High-Dimensional Multi-Model Estimation – Its Algebra, Statistics, and Sparse Representation

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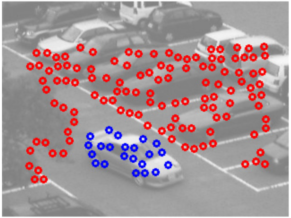
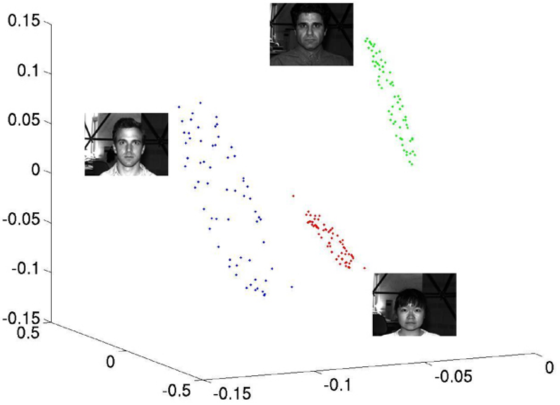
April 9, 2009. University of Texas, Austin

Segmentation of Multi-Model Distribution

parking-lot movie



Classification of Multi-Model Distribution



Face Recognition: *"Where amazing happens!"*

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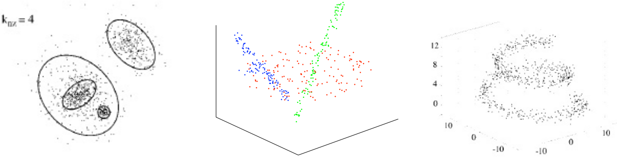
Figure: Steve Nash, Yao Ming, Tim Duncan.

Problem Statement: Seeking Compact Representation of HD Data

- Determine a **class** of models and the **number** of mixture.



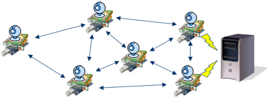
- **Curse of dimensionality**! [Bellman 1957]
  1. In the 1950s, beyond 10-D space is high.
  2. Until 2005, face recognition uses $<$ 100-D features.
  3. Today, applications beyond thousands of dimensions.

- Robust to high **noise** and **outliers**.

mixture model

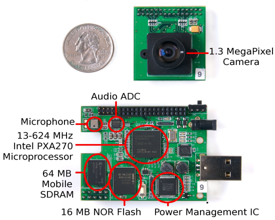
Distributed Sensing and Perception (DSP)

- In resource-constrained sensor environments

Centralized Perception	Distributed Perception
	
powerful processors	mobile processors
(virtually) unlimited memory	limited onboard memory
(virtually) unlimited bandwidth	band-limited communications
simple sensor management	complex sensor networks

CITRIC: Wireless Smart Camera Platform

- CITRIC platform



- Available library functions
  1. Full support **Intel IPP Library** and **OpenCV**.
  2. **JPEG compression**: 10 fps.
  3. **Edge detector**: 3 fps.
  4. **Background Subtraction**: 5 fps.
  5. **SIFT detector**: 10 sec per frame.

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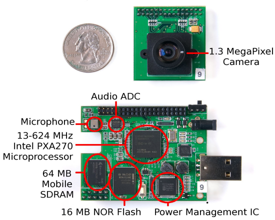

- Early adopters

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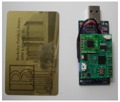
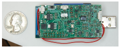
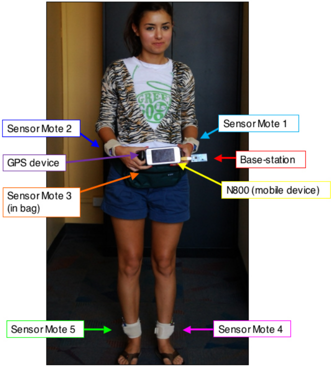
DexterNet: Wireless Body Sensor Network

- Heterogeneous body sensors

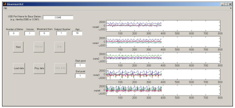




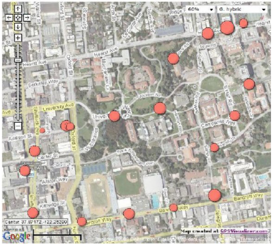
- Layout



- Applications
  1. Distributed wearable action recognition (d-WAR)

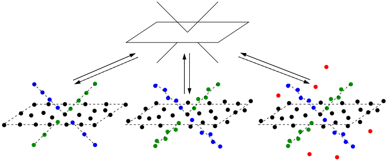


  2. Adopted by Oakland Children's Hospital and Vanderbilt Hospital



Outline

1. **Unsupervised** segmentation/clustering
   Partition samples drawn from $\mathcal{A} = S_1 \cup S_2 \cup \ldots \cup S_K$ in $\mathbb{R}^D$, and estimate model parameters.



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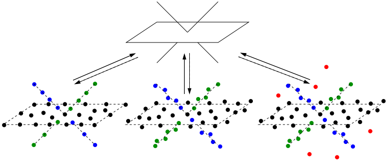
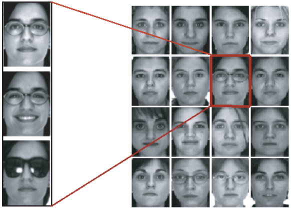


2. **Supervised** recognition
Assume training examples $\{A_1, \cdots, A_K\}$ for K models. Given a test sample $\mathbf{y}$, determine its membership label($\mathbf{y}$) $\in [1, 2, \cdots, K]$.



