Monte Carlo Solutions to specific PDEs

Yusuf Bugra Erol
MATH224B Project Report
May 9, 2013

1 Introduction

Following Fourier Transform will be used for our purposes.

\[ g(k,t) = \int_{-\infty}^{\infty} f(x,t) e^{-jkx} \, dx \] (1)

\[ f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k,t) e^{jkx} \, dk \] (2)

One property we will use often is

\[ f(x,t) \leftrightarrow g(k,t) \] (3)

\[ \frac{\partial f}{\partial x} (x,t) \leftrightarrow (jk)g(k,t) \] (4)

\[ \frac{\partial^n f}{\partial x^n} (x,t) \leftrightarrow (jk)^n g(k,t) \] (5)

The following identity will be useful in the derivations.

\[ \int_{-\infty}^{\infty} e^{jkx} \, dk = 2\pi \delta(x) \] (6)

2 Diffusion Equations

Let us consider the following model

\[ \frac{\partial f}{\partial t} = a \frac{\partial^n f}{\partial x^n} \] (7)

where \( x \in \mathbb{R} \) and \( t \geq 0 \), with the initial condition \( f(x,0) = f_0(x) \). Define \( g_0(k) \) as \( \mathcal{F}(f_0(x)) \).
2.1 First Order Case

We will consider the following model.

\[
\frac{\partial f}{\partial t} = a \frac{\partial f}{\partial x} \tag{8}
\]

Taking the Fourier Transform of both sides,

\[
\frac{\partial}{\partial t} g(k,t) = ajkg(k,t) \tag{9}
\]

The solution to the 1st order ODE is \(g(k,t) = g_0(k)e^{jakt}\). Taking the inverse Fourier Transform,

\[
f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_0(k) e^{jakt} e^{jkh} dk
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jakt} e^{jkh} dk \int_{-\infty}^{\infty} f_0(x') e^{-jkh} dx'
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(x') dx' \int_{-\infty}^{\infty} e^{j(kx+at-x')} dk
\]

\[
= \int_{-\infty}^{\infty} f_0(x') \delta(x + at - x') dx'
\]

(10)

Using the sifting property of impulse

\[
f(x,t) = f_0(x + at) \tag{11}
\]

Therefore, the solution is a travelling wave solution.

2.2 Second Order Case

We will consider the following model.

\[
\frac{\partial f}{\partial t} = a \frac{\partial^2 f}{\partial x^2} \tag{12}
\]

Taking the Fourier Transform of both sides,

\[
\frac{\partial}{\partial t} g(k,t) = ajkg(k,t) = -ak^2g(k,t) \tag{13}
\]
The solution to the 2nd order ODE is \( g(k,t) = g_0(k)e^{-ak^2t} \). Taking the inverse Fourier Transform,

\[
f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_0(k)e^{-ak^2t} e^{jkx} dk
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ak^2t} e^{jkx} dk \int_{-\infty}^{\infty} f_0(x')e^{-jkx'} dx'
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(x')dx' \int_{-\infty}^{\infty} e^{-ak^2t} e^{jk(x-x')} dk
\]

(14)

Using the following Fourier pair,

\[
e^{-\rho t^2} \leftrightarrow \sqrt{\pi} \rho e^{-\frac{k^2}{4\rho}}
\]

(15)

Therefore,

\[
\int_{-\infty}^{\infty} e^{-ak^2t} e^{jk(x-x')} = \sqrt{\pi} e^{-\frac{(x-x')^2}{4at}}
\]

(16)

Hence,

\[
f(x,t) = \int_{-\infty}^{\infty} f_0(x') \left\{ \frac{1}{\sqrt{4\pi at}} e^{-\frac{(x-x')^2}{4at}} \right\} dx'
\]

(17)

Above form can be considered as an expectation as \( \frac{1}{\sqrt{4\pi at}} e^{-\frac{(x-x')^2}{4at}} \) corresponds to the probability density function (pdf) of a Gaussian random variable with mean \( x \) and variance \( 2at \), i.e. \( N(x,2at) \). Thus; \( f(x,t) = \mathbb{E}[f_0(x')] \),

\[
f(x,t) \approx \frac{1}{N} \sum_{i=1}^{N} f_0(s_i); s_i \sim N(x,2at)
\]

(18)

For the special case of \( a = \frac{1}{2} \), the samples are supposed to be taken from \( N(x,t) \). This pdf corresponds to a Brownian Motion, \( b(t) \) that is initiated at \( x \). This can be written as \( f(x,t) = \mathbb{E} [f_0(b(t)) \mid b(0) = x] \). Hence, for this special case one needs to simulate different Brownian paths, and average over them according to the \( f_0(\cdot) \).

