi)}.
\]

(42)

Another well-known fact is that the operator \( L \) can readily be constructed from the knowledge of the \( \psi_i(t) \). Specifically (see, for example, Ince [14]), one can write

\[
Ly = \begin{vmatrix}
\psi_1(t) & \psi_1^*(t) & y(t) \\
\psi_1^{(1)}(t) & \psi_1^{*(1)}(t) & y^{(1)}(t) \\
\vdots & \vdots & \vdots \\
\psi_1^{(n)}(t) & \psi_1^{*(n)}(t) & y^{(n)}(t)
\end{vmatrix}
\]

(43)

Thus, if \( h(t, \xi) \) is given in the form (39) and it is known that the system is characterizable in the form (1), then \( L \) can be obtained at once from (43). Far less trivial is the more general case where the differential equation characterizing the system is of the form \( Ly = Mu \) or, more explicitly,

\[
[a_n(t)p^n + \cdots + a_0(t)]y = [b_m(t)p^m + \cdots + b_0(t)]u,
\]

(44)

where \( m < n \) and the \( a_i(t) \) and \( b_i(t) \) are initially unknown coefficients.\(^3\) In this case, the impulsive response reads

\[
h(t, \xi) = 1(t - \xi) \sum_{i=1}^{n} \psi_i(\xi)\theta_i(\xi)
\]

(45)

where the \( \psi_i(\xi), i = 1, \ldots, n \) are \( n \) linearly independent solutions of \( Ly = 0 \) and the \( \theta_i(\xi), i = 1, \ldots, n \), are given by

\[
\theta_i(\xi) = M^*[\psi_i^*(\xi)], \quad i = 1, \ldots, n,
\]

(46)

where the \( \psi_i^*(\xi) \) are \( n \) linearly independent solutions of the adjoint equation \( L^*y = 0 \) (satisfying \([41]\) and \([42]\)) and \( M^* \) is the adjoint of \( M \). If \( h(t, \xi) \) is given in this form, then from (45) one can find the \( \psi_i(\xi) \) and \( \theta_i(\xi) \).

Once the \( \psi_i(\xi) \) are known, it is a relatively simple matter to determine the \( \psi_i^*(\xi) \).\(^4\) Thus, the problem reduces to finding \( M^* \) from (46). On writing

\[
M^* = [b_m^*(t)p^m + \cdots + b_n^*(t)]
\]

(47)

where the \( b_k^*(t) \) are undetermined coefficients, (46) yields a set of \( n \) simultaneous equations in the \( b_k^*(t) \)

\[
\theta_i(t) = \sum_{k=0}^{n-1} b_k^*(t)\psi_i^{(k)}(t), \quad i = 1, \ldots, n.
\]

(48)

Once these \( n \) equations are solved for the \( m+1 \) coefficients \( b_k^*(t) \) in \( M^* \), \( M \) can be found from the relation between a differential operator and its adjoint.\(^2\) This completes the determination of \( L \) and \( M \) in (44) from the knowledge of \( h(t, \xi) \).

It is simpler and frequently more effective to derive from \( h(t, \xi) \), not a single differential equation of the form \( Ly = Mu \), but a system of \( n \) first-order differential equations of the form

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\]

(49)

together with the expression for the output

\[
y(t) = a_1(t)x_1(t) + \cdots + a_n(t)x_n(t) + \beta_0(t)u(t) + \cdots + \beta_{n-m}(t)u^{(m-n)}(t)
\]

(50)

\(^2\) The procedure for finding the operator \( M \) given in Batkov's paper is somewhat less straightforward than that given here.

\(^4\) See Ince [14], p. 124.
where \( x(t) = (x_1(t), \cdots, x_n(t)) \) is a state vector, \( A(t) \) and \( B(t) \) are matrices, and the \( a_i(t) \) and \( b_k(t) \) are scalar coefficients. These equations constitute a very useful mode of characterization of a system and will be referred to as its state equations. For a homogeneous equation \( Ly = 0 \), the state equation \( \dot{x} = Ax \) is simply a normal form of \( Ly = 0 \) (see, for instance, Ince [14]). Equations of the form (49) and (50) have been employed extensively in analytical dynamics and, more recently, in the theory of automatic control.

We shall discuss briefly three problems arising in connection with the state equations of a system: first, the problem of obtaining the representation (49) and (50) from \( h(t, \xi) \); second, the problem of obtaining (49) and (50) from \( Ly = 0 \); and third, the problem of obtaining (49) and (50) directly from the structure of a time-varying network.

Batkov [23] and, somewhat earlier, Darlington [25] have indicated how the first two of these problems can be solved. As for the third problem, Kinarawala [26] has shown how one can adapt Bashkov’s A-matrix technique [27] to time-varying networks and thereby obtain (49) for any \( R(t)L(t)C(t) \) network without the necessity of performing differentiations or integrations on the loop or node equations of the network.