Affine Motion Segmentation

Assume 3-D objects are far away from the camera

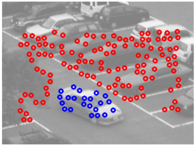
- Points $\mathbf{p}_1, \ldots, \mathbf{p}_N \in \mathbb{R}^3$ are tracked in F image frames.
- Image of $\mathbf{p}_i$ in jth frame:

$$\mathbf{m}_{ij} \doteq \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix}^T = A_j \mathbf{p}_i + \mathbf{b}_j \in \mathbb{R}^2, \quad j = 1, \ldots, F.$$

- Stack images of $\mathbf{p}_i$ in all F frames

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{m}_{i1} \\ \vdots \\ \mathbf{m}_{iF} \end{bmatrix} = \begin{bmatrix} A_1 & \mathbf{b}_1 \\ \vdots & \\ A_F & \mathbf{b}_F \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} \in \mathbb{R}^{2F}.$$

parking-lot movie



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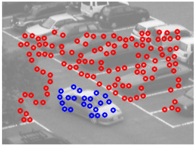


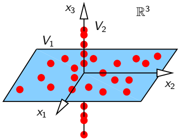
Challenge: Affine Motion Segmentation ⇒ Subspace Segmentation

Each motion satisfies a 4-D subspace model.

Generalized Principal Component Analysis (GPCA)

1. For a single subspace

- $V_1^{\perp} : (x_3 = 0)$
- $V_2^{\perp} : (x_1 = 0) \& (x_2 = 0)$



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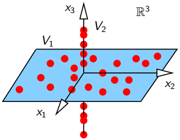

2. For $\mathcal{A} = V_1 \cup V_2$

$$\forall \mathbf{z} = (x_1, x_2, x_3)^T, \quad \mathbf{z} \in V_1 \cup V_2 \Leftrightarrow \{x_3 = 0\} | \{(x_1 = 0) \& (x_2 = 0)\}$$

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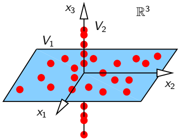

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3. By De Morgan's law

$$\{x_3 = 0\} | \{(x_1 = 0)\&(x_2 = 0)\} \Leftrightarrow (x_1 x_3 = 0)\&(x_2 x_3 = 0) \Leftrightarrow \begin{cases} x_1 x_3 = 0 \\ x_2 x_3 = 0 \end{cases}$$

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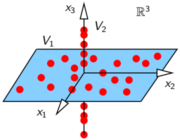

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4. **Vanishing polynomials**: $p_1 = x_1 x_3, p_2 = x_2 x_3$

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Subspace Properties

If $p_1, p_2 \in I_K(\mathcal{A})$, $p_1(\mathbf{x}) = 0$ and $p_2(\mathbf{x}) = 0$

1. Closed under **addition**: $(p_1 + p_2)(\mathbf{x}) = 0 \Rightarrow (p_1 + p_2) \in I_K(\mathcal{A})$.
2. Closed under **scalar multiplication**: $\forall a \in \mathbb{R}$, $ap_1(\mathbf{x}) = 0 \Rightarrow ap_1 \in I_K(\mathcal{A})$.

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- $I_K(\mathcal{A})$ is determined by a linearly-independent **polynomial basis**.
- **Closed-form solution:**

$$\dim(I_K(\mathcal{A})) = \sum_S (-1)^{|S|} \binom{i + D - 1 - c_S}{D - 1 - c_S},$$

where $c_S = \sum_{j \in S} c_j$ and the sum is over all $S \subseteq \{1, \ldots, n\}$.

Reference:

Yang, et al., *Estimation of subspace arrangements with applications in modeling and segmenting mixed data*, **SIAM Review**, 2008.

Estimation of Vanishing Polynomials

1. **Veronese embedding**: Given N samples $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^3$,

$$\begin{aligned} L_2 &\doteq [\nu_2(\mathbf{x}_1), \ldots, \nu_2(\mathbf{x}_N)] \in \mathbb{R}^{M_2^{[3]} \times N} \\ &= \begin{bmatrix} \cdots & (x_1)^2 & \cdots \\ \cdots & (x_1 x_2) & \cdots \\ \cdots & (x_1 x_3) & \cdots \\ \cdots & (x_2)^2 & \cdots \\ \cdots & (x_2 x_3) & \cdots \\ \cdots & (x_3)^2 & \cdots \end{bmatrix} \end{aligned}$$

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2. The null space of L_2 is $\begin{array}{l} \mathbf{c}_1 = [0,0,1,0,0,0] \\ \mathbf{c}_2 = [0,0,0,0,1,0] \end{array} \Rightarrow \begin{array}{l} p_1 = \mathbf{c}_1 \nu_2(\mathbf{x}) = x_1 x_3 \\ p_2 = \mathbf{c}_2 \nu_2(\mathbf{x}) = x_2 x_3 \end{array}$

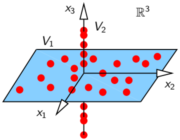

Figure: $2nd$-degree vanishing polynomials: $p_1 = x_1 x_3$, $p_2 = x_2 x_3$.

Calculate Subspace Basis Vectors using Polynomial Derivatives

1. $V_1^\perp, \cdots, V_K^\perp$ recovered by the *derivatives*

$$\nabla_{\mathbf{x}} P = [\, \nabla_{\mathbf{x}} p_1 \;\; \nabla_{\mathbf{x}} p_2 \,] = \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \\ x_1 & x_2 \end{bmatrix}.$$

2. Pick $\mathbf{z} = [1, 1, 0]^T \in V_1$, then $\nabla_{\mathbf{x}} P(\mathbf{z}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

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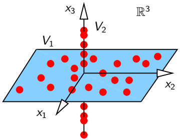

Figure: $P(\mathbf{x}) \doteq [p_1(\mathbf{x}) \;\; p_2(\mathbf{x})] = [x_1 x_3, x_2 x_3]$.

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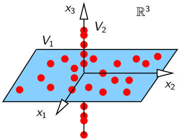
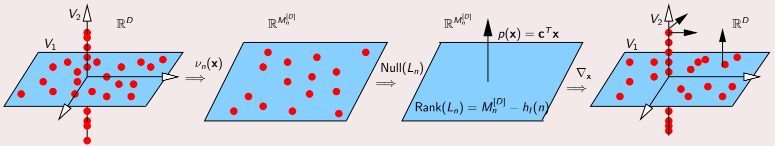


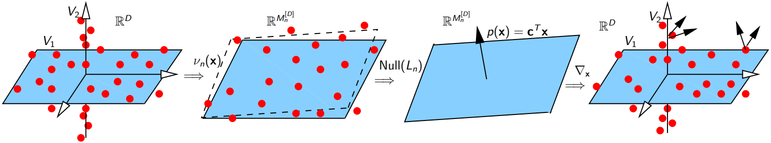
Figure: $P(\mathbf{x}) \doteq [p_1(\mathbf{x}) \ p_2(\mathbf{x})] = [x_1 x_3, x_2 x_3]$.