### 2.3 Higher Order Cases

We will consider the following models.

\[
\frac{\partial f}{\partial t} = a \frac{\partial^n f}{\partial x^n}
\]

(19)
Same methodology will lead to the Fourier Transform

\[ g(k, t) = g_0(k) e^{a(jk)^n t} \] (20)

For \( n = 3 \), taking the inverse transform

\[ f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_0(x') dx' \int_{-\infty}^{\infty} e^{jk(x-x')-jak^3t} dk \] (21)

Noticing the similarity to Airy function,

\[ Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(z\tau+\tau^3/3)} d\tau \] (22)

one can write the solution as,

\[ f(x, t) = \int_{-\infty}^{\infty} g_0(x') \left\{ \frac{1}{(3at)^{1/3}} Ai \left[ -\frac{x'-x}{(3at)^{1/3}} \right] \right\} dx' \] (23)

The Airy function exhibits negative values hence it is hard to interpret the above form as a probability. In [1], it is shown that for \( n = 4 \), the integral form does not have a probabilistic interpretation either.

### 2.4 First and Second Case Combined

We will consider the following models.

\[ \frac{\partial f}{\partial t} = a \frac{\partial f}{\partial x} + b \frac{\partial^2 f}{\partial x^2} \] (24)

Taking the Fourier Transform of both sides,

\[ \frac{\partial g}{\partial t} = a(jk)g(k, t) + b(jk)^2 g(k, t) = (ajk - bk^2)g(k, t) \] (25)

Therefore, \( g(k, t) = g_0(k)e^{(ajk-bk^2)t} \). Taking the inverse transform,

\[ f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_0(k) e^{(ajk-bk^2)t} e^{jtx} dx \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_0(x') dx' \int_{-\infty}^{\infty} e^{(ajk-bk^2)t} e^{-jk(x'-x)} dk \] (26)

Let us remember the Levy process. If \( \{X_t\}_{t \geq 0} \) is a Levy process, then by Levy-Khinchine formula, its characteristic function can be written as

\[ \mathbb{E} [e^{j\theta X_t}] = \exp \left( j\theta \right) \left[ \text{linear drift} \right] \left[ \text{Brownian Motion} \right] \left[ \text{Poisson process} \right] \]

\[ = \exp \left( \frac{j\theta t}{2} - \frac{\sigma^2 t^2}{2} \right) + t \int_{\{0\}} (e^{j\theta x} - 1 - j\theta \mathbb{I}_{|x| < 1}) W(dx) \] (27)
Levy process has three independent components: a linear drift, a Brownian motion and a superposition of Poisson processes with different jump sizes. \( W(dx) \) is the measure that represents the arrival rate of the Poisson process. The triplet \((\eta, \sigma, W)\), defines the Levy process completely. If we set \( W(dx) = 0 \), then

\[
\mathbb{E} \left[ e^{j\theta X_t} \right] = \exp \left( j\eta t \theta - \frac{1}{2} \sigma^2 t \theta^2 \right) \tag{28}
\]

The above form can be written as

\[
\int_{-\infty}^{\infty} e^{jkX_t} f_{X_t}(x_t) dx_t = \exp \left( j\eta t k - \frac{1}{2} \sigma^2 t k^2 \right) \tag{29}
\]

Therefore,

\[
\int_{-\infty}^{\infty} e^{(ajk - bk^2)t} e^{-jk(x'-x)} dk = 2\pi f_{X_t}(x' - x); \quad X_t \sim \text{Levy}(a, \sqrt{2b}, 0) \tag{30}
\]