The basic idea behind the methods of Darlington and Batkov is closely related to the partial-fractions technique of Lur’e [28] for casting into the diagonal or, more generally, the Jordan canonical form, the dynamical equations of the linear part of a nonlinear system. Specifically, we observe that each term in the expression for \( h(t, \xi) \)

\[
h(t, \xi) = (1 - \xi) \sum_{i=1}^{n} \psi_i(t)\theta_i(\xi) \tag{51}
\]

can be identified with the impulsive response of a first-order system. Thus, we can write

\[
y(t) = x_1(t) + \cdots + x_n(t) \tag{52}
\]

where \( x_i(t) \) is defined by

\[
x_i(t) = \int_{-\infty}^{t} \psi_i(t)\theta_i(\xi)u(\xi)d\xi. \tag{53}
\]

Differentiating both sides of (53), we get

\[
\dot{x}_i(t) = \psi_i(t)\theta_i(t)u(t) + \int_{-\infty}^{t} \psi_i(t)\theta_i(\xi)u(\xi)d\xi \tag{54}
\]

which can be written as

\[
\dot{x}_i(t) = \frac{\psi_i(t)}{\psi_i(t)} x_i(t) + \psi_i(t)\theta_i(t)u(t), \quad i = 1, \cdots, n. \tag{55}
\]

These equations, together with (52), form the state equations for the system, with

\[
A = \begin{bmatrix}
\psi_1/\psi_1 & 0 & 0 & \cdots & 0 \\
0 & \psi_2/\psi_2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \psi_n/\psi_n
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
\psi_1\theta_1 \\
\psi_2\theta_2 \\
\psi_n\theta_n
\end{bmatrix} \tag{56}
\]

This technique results in a diagonal \( A \) even in the case of a time-invariant system whose characteristics equation \( L(p) = 0 \) has repeated roots. However, the price for this is that one or more of the elements of \( A \) are functions of time rather than constants. If it is required that the elements of \( A \) be constants, then \( A \) can always be put into the Jordan canonical form but not, in general, the diagonal form.

---

\[ ^5 \text{See also B. V. Bulgakov, "On normal coordinates," Prikl. Mat. i Meh., vol. 10, no. 2, pp. 273–290; 1946.} \]
The methods described above furnish a straightforward way for effecting a transition from a single differential equation

$$L y = u, \quad (L \text{ of order } n) \quad (65)$$

to the state equations

$$\dot{x} = Ax + Bu, \quad A \text{ diagonal} \quad (66)$$

$$y = \alpha_1 x_1 + \cdots + \alpha_n x_n \quad (67)$$

provided one knows a set of $n$ linearly independent solutions of the homogeneous equation $Ly = 0$. Note that if $A$ is not required to be diagonal, then the transition can be effected in conventional ways without the knowledge of solutions of $Ly = 0$.

From (66) and (67), one can pass to a more explicit characterization

$$x(t) = G(t, t_0)x(t_0) + \int_{t_0}^{t} G(t, \xi)Bu(\xi)d\xi \quad (68)$$

$$y = \alpha_1 x_1 + \cdots + \alpha_n x_n \quad (69)$$

by using the standard form of solution of the vector equation (66). (See, for instance, Coddington and Levinson [24] and Laning and Battin [16].) The $n \times n$ matrix $G(t, t_0)$ appearing in (68) has as its $i$th column ($i = 1, \ldots, n$) the functions $\psi_i(t, t_0), \psi_i^{(')}(t, t_0), \ldots, \psi^{('i-1)}(t, t_0)$ where $\psi_i(t, t_0)$ is the solution of the $Ly = 0$ which satisfies the initial conditions

$$\left[ \frac{\partial \psi_i^{('k)}(t, t_0)}{\partial t^k} \right]_{t=t_0} = 0 \quad (70)$$

for $k \neq i - 1$ (with $k$ ranging over the set $0, \ldots, n - 1$) and

$$\left[ \frac{\partial \psi_i^{('k)}(t, t_0)}{\partial t^k} \right]_{t=t_0} = 1 \quad (71)$$

for $k = i - 1$. $G(t, t_0)$ is known by a variety of names of which the transition matrix is the more frequently used in engineering literature. (For a more detailed discussion of the properties of $G(t, t_0)$, see, for instance, Laning and Battin [34] and Kalman and Bertram [35].)

Now suppose that one does not know a set of $n$ linearly independent solutions of $Ly = 0$ or $\dot{x} = Ax$. In these circumstances, there is no general way of passing from (65) or (66) to (68). However, there are special cases for which $G(t, t_0)$ can readily be found. One such case has been discussed by Gauthier [36] and independently by Kinawala [26]. Specifically, this is the case where, as in the constant coefficient case, $G(t, t_0)$ can be expressed as

$$G(t, t_0) = \exp - \int_{t_0}^{t} A(\xi)d\xi. \quad (72)$$

It is easy to show that $G(t, t_0)$ can be written in this form if, and only if, the matrices $A(t)$ and $\int_{t_0}^{t} A(\xi)d\xi$ commute, i.e.,

$$A(t) \int_{t_0}^{t} A(\xi)d\xi \equiv \left( \int_{t_0}^{t} A(\xi)d\xi \right) A(t). \quad (73)$$

We turn next to an important problem on which the techniques discussed in this section have a significant bearing. This is the problem of factorization.

### III. Problem of Factorization

In dealing with stationary as well as nonstationary time series and, more particularly, with problems of prediction, detection and filtering, it is frequently expedient to employ so-called spectrum-shaping techniques which allow one to modify at will the spectral density and the correlation function of a random process. The spectrum-shaping technique was first used by Bode and Shannon [31] and independently by Zadeh and Ragazzini [32]. Its extension to time-varying systems was given by Laning and Battin [16] and Miller and Zadeh [33].