Diagram of GPCA



Robust GPCA

1. Stability: **Principal Component Analysis** (PCA)



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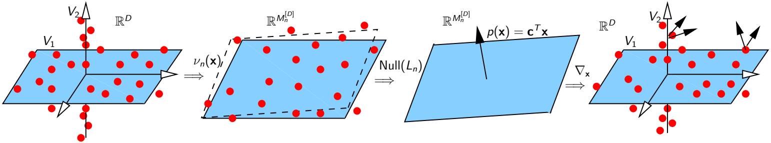
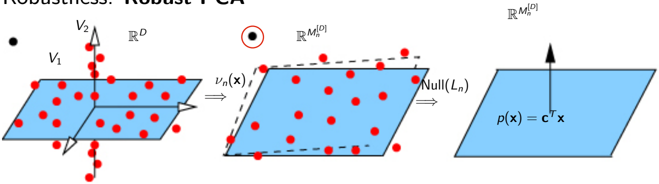


2. Robustness: **Robust PCA**



Robust GPCA

Outlier Elimination

Figure: Elimination of outliers.

References:

- Yang, *Estimation of subspace arrangements: Its algebra and statistics*, **Dissertation**, 2006.
- MATLAB toolboxes available on my website.

Summary: (Robust) GPCA

Advantages:

- Closed-form algebraic solution, not iterative.
- Segment subspaces with mixed dimensions.
- Robust to noise and outliers.

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  Question: How about mixture nonlinear manifolds?

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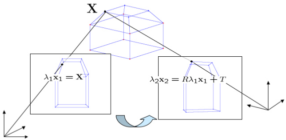
- **User provides correct subspace number and dimensions.**
  Question: How to select a good mixture model?

Perspective Motion Segmentation

Given two image correspondences $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$

- Epipolar



$$\mathbf{x}_2^T \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \mathbf{x}_1 = 0$$

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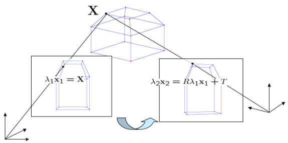

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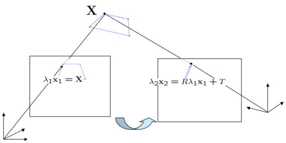
- Homography



$$\mathbf{x}_2 \times \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}_1 = 0$$

Each perspective constraint is linear w.r.t. $(\mathbf{x}_1, \mathbf{x}_2)$, but **in different form!**

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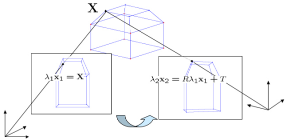

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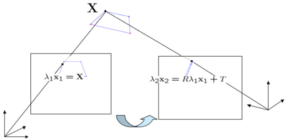

$$\mathbf{x}_2 \times \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}_1 = 0$$

Each perspective constraint is linear w.r.t. $(\mathbf{x}_1, \mathbf{x}_2)$, but **in different form!**

- Classical solution: **Random Sample Consensus** (RANSAC).

Quadratic Manifolds in Joint Image Space

Joint image space: Stack $\mathbf{x}_1 = (x_1, y_1, 1)^T$ and $\mathbf{x}_2 = (x_2, y_2, 1)^T$

$$\mathbf{y} = (x_1, y_1, x_2, y_2, 1)^T \in \mathbb{R}^5$$

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- Quadratic fundamental manifold (QFM)

$$\mathbf{y}^T A \mathbf{y} \doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & f_{11} & f_{21} & f_{31} \\ 0 & 0 & f_{12} & f_{22} & f_{32} \\ f_{11} & f_{12} & 0 & 0 & f_{13} \\ f_{21} & f_{22} & 0 & 0 & f_{23} \\ f_{31} & f_{32} & f_{13} & f_{23} & 2f_{33} \end{pmatrix} \mathbf{y} = 0. \quad (1)$$

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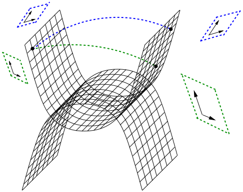
- Quadratic homograpy manifold (QHM)

$$\begin{aligned}
\mathbf{y}^T B_1 \mathbf{y} &\doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & 0 & h_{31} & -h_{21} \\ 0 & 0 & 0 & h_{32} & -h_{22} \\ 0 & 0 & 0 & 0 & 0 \\ h_{31} & h_{32} & 0 & 0 & h_{33} \\ -h_{21} & -h_{22} & 0 & h_{33} & -2h_{23} \end{pmatrix} \mathbf{y} = 0, \\
\mathbf{y}^T B_2 \mathbf{y} &\doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & -h_{31} & 0 & h_{11} \\ 0 & 0 & -h_{32} & 0 & h_{12} \\ -h_{31} & -h_{32} & 0 & 0 & -h_{33} \\ 0 & 0 & 0 & 0 & 0 \\ h_{11} & h_{12} & -h_{33} & 0 & 2h_{13} \end{pmatrix} \mathbf{y} = 0, \\
\mathbf{y}^T B_3 \mathbf{y} &\doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & h_{21} & -h_{11} & 0 \\ 0 & 0 & h_{22} & -h_{12} & 0 \\ h_{21} & h_{22} & 0 & 0 & h_{23} \\ -h_{11} & -h_{12} & 0 & 0 & -h_{13} \\ 0 & 0 & h_{23} & -h_{13} & 0 \end{pmatrix} \mathbf{y} = 0.
\end{aligned} \tag{2}$$