Hence,

\[
f(x, t) = \frac{1}{N} \sum_{i=1}^{N} f_0(s_i); s_i \sim X_t = \text{Levy}(a, \sqrt{2b}, 0) \mid X_0 = x \tag{31}
\]

### 3 Fractional Orders

Fractional derivatives has been introduced to the literature as some physical processes exhibits behavior that cannot be explained by standard derivatives. [2] The Riemann-Liouville fractional derivative is defined as

\[
\frac{\partial^\alpha}{\partial x^\alpha} f(x, t) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{x} \frac{f(y, t) dy}{(x-y)^{\alpha-n+1}} \tag{32}
\]

where \( \alpha = 1 + [\alpha], [\alpha] \) is the largest integer not greater than \( \alpha \). Following Fourier relations exist for the fractional derivatives as they do for ordinary derivatives.

\[
\frac{\partial^\alpha}{\partial x^\alpha} f(x, t) \leftrightarrow (jk)^\alpha f(k, t) \tag{33}
\]

\[
\frac{\partial^\alpha}{\partial (-x)^\alpha} f(x, t) \leftrightarrow (-jk)^\alpha f(k, t) \tag{34}
\]

\[
\frac{\partial^\alpha}{\partial |x|\alpha} f(x, t) \leftrightarrow -|k|^\alpha f(k, t) \tag{35}
\]

We will define stable distributions for our purposes. Stable distributions generalize Gaussian random variables to distributions with infinite variance, infinite mean or both.
Definition 1. A random variable $X$ is said to have a stable distribution if for any positive numbers $A$ and $B$, there are real numbers $C$, $D$ and $C > 0, \alpha \in (0, 2]$ such that $C^\alpha = A^\alpha + B^\alpha$ and

$$AX_1 + BX_2 = CX + D,$$  \hspace{1cm} (36)

where $X_1$ and $X_2$ are i.i.d. copies of $X$.

Notice that for $\alpha = 2$, we obtain Gaussian distribution. Explicit expressions for stable densities are not known in general. However, the characteristic functions (i.e. the Fourier representations) are known.

Given four parameters $\{\alpha, \sigma, \beta, \mu\}$, a stable distribution $X = S_\alpha(\sigma, \beta, \mu)$ has the following characteristic function.

$$p_X(x) \leftrightarrow \begin{cases} \exp \{ -\sigma^\alpha |\omega|^\alpha (1 + j\beta \text{sgn}(\omega) \tan \frac{\alpha \pi}{2}) + j\mu \omega \}, & \alpha \neq 1 \\ \exp \{ -\sigma |\omega| (1 + j\beta^2 \frac{\pi}{2} \text{sgn}(\omega) \ln(\omega)) + j\mu \omega \}, & \alpha = 1 \end{cases}$$  \hspace{1cm} (37)

The absolute value of $\beta \in [-1, 1]$ controls the asymmetry of the density (i.e. skewness). Notice that, when $\alpha = 2$, $\beta$ becomes irrelevant and

$$S_2(\sigma, \beta, \mu) = N(\mu, 2\sigma^2)$$  \hspace{1cm} (38)

We will consider following fractional diffusion model. Let us define the parameters $\alpha \in (0, 2) \setminus \{1\}$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$:

$$c = \cos \left( \frac{\alpha \pi}{2} \right), \quad d = \sin \left( \frac{\alpha \pi}{2} \right)$$  \hspace{1cm} (39)

The PDE is

$$\frac{\partial f}{\partial t} = -\frac{1 + \beta}{2c} \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x, t) - \frac{1 - \beta}{2c} \frac{\partial^{\alpha}}{\partial (-x)^{\alpha}} f(x, t) + \mu \frac{\partial}{\partial x} f(x, t) t$$  \hspace{1cm} (40)

Taking the Fourier transform

$$\frac{\partial}{\partial t} g(k, t) = -\frac{1 + \beta}{2c} (jk)^\alpha g(k, t) - \frac{1 - \beta}{2c} (-jk)^\alpha g(k, t) + (j\mu k) g(k, t)$$  \hspace{1cm} (41)