In the case of nonstationary processes, a typical problem in spectrum shaping is the following. Given a nonstationary process $\{u(t), -\infty < t < \infty\}$ with covariance function $R(t, \tau)$, find a linear network $B$ such that the covariance function of the process resulting from acting on $\{u(t)\}$ by $B$ is $\delta(t-\tau)$ (or, more generally, a linear combination of $\delta$ functions of various orders). In other words, we wish to find a $B$ such that $\{y(t), -\infty < t < \infty\}$, where $y=Bu$, has white spectrum. Furthermore, we wish to characterize $B$ by a single differential equation.

This problem has an important bearing on the solution of an integral equation which is encountered very frequently in the prediction, detection and filtering of nonstationary processes. The equation in question reads

$$f(t) = \int_{a}^{b} R(t, \tau)x(\tau)d\tau, \quad a \leq t \leq b \quad (74)$$

where $R(t, \tau)$ is a covariance function, $f(t)$ is a given function on $[a, b]$, and $x(t)$ is an unknown function. It is easy to show (see, for instance, Miller and Zadeh [33]) that the solution of this equation can be reduced to the factorization of $R(t, \tau)$. In this connection, it is of interest to note that the methods given by Laning and Battin [16], Zadeh and Miller [33], Shinbrot [37], [38], Pugachev [39], [40], and others, for the solution of this equation are either explicitly based on the factorization of $R(t, \tau)$ or make an implicit use of it.

For our purposes, it will be somewhat more convenient to discuss a converse problem, namely, that of finding a spectrum-shaping network $B$ such that the covariance
function of the process resulting from acting with $B$ on white noise is $R(t, \tau)$. If the impulsive response of $B$ is denoted by $h(t, \xi)$, then it is easy to verify that

$$R(t, \tau) = \int_{-\infty}^{\infty} h(t, \xi) h(\tau, \xi) d\xi$$

(73)

Thus, in this case the factorization problem becomes that of solving the integral equation (73) for $h(t, \xi)$. [Note that if $h(t, \xi)$ is assumed to be nonanticipative, then the upper limit in (73) becomes $\min (t, \tau)$.]

If $B$ is characterizable by a differential equation of order $n$, then its impulsive response is of the form (45)

$$h(t, \xi) = (t - \xi) \sum_{i=1}^{n} \psi_i(t) \theta_i(\xi)$$

(74)

where the meaning of the $\psi_i$ and $\theta_i$ was discussed previously. On substituting this expression into (73), one gets, after some straightforward manipulations

$$R(t, \tau) = \sum_{i=1}^{n} \psi_i(t) \gamma_i(\tau), \quad t > \tau$$

(75)

$$= \sum_{i=1}^{n} \psi_i(\tau) \gamma_i(t), \quad t < \tau$$

(76)

where

$$\gamma_i(t) = \sum_{j=1}^{n} \psi_j(t) \int_{-\infty}^{t} \theta_j(\xi) \theta_i(\xi) d\xi; i = 1, \ldots, n.$$  

Thus, if $R(t, \tau)$ is given in this form, then the problem of finding the differential equation can be solved by first finding $h(t, \xi)$ [based on (74)-(76)] and then using $h(t, \xi)$ to determine the differential equation by the techniques described in the preceding section. Essentially, this approach to the determination of the spectrum-shaping network was developed by Darlington [29] and, independently, by Batkov [41]. Darlington, in particular, has considered the questions of existence and uniqueness of a nonanticipative (physically realizable) impulsive response $h(t, \xi)$ satisfying the integral equation (73). However, neither Darlington nor Batkov have proved that any given real-valued covariance function (satisfying the conditions of symmetry and positive definiteness) can be expressed in the form (73), with $h(t, \xi)$ being a real-valued function vanishing for $\xi > t$.

As is well known for the time-invariant case, a rational spectral-density function $S(\omega)$ [$S(\omega)=$Fourier transform of $R(t)$, with $R(t) = R(t, 0) =$ autocorrelation function] can be expressed in an infinity of ways as a product of factors $H_1(j\omega)$ and $H_2(j\omega)$

$$S(\omega) = H_1(j\omega) H_2(j\omega)$$

(76a)

if no restrictions are imposed on $H_1(j\omega)$ and $H_2(j\omega)$ other than that $H_1$ and $H_2$ be rational and real functions of $j\omega$. However, if 1) $H_1(j\omega)$ is required to be minimum phase (no poles or zeros in the right-half plane), which is equivalent to requiring that the impulse responses $h_1(t)$ and $h_1^{-1}(t)$, corresponding to $H_1(j\omega)$ and $1/H_1(j\omega)$, respectively ($h_1=$inverse Fourier transform of $H_1$, $h_1^{-1}=$inverse Fourier transform of $H_1^{-1}$), be nonanticipative, and 2) $H_2(j\omega)$ is required to be maximum phase (no poles or zeros in the left-half plane), which is equivalent to requiring that $h_2$ and $h_2^{-1}$ be purely anticipative, i.e., $h_2(t) = h_2^{-1}(t) = 0$ for $t > 0$, then the factorization (76a) is essentially unique.