Segmentation of Quadratic Manifolds

- Convert mixture perspective motion as **segmentation of mixture quadratic manifolds** defined by

$$p_j(\mathbf{y}) \doteq \mathbf{y}^T Q_j \mathbf{y} = 0. \tag{3}$$



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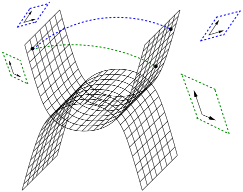

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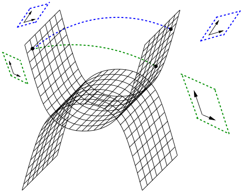

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Robust Algebraic Segmentation

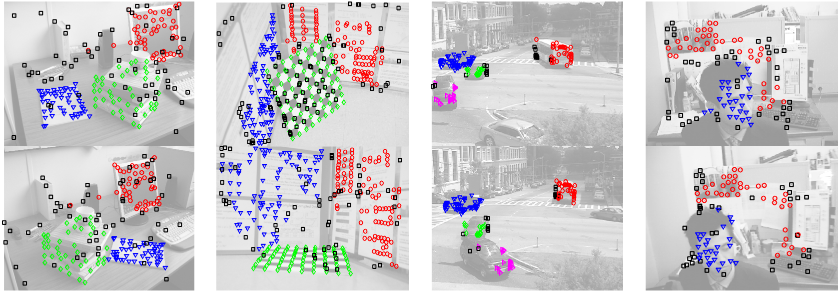
$$Y = \{\mathbf{y}_1, \cdots, \mathbf{y}_n\} \Rightarrow I_{2K}(\mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \{S_1, \cdots, S_K\}$$

Reference:

Yang, et al., *Robust Algebraic Segmentation of Mixed Rigid-Body and Planar Motions*, **IJCV (submitted)**, 2008.

Experiment

1. Visualization



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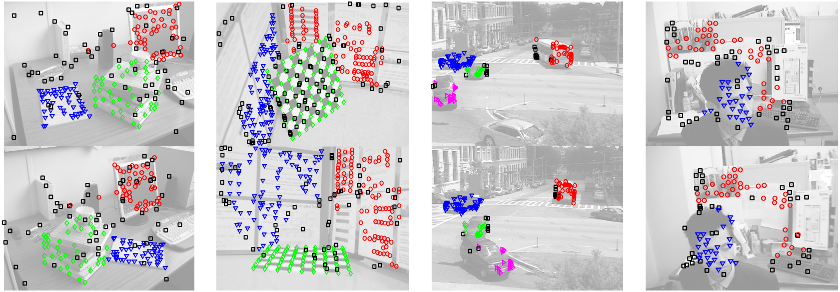
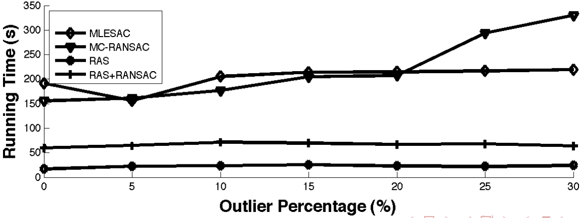


2. Faster than RANSAC!



boxes	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	9.24%	**0.84%**	1.68%	**0.84%**
VR	36.97%	84.87%	100%	87.39%
carsnbus3	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	45.75%	12.55%	2.83%	**1.62%**
VR	83.81%	90.28%	97.17%	85.83%
deliveryvan	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	23.23%	10.63%	5.91%	**0.39%**
VR	97.64%	96.85%	100%	94.09%
desk	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	9.00%	2.50%	3.00%	**0.50%**
VR	55.50%	93.50%	91.50%	93.50%
lightbulb	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	39.52%	**0.00%**	**0.00%**	**0.00%**
VR	76.19%	82.86%	100%	99.52 %
manycars	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	30.56%	22.22%	**0.00%**	**0.00%**
VR	90.28%	95.83%	100%	88.89%
man-in-office	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	20.56%	34.58%	20.56%	**11.21%**
VR	89.72%	95.33%	84.11%	82.24%
nrbooks3	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	12.38%	9.05%	5.48%	**0.95%**
VR	41.19%	65.48%	94.29%	88.33%
office	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	2.28%	0.33%	10.42%	**0.00%**
VR	89.59%	90.55%	86.97%	93.49%
parking-lot	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	7.86%	5.00%	3.57%	**2.86%**
VR	98.57%	96.43%	100%	97.86%
posters-checkerboard	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	20.58%	1.06%	9.23%	**0.00%**
VR	49.87%	97.36%	70.71%	95.25%
posters-keyboard	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	8.59%	**0.25%**	10.61%	0.51%
VR	56.06%	83.33%	78.03%	88.13%
toys-on-table	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	38.10%	38.10%	15.08%	**7.94%**
VR	91.27%	92.86%	81.75%	77.78%

Summary: Robust Algebraic Segmentation

Advantages:

- Extends the algebraic framework of GPCA.
- Closed-form solution to segment quadratic manifolds with mixed dimensions.
- More accurate than RANSAC, two to three times faster.

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Advantages:

- Extends the algebraic framework of GPCA.
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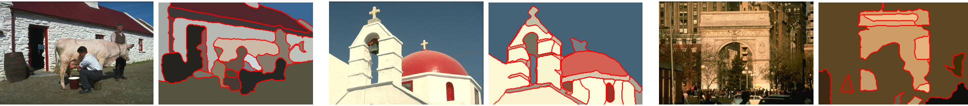
Limitations:

- **Algorithm grows exponentially with model number and dimensions.**
- **How to determine an optimal mixture model?**