For simplicity let us assume that $f(x, 0) = \delta(x)$. Then the solution is

$$g(k, t) = \exp \left\{ -\frac{1 + \beta}{2c} (jk)^\alpha t - \frac{1 - \beta}{2c} (-jk)^\alpha t + j\mu kt \right\}$$  \hspace{1cm} (42)

Noticing the following identity

$$(jk)^\alpha = |k|^\alpha e^{j\text{sgn}(k) \frac{\alpha \pi}{2}} = |k|^\alpha (c + j\text{sgn}(k)s)$$  \hspace{1cm} (43)
Then one can rewrite Equation (42) as
\[ g(k,t) = \exp \left\{ -\frac{1+\beta}{2c}|k|^\alpha t (c + j\text{sgn}(k)s) - \frac{1-\beta}{2c}|k|^\alpha t (c - j\text{sgn}(k)s) + j\mu kt \right\} \]
\[ = \exp \left\{ -|k|^\alpha t - j\beta \text{sgn}(k) \tan \left( \frac{\alpha\pi}{2} \right) |k|^\alpha t + j\mu kt \right\} \]  
(44)

Notice that this is the same characteristic function for a stable density. Thus, for \( f(x,0) = \delta(x) \), the solution is given by the stable density

\[ f(x,t) = S_{\alpha} \left( t^{1/\alpha}, \beta, \mu t \right) \]  
(45)

4 Second Order Multivariate Case

A multivariate Gaussian random vector of dimension \( d \) with mean \( \mu \) and covariance matrix \( \Sigma \) has the following density.

\[ f_X(x) = \frac{1}{\sqrt{(2\pi)^d|\Sigma|}} \exp \left\{ -\frac{1}{2}(x - \mu)^T\Sigma^{-1}(x - \mu) \right\} \]  
(46)

Let us consider the following model.

\[ \frac{\partial f}{\partial t} = a\nabla^2 f \]  
(47)

where the initial condition is \( f(x,0) = f_0(x) \). The solution in this case can be found by Fourier Transform methods. The solution is

\[ f(x,t) = \int_{\mathbb{R}^d} f_0(x') \left\{ \frac{1}{\sqrt{(4\pi at)^d}} \exp \left[ -\frac{(x' - x)^T(x' - x)}{4at} \right] \right\} dx' \]  
(48)

The expression inside the curly brackets corresponds to a multivariate Gaussian density which has a mean of \( x \) and a covariance matrix of \( 2atI_{d\times d} \), i.e. \( N(x,2atI_{d\times d}) \). One can approximate above integral as a Monte Carlo sum as follows.

\[ f(x,t) = \frac{1}{N} \sum_{i=1}^{N} f_0(s_i); \ s_i \sim N(x,2atI_{d\times d}) \]  
(49)

For \( a = 1/2 \) this corresponds to a multivariate Brownian motion which is initiated at \( x \) at \( t = 0 \). A proof for \( d = 2 \) is given below.

Proof. Let us consider the following model.

\[ \frac{\partial f}{\partial t} = a \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \]  
(50)
Taking the Fourier Transform of both sides,

\[
\frac{\partial g(k_1, k_2, t)}{\partial t} = a ((jk_1)^2 g(k_1, k_2, t) + (jk_2)^2 g(k_1, k_2, t))
\]

\[
= a \left( - (k_1^2 + k_2^2) g(k_1, k_2, t) \right)
\]

(51)

The solution to this first order ODE is

\[
g_0(k_1, k_2) e^{-a(k_1^2 + k_2^2)t}
\]

Taking the inverse transform

\[
f(x_1, x_2, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_0(k_1, k_2) e^{-a(k_1^2 + k_2^2)t} e^{jk_1 x_1} e^{jk_2 x_2} dk_1 dk_2
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(k_1^2 + k_2^2)t} e^{jk_1 x_1} e^{jk_2 x_2} dk_1 dk_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(x_1', x_2') e^{-jk_1 x_1'} e^{-jk_2 x_2'} dx_1' dx_2'
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(x_1', x_2') dx_1' dx_2' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(k_1^2 + k_2^2)t} e^{jk_1 (x_1 - x_1')} e^{jk_2 (x_2 - x_2')} dk_1 dk_2
\]