Now in the time-varying case, the algebraic decomposition (76a) which corresponds to the convolution

$$R(t) = \int_{-\infty}^{\infty} h_1(t - \xi) h_2(\xi) d\xi$$

(76b)

is replaced by

$$R(t, \tau) = \int_{-\infty}^{\infty} h_1(t, \xi) h_2(\xi, \tau) d\xi$$

(76c)

which is the composition* of impulsive responses $h_1$ and $h_2$. Equivalently, (76c) may be written as

$$R(t, \tau) = \int_{-\infty}^{\infty} h_1(t, \xi) h_2^{*}(\tau, \xi) d\xi$$

(76d)

where $h_2^*$ is the adjoint of $h_2$ [i.e., $h_2^*(t, \xi) = h_2(\xi, t)$].

Thus, in the time-varying case, the algebraic problem of decomposition of $S(\omega)$ into two factors $H_1(j\omega)$ and $H_2(j\omega)$ satisfying 1) and 2) becomes that of solving the integral equation (76c) for $h_1$ and $h_2$, under the condition that $h_1(t, \xi)$ and $h_1^{-1}(t, \xi)$ vanish for $t < \xi$, and $h_2(t, \xi)$ and $h_2^{-1}(t, \xi)$ vanish for $t > \xi$.

If $h_1$ and $h_2$ satisfy these conditions and $R(t, \tau)$ is symmetric in its arguments [which in the stationary case corresponds to the condition that $S(\omega)$ is an even function of $\omega$], then $h_1 = h_2^*$ and (76d) becomes

$$R(t, \tau) = \int_{-\infty}^{\min (t, \tau)} h_1(t, \xi) h_2^*(\tau, \xi) d\xi$$

(76e)

which is equivalent to (73). Now if: a) $h(t, \xi)$ is regarded as the impulsive response of a system characterized by a differential equation

$$L y = M u$$

(76f)

in which $L$ and $M$ are undetermined differential operators, and b) $R(t, \xi)$ is given in the form (75), (76), then by using the techniques of Darlington and Batkov it is

Essentially, if $B_1$ is a system whose impulsive response is $h_i(t = 1, 2, 3)$, then $h_0$ is the impulsive response of $B_1B_2$, that is, the tandem combination of $B_1$ and $B_2$, with $B_1$ operating on the output of $B_2$. 

* The composition of $h_1(t, \xi)$ and $h_2(t, \xi)$ is defined as

$$h_0(t, \xi) = \int_{-\infty}^{\infty} h_1(t, \lambda) h_2(\lambda, \xi) d\lambda.$$
From this and the preceding equations, it can be deduced that \( \alpha(t, \xi) \) satisfies a Fredholm integral equation of the second kind

\[
\alpha(t, \xi) = \eta(t, \xi) - \int_0^t \eta(\lambda, \xi) \alpha(t, \lambda) d\lambda, \quad t > \xi. \tag{81}
\]

Now the function \( \eta(t, \xi) \) in (81) can be expressed explicitly in terms of \( R(t, \tau) \) and its derivatives by expressing the two factors in the denominator of \( \eta(t, \xi) \) [see (77)] in terms of \( R(t, \tau) \). For example,

\[
\frac{\partial^{n-m-1} h(\theta, \lambda)}{\partial \theta^{n-m-1}} \bigg|_{\lambda = \theta} = \sqrt{(-1)^{n-m-1} R_{2n-2m-1}(\theta, \theta)}. \tag{82}
\]

Thus, (81) can be solved, in principle, for \( \alpha(t, \xi) \) in terms of \( \eta(t, \xi) \). Then, knowing \( \alpha(t, \xi) \), one can determine \( \beta(t, \xi) \) from (80). Finally, \( h(t, \tau) \) is found from the relation (note the interchange in arguments in \( h \))

\[
h(\tau, t) = \gamma(t, \tau) - \int_0^t \alpha(\lambda, \gamma(\lambda, \tau) d\lambda, \quad t < \tau \tag{83}
\]

and the denominator of \( \gamma(t, \tau) \) is expressible in terms of \( R(t, \tau) \) in the manner cited previously. In summary, by the use of this technique, the solution of the integral equation (73) for \( h(t, \xi) \) is reduced to the considerably simpler problem of solving the Fredholm equation (81) for \( \alpha(t, \lambda) \).

The problem of factorization is encountered also in the analysis of multipath communication systems [43], [44], where it leads to integral equations of the form (73) defined on a finite interval. Such equations have been discussed by Kailath [43] and Middleton [44].

IV. RANDOM-VARYING SYSTEMS

Linear randomly-varying systems play an important role in problems encountered in the study of such varied phenomena as propagation through time-varying media, reflection from fluctuating targets, turbulence, scattering, amplitude and phase modulation, magnetohydrodynamics and plasma.