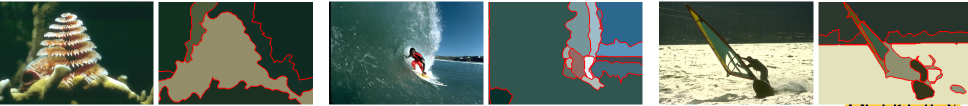
Natural Image Segmentation

(a) Nature



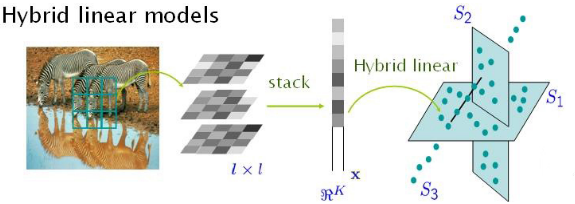
(b) Urban

(c) Portraits



(d) Water

Lossy Minimum Description Length



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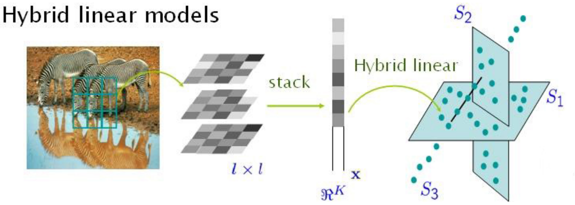

1. **Lossy coding length** $L_\epsilon(V, \mathcal{A})$:
   Quantize $V = (v_1, \cdots, v_N) \in \mathbb{R}^{D \times N}$ as **a sequence of binary bits** up to a distortion
   $$\mathbb{E}[\|v_i - \hat{v}_i\|^2] \leq \epsilon^2.$$
2. **Lossy MDL**
   $$\mathcal{A}^*(\epsilon) = \arg\min\{L_\epsilon(V, \mathcal{A}) + \text{Overhead}(\mathcal{A})\}.$$

Lossy MDL for Mixture Subspaces

1. Model V_i as a (degenerate) Gaussian model

$$\textbf{Bit rate: } R(V_i) = \frac{1}{2}\log_2 \det(I + \frac{D}{\epsilon^2 N_i} V_i V_i^T).$$

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2. **Coding length** for V_i of N_i samples

$$L(V_i) = N_i R(V_i) + \frac{D}{2}\log_2 \det(1 + \frac{1}{\epsilon^2}\mu_i \mu_i^T) + N_i(-\log_2(N_i/N)).$$

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3. **Total coding length**: $L^s(V_1, \cdots, V_K) = \sum_i L(V_i)$.

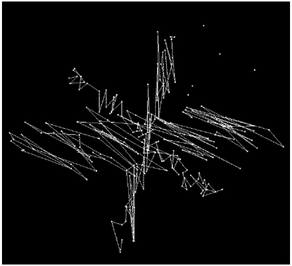
Ideal Optimization is NP-Hard

The process must exhaust all possible combinations of N samples.

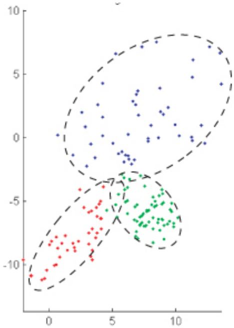
A Greedy Optimization

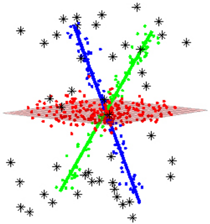
1. **Initialize**: Assume N samples as individual groups.
2. **Iteration**: Merge two groups that reduces largest coding length.
3. **Stop**: If any further merging cannot reduces L^s.
4. **Output**: Estimation of K and the grouping.

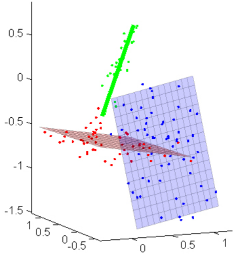
animation



Simulation







Visualization

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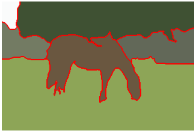

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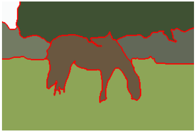

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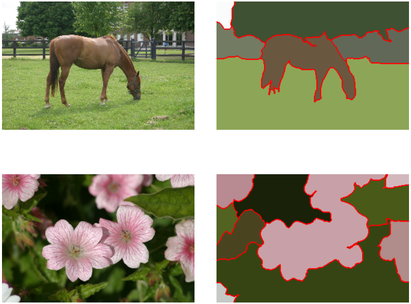

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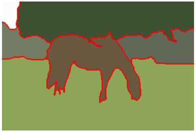

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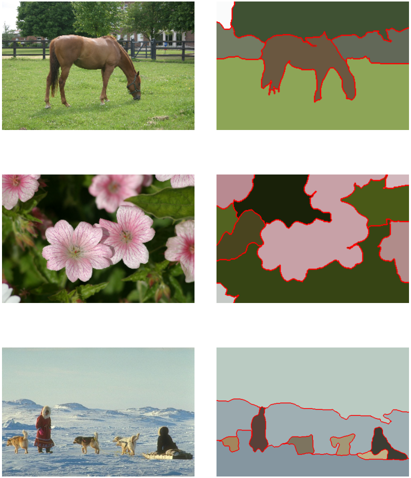


Quantitative Comparison on Berkeley Segmentation Dataset

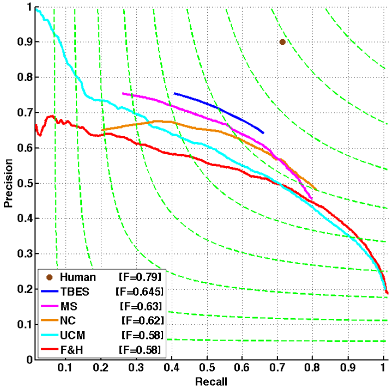


Figure: Precision vs Recall for texture region boundaries.

References:

- Yang et al., *Unsupervised Segmentation of Natural Images via Lossy Data Compression*, **CVIU**, 2008.
- MATLAB implementation available on my website.