(52)

Using the following Fourier transform pair,

\[
e^{-\frac{\rho}{2} ||x||^2} \leftrightarrow \left( \sqrt{\frac{\pi}{\rho}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{|k|^2}{4\rho}}
\]

(53)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(k_1^2 + k_2^2)t} e^{jk_1 (x_1 - x_1')} e^{jk_2 (x_2 - x_2')} dk_1 dk_2 = \left( \sqrt{\frac{\pi}{at}} \right)^2 e^{-\frac{1}{4\pi t} ((x_1 - x_1')^2 + (x_2 - x_2')^2)}
\]

(54)

Thus,

\[
f(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0(x_1', x_2') \left\{ \frac{1}{4\pi at} e^{-\frac{1}{4\pi t} ((x_1 - x_1')^2 + (x_2 - x_2')^2)} \right\} dx_1' dx_2'
\]

\[
= \int_{\mathbb{R}^2} f_0(x') \left\{ \frac{1}{4\pi at} e^{-\frac{1}{4\pi t} ||x' - x||^2} \right\} dx'
\]

\[
N(x, 2atI_2 \times I_2)
\]

(55)

\[
\square
\]

5 Feynman-Kac

Feynman-Kac technique no longer applies to the following variant of the heat equation.

\[
\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + V(x, t) f
\]

(56)

with the initial condition \(f(x, 0) = f_0(x)\). The Feynman-Kac solution gives the solution the heat equation as

\[
f(x, t) = \mathbb{E} \left[ e^{\int_{0}^{t} V(x + b(s), t-s) ds} f_0(x + b(t)) \right]
\]

(57)

where \(b(t)\) is the standard Brownian Motion. Notice that when \(V(x, t) = 0\), i.e. the heat equation Fourier considered, Feynman-Kac formula reduces to \(\mathbb{E} [f_0(x + b(t))]\) or equivalently \(\mathbb{E} [f_0(b(t)) \mid b(0) = x]\) which is what we have found in Section 2.2.
6 KdV

Let us consider the following
\[
\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x}
\]  
(58)

The solitary wave solution is given by
\[
u(x,t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x - ct - a) \right)
\]  
(59)

where \(a\) is an arbitrary constant and \(c\) is the phase speed.

\textbf{Proof.} Let us assume that the solution is of the form \(u(x,t) = f(x - ct)\). Define \(z = x - ct\).

One can rewrite the KdV equation as
\[
-c \frac{\partial f}{\partial z} = -\frac{\partial^3 f}{\partial z^3} + 6f \frac{\partial f}{\partial z} = -\frac{\partial^3 f}{\partial z^3} + 3 \frac{\partial f^2}{\partial z}
\]

Integrating both sides with respect to \(z\) once
\[
-cf = -\frac{\partial^2 f}{\partial z^2} + 3f^2 + K_1
\]

Multiplying both sides with \(\frac{\partial f}{\partial z}\) and integrating both sides will lead to
\[
-\frac{c}{2} f^2 = -\frac{1}{2} \left( \frac{\partial f}{\partial z} \right)^2 + f^3 + K_1 f + K_2
\]
which can be rewritten as
\[
\left( \frac{\partial f}{\partial z} \right)^2 = 2f^3 + cf^2 + K_1 f + K_2
\]

It is known that cnoidal function (one of the Jacobi elliptic functions) solves the above differential equation. One can easily show that, Equation (59) satisfies the differential equation. \(\square\)

Figure 1 shows the solitary wave solution with \(c = 1\) and \(a = 0\).