Despite their importance, few if any attempts at studying the behavior of randomly-varying systems on a theoretical level were made prior to 1950. The papers published since then deal almost entirely with systems in which the random variations in parameters are stationary. This restriction is an essential one, since from a statistical viewpoint a stationary randomly-varying system behaves in some respects like a time-
invariant system. For example, if one is observing the input and output of a time-varying black box \( B \) and it is not known whether \( B \) is linear or nonlinear, then there is no way of deciding between the two alternatives if \( B \) is nonstationary. On the other hand, if \( B \) is stationary, then it can be shown that \( B \) is linear if it has the following extended superposition property. Let \( \{ u(t) \}, \quad -\infty < t < \infty \) and \( \{ v(t) \}, \quad -\infty < t < \infty \) be two independent stationary processes which are independent also of the random processes governing the behavior of \( B \). Let \( R_u(\tau) \) and \( R_v(\tau) \) be the correlation functions of the processes resulting from operating with \( B \) on the \( \{ u \} \) and \( \{ v \} \) processes, respectively. Let \( R_{au+bv}(\tau) \) be the correlation function of the process resulting from operating with \( B \) on the process \( \{ au+\beta v \} \), where \( \alpha \) and \( \beta \) are arbitrary real constants. If

\[
R_{au+bv}(\tau) = \alpha^2 R_u(\tau) + \beta^2 R_v(\tau)
\]  

(85)

for all \( \alpha, \beta, \{ u \} \) and \( \{ v \} \), then we shall say that \( B \) has the superposition property for correlation functions. Clearly, any stationary linear system will have this property, and the converse can also be demonstrated to be true. Consequently, (85) can be used as a basis for determining whether \( B \) is linear or nonlinear by observing the input and output of \( B \) over periods of time sufficiently long to enable the observer to obtain accurate estimates of the correlation functions involved in (85). It is tacitly understood, of course, that \( B \) is such that in the absence of input its state reverts to the ground (unexcited) state as \( t \to \infty \).

The correlation function of the output \( y \) of a stationary system \( B \) which is subjected to a stationary input process \( \{ u \} \) can be conveniently expressed in terms of the correlation function of \( B \). Specifically, the correlation function of a stationary randomly-varying system \( B \) is defined as follows (see Zadeh [45], [46]):

\[
R(j\omega, \tau) = E\{ H(j\omega, \tau)RH(-j\omega, t+\tau) \}
\]

(86)

where \( H(j\omega, t) \) is the frequency-response function of \( B \), and \( E \) denotes the expectation operator. (It is understood that, for each real \( \omega \), \( \{ H(j\omega, t) \}, \quad -\infty < t < \infty \) is a stationary random process.)

If \( \{ u(t) \} \) and \( \{ H(j\omega, t) \} \) are independent, then it can readily be shown that the correlation function of the output process, \( R_y(\tau) \), is related to \( R(j\omega, \tau) \) and the correlation function of the input process \( R_u(\tau) \) through

\[
R_y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(j\omega, \tau)F[R_u(\tau)]e^{i\omega\tau}d\omega
\]

(87)

where \( F[R_u(\tau)] \) is the Fourier transform of the input correlation function. This relation is of the same form as the equation expressing the output of a time-varying network with frequency-response function \( R(j\omega, \tau) \), with the input being \( R_u(\tau) \). [Compare (3).] Relations of the form (87) have been applied by Bugnolo [47], [48] to the analysis of scattering by randomly-varying media.

In the special but significant case of a system characterized by an input-output relationship of the form

\[
y(t) = u(t - \alpha(t))
\]

(88)

the frequency-response function reads

\[
H(j\omega, \tau) = e^{-j\omega\tau}
\]

(89)

where \( \alpha(t) \) plays the role of a variable time delay. If \( \{ \alpha(t) \} \) is assumed to be a stationary Gaussian process, then the expression for \( R(j\omega, \tau) \) is

\[
R(j\omega, \tau) = \exp \{ \omega^2 [ R(\tau) - R(0)] \}
\]

(90)

where \( R(\tau) \) is the correlation function of \( \{ \alpha(t) \} \). This result was extended by Price [50] to systems for which \( H(j\omega, \tau) \) is of the form

\[
H(j\omega, \tau) = \sum I a_i e^{j\omega\alpha_i(t)}
\]

(91)

where the \( \{ \alpha_i(t) \} \) are independent stationary Gaussian processes. Such systems are encountered in connection with the detection of fluctuating targets—a subject that is discussed at length by Price [51] and Price and Green [52]. A more detailed account of the results obtained by Price, Price and Green, Kailath [52], [53], and others, in connection with the identification of randomly-varying systems will be presented in Part II of this paper.

Although the correlation function \( R(j\omega, \tau) \) of a stationary randomly-varying system \( B \), or more generally, the function \( R(j\omega, j\omega', \tau) \)

\[
R(j\omega, j\omega', \tau) = E\{ H(j\omega, \tau)H(-j\omega', t+\tau) \}
\]

(92)

which is the expectation of the product of \( H(j\omega, \tau) \) and \( H(j\omega', t+\tau) \), conveys considerable information concerning the statistical characteristics of \( B \), its application is limited to those systems which are characterized by their frequency-response functions. If—as is frequently the case—\( B \) is characterized by a differential equation with randomly-varying coefficients, then it is generally simpler to employ more direct techniques for studying the behavior of \( B \) in statistical terms. For example, if in

\[
a_i(t) = \frac{d^n y}{dt^n} + \cdots + a_0(t)y = u(t)
\]

(93)

each \( a_i(t) \) can be expressed as

\[
a_i(t) = \bar{a}_i(l) + \epsilon_i(t)
\]