Multi-Model Classification

- Applications

Figure: Face Recognition

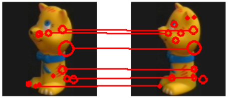


Figure: Object Recognition

Figure: Action Recognition

Multi-Model Classification

- Applications

Figure: Face Recognition

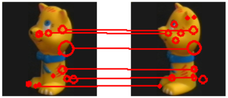
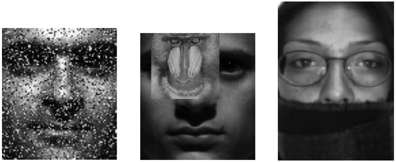


Figure: Object Recognition

Figure: Action Recognition

- Face recognition under **distortion, occlusion, & disguise**



Sparse Representation

Sparsity

A signal is sparse if most of its coefficients are (approximately) zero.

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1. Sparsity in frequency domain

Figure: 2-D DCT transform.

2. Sparsity in spatial domain

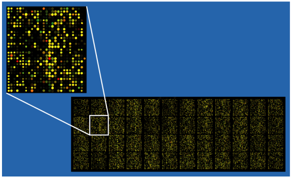
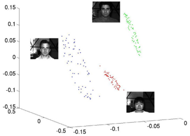


Figure: Gene microarray data.

Classification of Mixture Subspace Model

1. Face-subspace model: Assume $\mathbf{y}$ belongs to Class i in K classes.



$$\begin{aligned} \mathbf{y} &= \alpha_{i,1}\mathbf{v}_{i,1} + \alpha_{i,2}\mathbf{v}_{i,2} + \cdots + \alpha_{i,n_1}\mathbf{v}_{i,n_i}, \\ &= A_i\alpha_i, \end{aligned}$$

where $A_i = [\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \cdots, \mathbf{v}_{i,n_i}]$.

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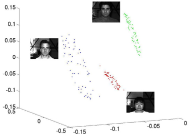

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2. Nevertheless, Class i is the **unknown** variable we need to solve:

Sparse representation $\mathbf{y} = [A_1, A_2, \cdots, A_K] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{bmatrix} = A\mathbf{x}.$

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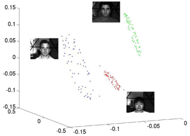

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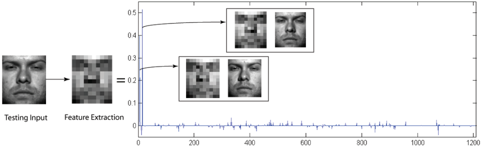
3. $\mathbf{x}^* = [\, 0 \cdots 0 \ \alpha_i^T \ 0 \cdots 0 \,]^T \in \mathbb{R}^n.$



Sparse representation $\mathbf{x}^*$ encodes membership!

ℓ^1-Minimization

1. Ideal solution: ℓ^0-Minimization (**NP-hard**)

$$(P_0) \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ s.t. } \mathbf{y} = A\mathbf{x}.$$

$\|\cdot\|_0$ simply counts the number of nonzero terms.

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2. **Compressive Sensing**: Under mild condition, ℓ^0-minimization is equivalent to

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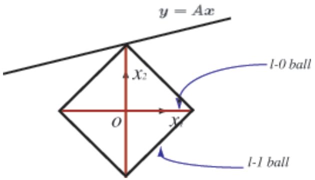
3. ℓ^1-Ball

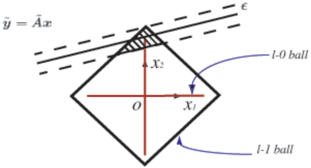
- Solution equal to ℓ^0-minimization.
- ℓ^1-Minimization is convex.
  1. **Matching pursuit** [Mallat 1993]
  2. **FOCUSS** [Rao 1995]
  3. **Lasso** [Tibshirani 1996]
  4. **Basis pursuit** [Chen 1998]
  5. **Reweighted** ℓ^1 [Candes 2007]



Stability of ℓ^1-Minimization

- ℓ^1 near solution

$$\mathbf{y} = A\mathbf{x} + \mathbf{e} \quad \text{s.t. } \|e\|_2 < \epsilon.$$



Stability of ℓ^1-Minimization

- ℓ^1 near solution

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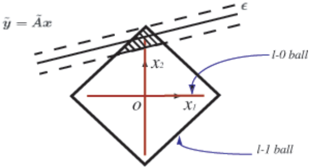
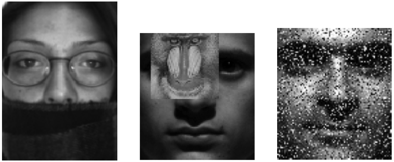


- **Bounded data noise produces bounded ℓ^1 solution**

$$(P_1') \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \|\mathbf{y} - A\mathbf{x}\|_2 < \epsilon.$$

Restricted Isometry Property [Candès, Romberg, Tao 2004]: $\|\mathbf{x}^* - \mathbf{x}_0\|_2 < C\epsilon$.

Occlusion Compensation: Unbounded Data Noise



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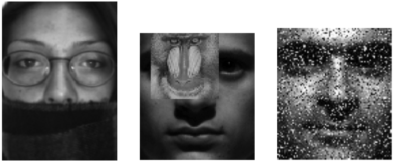
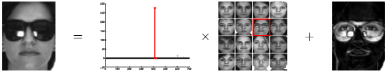


1. Sparse representation + sparse innovation

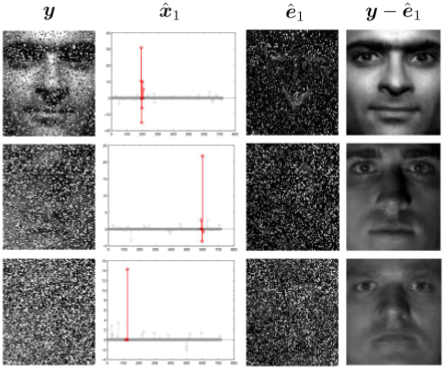
$$\mathbf{y} = A\mathbf{x} + \mathbf{e}$$

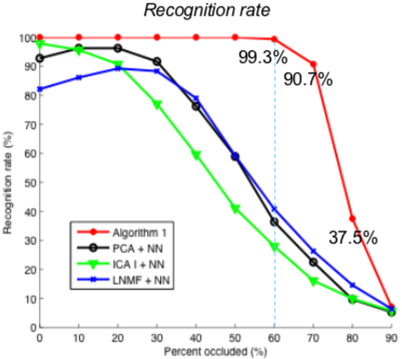


2. Occlusion compensation

$$\mathbf{y} = \begin{bmatrix} A & \mid & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = B\mathbf{w}$$

Dense Error Correction: High-Dimensional Regime





Reference:

Yang et al., *Robust face recognition via sparse representation*, **PAMI**, 2009.