The soliton solution is characterized by the coefficients \(\{m_i, \xi_i\}_{1 \leq i, j \leq N}\). Define \(A(x,t)\);
\[
a_{i,j}(x,t) = \frac{\sqrt{m_i m_j}}{\xi_i + \xi_j} \exp \left( 4(\xi_i^3 + \xi_j^3)t - (\xi_i + \xi_j)x \right)
\]  
(60)
Solution to KdV is given by;

\[ u(x, t) = -2 \frac{\partial^2}{\partial x^2} \log (\det (I + A(x, t))) \] (61)

Define;

\[ X(y, t) = \sum_{i=1}^{N} \exp \left( 4(\xi_i^3 t - \xi_i y) \right) \sqrt{m_i} X_i; \text{ where } X_i \sim N(0, 1) \text{ independent} \] (62)

Then, the solution to the KdV can be written in terms of an expectation,

\[ u(x, t) = 4 \frac{\partial^2}{\partial x^2} \log \mathbb{E} \left[ \exp \left\{ -\frac{1}{2} \int_{x}^{\infty} X(y, t)^2 dy \right\} \right] \] (63)

The behavior for the 2-soliton case with the coefficients \( m_1 = 0.2, m_2 = 3, \xi_1 = 1 \) and \( \xi_2 = 0.5 \) is given in Figure 2.

**Proof.** Notice that \( a_{ij} \) can be rewritten as

\[ a_{i,j}(x, t) = \frac{\sqrt{m_i m_j}}{\xi_i + \xi_j} \exp \left( 4(\xi_i^3 + \xi_j^3) t - (\xi_i + \xi_j) x \right) \]

\[ = \frac{\sqrt{m_i m_j}}{\xi_i + \xi_j} \int_{x}^{\infty} \exp \left( 4(\xi_i^3 + \xi_j^3) t - (\xi_i + \xi_j) y \right) dy \]

Notice that, \( A(x, t) \) is a positive-definite matrix.
Recall Equation (46). One can write the determinant of the covariance matrix for a multivariate Gaussian density as

\[ \sqrt{|\Sigma|^{-1}} = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} \exp \left\{ -\frac{1}{2} < \Sigma^{-1}(X - \mu), (X - \mu) > \right\} dX \]

Thus for any positive semi-definite symmetric matrix,

\[ \det(I + A)^{-\frac{1}{2}} = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} \exp \left\{ -\frac{1}{2} < AX, X > \right\} \exp \left\{ -\frac{1}{2} < X, X > \right\} dX \]

\[ = \mathbb{E} \left[ \exp \left\{ -\frac{1}{2} < AX, X > \right\} \right], \quad X \sim N(0, I_{d \times d}) \]

Moreover, we have

\[ < A(x, t)X, X > = \int_{x}^{\infty} \left( \sum_{i=1}^{n} \exp \left( 4(\xi_i^3 t - \xi_i y) \right) \sqrt{m_i} X_i \right)^2 dy \]

where \( X = (X_1, X_2, \ldots, X_n) \) and \( \{X_i\}_{1 \leq i \leq n} \) are i.i.d \( N(0, 1) \). Let us introduce the following Gaussian process.

\[ X(y, t) = \sum_{i=1}^{n} \exp \left( 4(\xi_i^3 t - \xi_i y) \right) \sqrt{m_i} X_i \]

Then immediately,

\[ u(x, t) = 4 \frac{\partial^2}{\partial x^2} \log \mathbb{E} \left[ \exp \left\{ -\frac{1}{2} \int_{x}^{\infty} X(y, t)^2 dy \right\} \right] \]
7 Experiments

We will give experimental results on the following cases.

- Second Order Diffusion: The accuracy of the results will compared with respect to the number of Monte Carlo samples used in the approximation
- Second Order Multivariate Diffusion

7.1 Second Order Diffusion

Figure 3(a), 3(b) and 3(c) shows the results for the second order heat equation. Notice that as number of Monte Carlo samples increase, the approximation error reduces with a rate of $\frac{1}{\sqrt{N}}$ due to the central limit theorem.
7.2 Second Order Multivariate Diffusion

The diffusion pattern for the two dimensional case is shown in Figure 4 for $N = 10000$. Notice that due to the approximation error the pattern is not exactly circular as one would expect.

8 Conclusion

Different differential models are considered throughout the paper and randomized Monte Carlo solutions are introduced. It is important to note that there is huge relation between the differential domain and the stochastic domain. One may use the differential analog when the stochastic techniques are not easy to implement or vice versa.

References