(94)

where \( \bar{a}_i(t) \) is the expected value of \( a_i(t) \), and \( \epsilon_i(t) \) is in some suitable sense small compared with \( \bar{a}_i(t) \), then (93) can be solved by the usual perturbation techniques, with \( y \) expressed as the sum of a nonrandom term and
a random term due to the $\epsilon(t)$. $[u(t)$ is assumed to be nonrandom.] This approach and elaborations on it have been employed by, among others, Sverdrup [54], Bugnolo [47], [48], and Janos [55]. The key assumption in Janos' work, which was motivated by applications to the multipath problem, is that the coefficients $a(t)$ are stationary and stationarily correlated Gaussian processes. In Bugnolo's work, the perturbation technique is used mainly to yield an approximation to the correlation function of a system comprising a randomly-varying propagation medium.

An important problem in the case of a stationary randomly-varying system whose output is a nonstationary process (e.g., due to the effect of initial conditions and/or the application of a nonstationary input) is that of determining the asymptotic behavior of various moments of the output as $t \to \infty$. The analysis of a first-order system

$$\dot{y} + a(t)y = u(t), \quad (95)$$

in which $a(t)$ is a randomly-varying coefficient and $u$ is a nonrandom input, was carried out by Rosenbloom [56] for the case where $\{a(t)\}$ is a Gaussian process, not necessarily stationary. Rosenbloom considered the question of stability of such systems and noted, in particular, that under certain conditions the unit step response of a first-order randomly-varying system can tend to 1 in probability, and yet the mean value of the response may approach $-\infty$ as $t \to \infty$.

A more general analysis of first-order systems in which both $\{a\}$ and $\{u\}$ are stationary and stationarily correlated Gaussian processes was given by Tikhonov [57]. An interesting feature of Tikhonov's method is a simple and yet highly effective artifice for avoiding the difficulties arising out of the dependence of $\{a(t)\}$ and $\{u(t)\}$ processes. Specifically, the general solution of (95) reads

$$y(t) = y(t_0) \exp \left\{ - \int_{t_0}^t a(\lambda) \, d\lambda \right\} + \int_{t_0}^t d\xi \exp \left\{ - \int_{\xi}^t a(\lambda) \, d\lambda \right\} u(\xi). \quad (96)$$

In this form, the calculation of various moments of $y(t)$ is complicated by the fact that the second term on the right involves the product of

$$\exp \left\{ - \int_{\xi}^t a(\lambda) \, d\lambda \right\}$$

and $u(\xi)$. These two processes are not independent and, furthermore,

$$\left\{ \exp \left\{ - \int_{\xi}^t a(\lambda) \, d\lambda \right\} \right\}$$

is not a Gaussian process.

As noted by Tikhonov, this apparent difficulty can be resolved by writing (96) as

$$y(t) = y(t_0) \exp \left\{ - \int_{t_0}^t a(\lambda) \, d\lambda \right\} - \frac{\partial}{\partial \gamma} \left\{ \int_{t_0}^t \exp \left\{ - \int_{\xi}^t (\lambda) \, d\lambda \right\} e^{-\gamma u(\xi)} \, d\xi \right\}_{\gamma=0} \quad (97)$$

where $\gamma$ is a dummy parameter. In this expression, the product of

$$\exp \left\{ - \int_{\xi}^t a(\lambda) \, d\lambda \right\}$$

and $u(\xi)$ is replaced by

$$\exp \left\{ - \int_{\xi}^t a(\lambda) \, d\lambda - \gamma u(\xi) \right\},$$

which is much easier to deal with. By using this artifice, Tikhonov succeeded in obtaining an explicit expression for the correlation function of $\{y(t)\}$, which enabled him to study the asymptotic behavior of the expectation of $y(t)$ and the variance of $y(t)$ for large $t$.

For higher-order systems, concrete results concerning the mean-square stability were obtained by Samuels and Eringen [58], [59]. In particular, for a system of the form

$$Ly + \alpha_k(t) \frac{dy}{dt} = u(t) \quad (98)$$

in which $L$ is a differential operator with constant coefficients, $\alpha_k$ is a randomly-varying parameter with correlation function $\delta(r)$, and $u$ is a randomly-varying input, Samuels and Eringen obtained an explicit expression for the expectation $E\{y^2(t)\}$ as a function of $t$. This permitted them to determine if a system of the type under consideration is stable in mean square, that is, if

$$\lim_{t \to \infty} E\{y^2(t)\} < M,$$

where $M$ is a finite constant. Similar results have been derived independently by Bergen [60].