Summary: Sparse Representation

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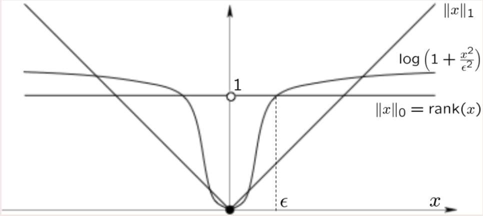
4. Extensions abound in **object recognition**, **action recognition**, **super-resolution**, and **sensor networks**.
   - Yang et. al., *Multiple-view object recognition in band-limited distributed camera networks*, (submitted) **ICDSC**, 2009.
   - Yang et. al., *Distributed sensor perception via sparse representation*, (submitted) **Proceedings of IEEE**, 2009.
   - Yang et. al., *Distributed compression and fusion of nonnegative sparse signals for multiple-view object recognition*, **Fusion**, 2009.
   - Yang et. al., *Distributed Recognition of Human Actions Using Wearable Motion Sensor Networks*, **JAISE**, 2009.

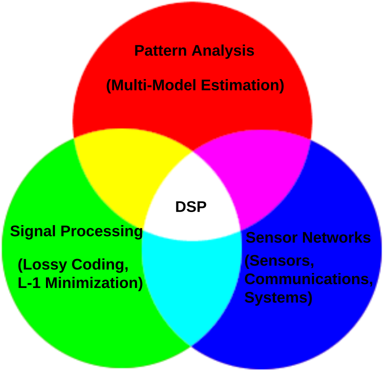
Take-Home Messages: Compact Representation of Complex, HD Data

1. **Goal:** Recover low-dim, mixture model (subspaces, Gaussians, and manifolds).
2. **Confluence of Algebra, Statistics, and Sparse Representation**
   - **Algebra**: Minimizing discrete dimensions via vanishing polynomials.
   - **Information Theory**: Lossy coding length as sub-optimal smooth approximation.
   - **Sparse Representation**: ℓ^1-minimization is convex, and exact under mild conditions.

Seek best proxy (surrogate, relaxation) to approximate HD mixture models suboptimally.



Bigger Picture: Distributed Sensing and Perception



References:
Yang et al., *CITRIC: A low-bandwidth wireless camera network platform*, **ICDSC**, 2008.
Yang et al., *DexterNet: An open platform for heterogeneous body sensor networks and its applications*, **BSN**, 2009.

Massive, Distributed Data in Computer, Sensor, Social Networks

1. Opportunistic Sensing

2. Perception in Smartphone Networks

3. Protect Renewable Power Networks

Acknowledgments

Collaborators

- **Berkeley**: Dr. Shankar Sastry, Dr. Ruzena Bajcsy, Dr. Trevor Darrell
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- **Cornell**: Philip Kuryloski
- **JHU**: Dr. René Vidal
- **UMich**: Dr. Harm Derksen
- **UT Dallas**: Dr. Roozbeh Jafari
- **Tampere University of Technology:** Ville-Pekka Seppa
- **Telecom Italia**: Dr. Marco Sgnoi, Roberta Giannantonio, Raffaele Gravina

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- ARO MURI: Adaptive Coordinated Control of Intelligent Multi-Agent Teams
- NSF TRUST Center at UC Berkeley

Robust Statistics

Three approaches to eliminate "**outliers**"

1. Probability-based: small-probability samples.
   Probability plots: [Healy 1968, Cox 1968]
   PCs: [Rao 1964, Ganadesikan & Kettenring 1972]
   M-estimators: [Huber 1981, Campbell 1980]
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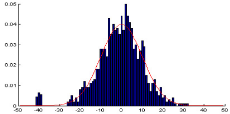

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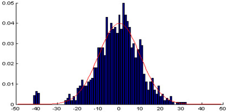
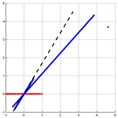


2. Influence-based: large influence on model parameters.
   Parameter difference with and without a sample: [Hampel et al. 1986, Critchley 1985]



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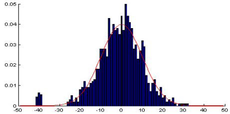

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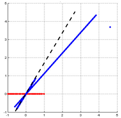
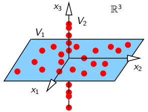


3. Consensus-based: not consistent with models of high consensus.
   Hough: [Ballard 1981, Lowe 1999]
   RANSAC: [Fischler & Bolles 1981, Torr 1997]
   Least Median Estimate (LME): [Rousseeuw 1984, Steward 1999]



Dynamic Texture

1. Dynamic texture as **ARMA model** for image $I(t)$ of D pixels

$$x(t+1) = Ax(t) + Bu(t) \in \mathbb{R}^d$$
$$I(t) = Cx(t) + w(t) \in \mathbb{R}^D$$

2. Subspace constraint on F frames, $t = 1, \cdots, F$

trackers movie

$$L = [I(1) \cdots I(F)] = C[x(1) \cdots x(F)] \in \mathbb{R}^{D \times F};$$
$$\Rightarrow \qquad \mathrm{Rank}(\mathbf{L}) = \mathbf{d}.$$

Mixture of two dynamic texture regions

$$[i(1) \cdots i(F)] \in \mathrm{Sys}(A_1, B_1, C_1) \quad \text{or} \quad [i(1) \cdots i(F)] \in \mathrm{Sys}(A_2, B_2, C_2)$$