On a more general level, an extension of Lyapounov's second method to randomly-varying, not necessarily linear, systems was made by Bertram and Sarachik [61]. Essentially, Bertram and Sarachik extended to systems characterized by vector differential equations of the form

$$\dot{x} = f(x, \omega, t) \quad (99)$$

where $\omega$ denotes an element of a probability space $\Omega$, the basic notions of stability, asymptotic stability, asymptotic stability in the large, etc., and derived sufficient conditions a la Lyapounov for various types of stability. For example, the equilibrium solution
$x \equiv 0$ of (99) is said to be stable in the mean if for any $\epsilon > 0$, $t_0$ exists a $\delta(\epsilon, t_0) > 0$ such that for any initial $x$ (at $t = t_0$) satisfying $\|x(t_0)\| < \delta(\epsilon, t_0)$ ($\|x\|$ denotes the norm of $x$), the expectation $E[\|x(t)\|]$ is smaller than $\epsilon$ for all $t \geq t_0$. Then—ina way paralleling that used in the second method of Lyapounov—it can be shown that if there exists a Lyapounov function $V(x, t)$ defined on the state space such that

1) $V(x, t) = 0$ for all $t$.
2) $V(x, t)$ is continuous in both $x$ and $t$ and the first partial derivatives in these variables exist.
3) For all $x$ in the state space, $V(x, t) \geq a\|x\|$ for some fixed $a > 0$.
4) $E[dV(x, t)/dt] \leq 0$ along a solution.

Then the solution $x(t) = 0$ is stable in the mean.

Unfortunately, this and related results do not provide an effective means of determining whether a given randomly-varying system is stable or unstable, since there are no general techniques for finding a Lyapounov function for a given time-varying, much less randomly-varying, system.

Indeed, the only fairly general type of randomly-varying system for which effective stability criteria have been developed is the piecewise constant system characterized by differential equations of the form

$$\dot{x} = A_kx, \quad t_{k-1} \leq t < t_k, \quad k = 1, 2, \cdots$$

where $x$ denotes the state vector, $A_k$ is a constant matrix, and either the $A_i$ or the $t_i$ or both are random variables, with the $A_i$ ranging over a finite set of constant matrices. Various special cases of such systems were studied by Bellman [62], Bellman, Harris, and Shapiro [63], Karlin [64], Kalman [65], Bergen [66], and Bharucha [67].

The principal contribution to the stability theory of such systems was made by Bellman, who was the first to point out that the Kronecker product of matrices furnishes a natural way for attacking the problem of stability of systems of this type. Specifically, Bellman considered the asymptotic behavior of products of matrices of the form

$$X_N = Z_N \cdots Z_1$$

where $Z_i, i = 1, 2, \cdots, N$, are the elements of a sequence of independent, identically-distributed random matrices. For simplicity, the $Z_i$ are assumed to range over two fixed $2 \times 2$ matrices $A$ and $B$, with $Pr \{Z_i = A\} = Pr \{Z_i = B\} = 1/2$.

In effect, the case considered by Bellman is the discrete-time version of (100), with $t_k = k$. Thus,

$$x_t = Z_kx_{k-1}, \quad k = 1, 2, \cdots$$

and

$$x_N = (Z_N \cdots Z_1)x_0.$$  

If the two components of $x_N$ are denoted by $x_{1N}$ and $x_{2N}$, and the norm of $x$ is defined by

$$\|x\| = x_1^2 + x_2^2,$$

then

$$\|x_N\| = x_0'Z_1' \cdots Z_N'Z_Nx_0$$

where the prime signifies transposition. We are interested in the behavior of the expected value of $\|x_N\|$ as $N \rightarrow \infty$.

The expected value of $\|x_N\|$ could readily be found from (104) if the $Z_i$ commuted with one another and their transposes, for (104) could then be written as

$$\|x_N\| = x_0'Z_1'Z_2' \cdots Z_N'Z_Nx_0,$$

and, on taking the expected value of both sides of (105), we would have

$$E[\|x_N\|] = x_0'\{E(Z_1'Z_2)\}'x_0$$

since $E(Z_1'Z_2) = E(Z_2'Z_1) = \cdots = E(Z_N'Z_N)$, and the $Z_i$ are mutually independent. Unfortunately, the $Z_i$, in general, do not commute and as a consequence the expectation of $\|x\|$ cannot be obtained in the simple fashion of (105) and (106).

To circumvent this difficulty, Bellman employed the notion of the Kronecker product (see, for instance, MacDuffie [68]), which is defined by the relation

$$A \otimes B = \begin{bmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ a_{11} & b_{21} & a_{12} & b_{22} \\ a_{11} & b_{21} & a_{12} & b_{22} \\ a_{11} & b_{21} & a_{12} & b_{22} \end{bmatrix}$$

where $A$ and $B$ are, for simplicity, taken to be $2 \times 2$ matrices with elements $a_{ij}$ and $b_{ij}$, respectively. Similarly, the $n$th Kronecker power of $A$ is defined as

$$A_{[n]} = A \otimes A \otimes \cdots \otimes A.$$  

Thus, the second power of a column 2-vector $x$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2x_1 \\ x_2^2 \end{bmatrix}.$$  

A key property of the Kronecker powers of a matrix is expressed by the following identity.

$$AB_{[n]} = A_{[n]}B_{[n]}$$

which means that the $n$th Kronecker power of the product of $A$ and $B$ is the product of $n$th Kronecker powers of $A$ and $B$. Based on this identity, we can derive from

$$x_t = Z_kx_{k-1}, \quad k = 1, 2, \cdots, N.$$
the relation
\[ x(t) = Z(t) x(t - 1), \quad k = 1, \ldots, N \] (112)
and hence,
\[ x(t) = Z(t) x(t - 1) \cdots Z(t_1) x(t_0). \] (113)

Now we can take the expectation of both sides of (113) without the problems which were encountered in (104) as a result of the noncommutativity of the \( Z \).

Thus, we obtain
\[ E[x(t)] = E[Z(t)] E[x(t_0)]. \] (114)

Comparing \( x(t) \) with \( x(t_0) \), it is clear that if \( x(t_0) \rightarrow 0 \) as \( N \rightarrow \infty \), so does \( x(t) \), and vice versa. Consequently, the stability in the mean of the system described by (102) is governed by the behavior of \( E[Z(t)] \) as \( N \rightarrow \infty \). Specifically, if the eigenvalues of the matrix \( E[Z(t)] \) lie inside the unit circle, then \( E[x(t)] \rightarrow 0 \) as \( N \rightarrow \infty \) (assuming \( E[x(t)] < \infty \)), and hence, the system is asymptotically stable in the large (in the mean norm). If one or more eigenvalues lie outside the unit circle, the system is unstable in the mean norm. These and other cases are discussed in greater detail by Bharucha [67].

The case where the \( t_k \) are random variables with \( t_k - t_{k-1}, \quad k = 1, 2, \ldots \), being mutually independent random variables, was first studied by Kalman [69]. Kalman, too, used Kronecker powers of matrices, but his formulas are somewhat complicated by the fact that they relate the expected value of \( \|x(t)\|^2 \) to \( x(t) \), rather than the expected value of \( x(t) \) to that of \( x(t_0) \), as in (114).

The foregoing discussion dealt with the case of free (unforced) systems, that is, systems subjected to zero input. For forced systems, Kalman showed that if the input is bounded, then the response is bounded (in mean square) if and only if the system is stable in the mean square.

Another theorem relating the stability of the free system to the boundedness of the response to bounded inputs was given by Bharucha [69]. Bharucha’s theorem is essentially an extension to randomly-varying systems of a theorem due to Perron [70] and Malkin [71]. The statement of Malkin’s theorem is as follows: consider a system \( B \) characterized by the vector equation
\[ x = A(t) x + u \] (115)
where \( x \) is the state vector, \( u \) is the input vector, and \( A(t) \) is a bounded \((|A(t)| \leq M < \infty, \quad 0 \leq t \leq \infty)\) and continuous matrix. Then the response of \( B \) to any bounded input is bounded if, and only if, there exist positive numbers \( \alpha \) and \( \beta \) such that the norm of every solution \( x(t) \) of the free system
\[ \dot{x} = A(t) x, \] (116)
satisfies the inequality
\[ \|x(t)\| \leq \alpha e^{-\beta(t-t_0)} \|x(t_0)\|, \quad 0 \leq t_0 < t \leq \infty. \] (117)

In the case of a randomly-varying system, \( |A(t)| \) is a random process. For such systems, Bharucha shows that:

1. \( \sup_{t} \|A(t)\| \) is almost surely bounded, and \( A(t) \) is almost surely continuous.

2. \( E \|G(t, \xi)\| \leq \alpha e^{-\beta(t-t_0)}, \quad \alpha > 0, \quad \beta > 0 \) where \( G(t, \xi) \) is the transition matrix of (115).

3. The input process is bounded and independent of \( A(t) \), then \( \sup_{t} E \|x(t)\| \) is bounded for every bounded \( \|x(t_0)\| \).

In the analyses of the stability in the mean of systems defined by (102), it is usually assumed that the \( Z \) form a sequence of independent random matrices. This assumption is much too restrictive for many practical purposes. For example, in the case of discrete channels with fading, the successive \( Z \) have a high degree of dependence. As an approximation to systems of this type, it is natural to assume that the \( Z \) form a Markov process. The stability in the mean square of such systems was studied by Bharucha, but little if anything is known concerning their behavior when subjected to random inputs.

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ADDITIONAL PAPERS


Compatible Single Sideband*

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Summary—A compatible single-sideband (CSSB) wave is a new type of modulated wave which is compatible with existing AM receivers. Spectrum analysis and measurement indicate that if the CSSB system is applied to a conventional AM broadcast transmitter, a desired-to-undesired sideband ratio of slightly better than 30 db will be achieved under normal modulation conditions. Also described is a beat frequency problem which introduces a special type of undesired sideband component. This component falls extremely close to the carrier and should not be present during the vast majority of program conditions. Thus, the technique meets the requirement of theoretically distortion-free envelope characteristics with a good desired-to-undesired sideband ratio.

Measurements are described which show the advantages of the system. It appears that the main advantage of the technique is to reduce co- and adjacent-channel interference effects. CSSB also provides a higher fidelity signal when received by conventional inexpensive broadcast receivers. The on-the-air tests indicated good listener acceptance of the new system.

The system also appears to have applications in communications service where cost and size bar the use of conventional sideband techniques. The technique does not suffer from Doppler shift difficulties.

INTRODUCTION

The fact that there is a severe shortage of spectrum space available to the broadcaster and to the high-frequency communicator has been stressed in a number of persuasive papers.1,2

One method of greatly easing this critical shortage is the use of the SSB. The number of SSB transmitters in operation has greatly increased during the past decade because of this serious spectrum shortage and because of a number of other operational advantages of SSB. However, there are many services that cannot justify the expense and complexity of conventional SSB equipment. These services include broadcasting and many mobile communication systems. The purpose of this paper is to describe a new type of SSB wave called compatible single sideband (CSSB), which appears to be

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