ℓ^1-Minimization Routines

- **Matching pursuit** [Mallat 1993]
  1. Find most correlated vector $\mathbf{v}_i$ in A with $\mathbf{y}$: $i = \arg\max \langle \mathbf{y}, \mathbf{v}_j \rangle$.
  2. $A \leftarrow A^{(i)}$, $x_i \leftarrow \langle \mathbf{y}, \mathbf{v}_i \rangle$, $\mathbf{y} \leftarrow \mathbf{y} - x_i \mathbf{v}_i$.
  3. Repeat until $\|\mathbf{y}\| < \epsilon$.
- **Basis pursuit** [Chen 1998]
  1. Start with number of sparse coefficients $m = 1$.
  2. Select m linearly independent vectors B_m in A as a basis
     $$\mathbf{x}_m = B_m^\dagger \mathbf{y}.$$
  3. Repeat swapping one basis vector in B_m with another vector not in B_m if improve $\|\mathbf{y} - B_m \mathbf{x}_m\|$.
  4. If $\|\mathbf{y} - B_m \mathbf{x}_m\|_2 < \epsilon$, stop; Otherwise, $m \leftarrow m + 1$, repeat Step 2.
- **Quadratic solvers**: $\mathbf{y} = A\mathbf{x}_0 + \mathbf{z} \in \mathbb{R}^d$, where $\|\mathbf{z}\|_2 < \epsilon$

$$\mathbf{x}^* = \arg\min\{\|\mathbf{x}\|_1 + \lambda \|\mathbf{y} - A\mathbf{x}\|_2\}$$

[LASSO, Second-order cone programming]: Much more expensive.

Matlab Toolboxes for ℓ^1-Minimization

- **ℓ^1-Magic** by Candes
- **SparseLab** by Donoho
- **cvx** by Boyd

Mild Conditions for ℓ^1/ℓ^0 Equivalence

$$(P_1) \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} = A\mathbf{x}$$

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$$\text{Asmyptotically with } \frac{k \uparrow}{d \uparrow} < 0.5$$

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- Long answers
  1. **(In)-coherence** [Gribvonel & Nielsen 2003, Donoho & Elad 2003]:

$$\mu(A, B) \doteq \sup_{\mathbf{a}\in A, \mathbf{b}\in B} \frac{|\langle \mathbf{a}, \mathbf{b}\rangle|}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

$\|\mathbf{x}\|_0 \leq \frac{1}{2}(1 + \frac{1}{\mu(A,B)})$ suffices. A **and** B **have to be incoherent.**

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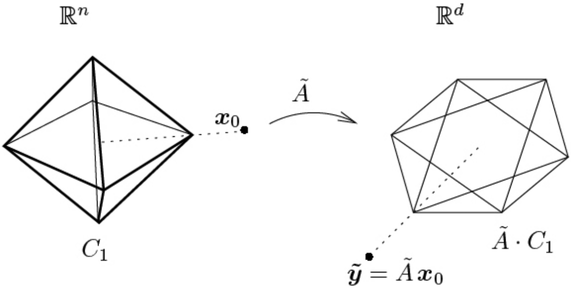
  2. **Restricted Isometry** [Candes & Tao 2005]:
     Define $\delta_k(A) \doteq min\delta$ such that

$$(1-\delta)\|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1+\delta)\|\mathbf{x}\|_2^2 \quad \forall k\text{-sparse } \mathbf{x}.$$

     $\delta_{2k}(A) \leq \sqrt{2} - 1$ suffices. **The columns of A should be uniformly well-spread.**

k-Neighborlyness [Donoho 2006]



- Define **cross polytope** C and **quotient polytope** P such that $P = AC$.

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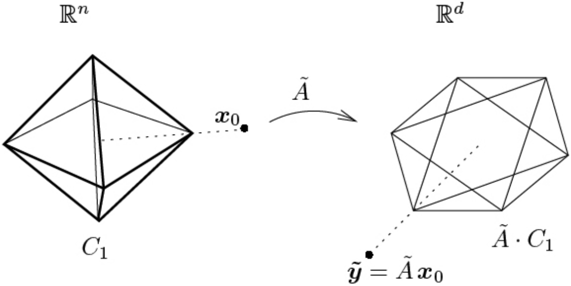

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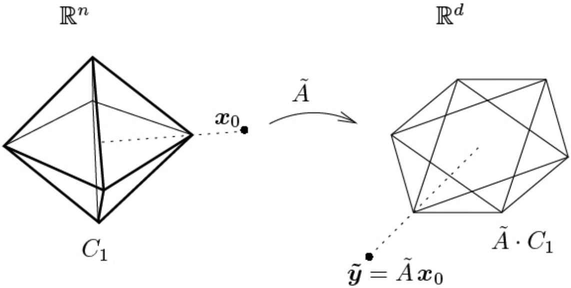

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- If $\mathbf{x}$ is k-sparse, $\mathbf{x}$ lie in a $(k-1)$-face of C in $\mathbb{R}^n$.
- **Necessary and Sufficient:** If ℓ^1/ℓ^0 holds for all k-sparse $\mathbf{x}$, all $(k-1)$-faces of C must be the faces of P on the boundary.

Sparse Representation in Classification: a Cross-and-Bouquet Model

- Traditional compressive sensing focuses on

$$\mathbf{y} = A\mathbf{x} + \mathbf{e}$$

1. A is component-wise Gaussian.
2. A is sparse Bernoulli.
3. A is megadictionary $[I|F]$, where F is Fourier or wavelets.

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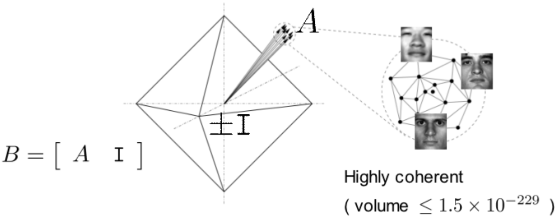
- Solving sparse representation for recognition purpose represents a special model

$$\mathbf{y} = [A \quad | \quad I] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$



Reference:

John Wright and Yi Ma, *Dense Error Correction via l1 Minimization*.