CS294-6 Lecture 12.

Last time.

1. Ambiguities on perspective projection.

$$\lambda x' = K \Pi_0 g X$$
$$= K R_0^{-1} R_0 \Pi_0 H^{-1} H g g_w^{-1} g_w X$$

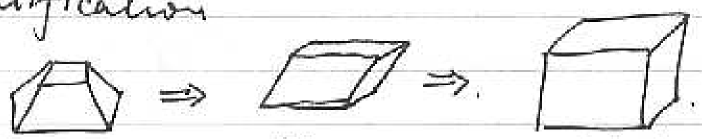
2. Eliminating the ambiguity on g_w:

fix the world coord. system on the first camera.

$$\begin{cases} \lambda_1 x_1' = K_1 \Pi_0 g_{1e} X_e \\ \lambda_2 x_2' = K_2 \Pi_0 g_{2e} X_e. \end{cases}$$

$$\Rightarrow (g_{1e} = [I, 0]) \quad \begin{cases} \lambda_1 x_1' = K_1 \Pi_0 X_e = K_1 \Pi_0 H^{-1} H X_e. \\ \lambda_2 x_2' = K_2 \Pi_0 g_{2e} H^{-1} H X_e \end{cases}$$

3. Stratification

① 

proj. recon. affine recon. Euclidean recon.

Let $H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} \in \mathbb{R}^{4\times4}$, then

$$\begin{cases} \lambda_1 x_1' = [I, 0] X_e \\ \lambda_2 x_2' = K_2 \Pi_0 g_{2e} H^{-1} H X_e \end{cases}$$
$$= K_2 \Pi_0 g_{2e} \underbrace{\begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix}}_{H_a^{-1}} \underbrace{\begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}}_{H_p^{-1}} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}^{-1} \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix}^{-1} X_e.$$

$$\therefore \Pi_{2p} = K_2 \Pi_0 g_{2e} H_a^{-1} H_p^{-1}, \quad X_p = H_p \underbrace{H_a X_e}_{X_a} = H_p X_a.$$

$$\Rightarrow \begin{cases} \lambda_1 x_1' = \Pi_0 X_p \\ \lambda_2 x_2' = \Pi_{2p} X_p. \end{cases}$$

Summary:
- projective camera : $\Pi_{ip} \doteq K_i \Pi_0 g_{ie} H_a^{-1} H_p^{-1}$
- affine camera : $\Pi_{ia} \doteq K_i \Pi_0 g_{ie} H_a^{-1}$
- Euclidean camera : $\Pi_{ie} \doteq K_i \Pi_0 g_{ie}$.

② Projective reconstruction.

Goal: Given point correspondences $\{(x_1^i, x_2^i)\}$, recover $\{X_p\}$ and the projection matrix Π_{ip}.

Recall we already know how to solve for F.

$$x_2^{iT} F x_1^i = 0 \Leftrightarrow (x_1^i \otimes x_2^i)^T F^s = 0$$

Then project $\sigma_3(F) = 0$.

Also recall such F is not unique (4-parameter family)

$$F = \widehat{T'}(KRK^{-1} + v_4 T' v^T).$$

Theorem 6.3 (Π_{1p}, Π_{2p}) and $(\Pi_{1p}, \tilde\Pi_{2p})$ yield the same F matrix iff $\exists$ a nonsingular transformation $H_p \in \mathbb{R}^{4\times 4}$, such that $\tilde\Pi_{2p} \sim \Pi_{2p} H_p$.

proof: first, remember we assume $\Pi_{1p} = [I, 0]$

denote $\Pi_{2p} = [\tilde R, \tilde T] \in \mathbb{R}^{3\times 4}$, $\tilde\Pi_{2p} = [\tilde R', \tilde T']$.

Then by assumption

$$\hat{\tilde T}'\tilde R' \sim \hat{\tilde T}\tilde R$$

That is $A \sim \lambda_4 A$ for some $\lambda_4 \in \mathbb{R}$.

Notice $\tilde T'$ is in the left null of A.

and $\tilde T$ is in the left null of $\lambda_4 A$

$\Rightarrow \tilde T \sim \tilde T'$

$\therefore \tilde R' \sim (\tilde R + \tilde T v^T)$ for some $v \in \mathbb{R}^3$.

$$\Rightarrow [\tilde R', \tilde T'] \sim [\tilde R \ \tilde T]\underbrace{\begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}}_{H_p \text{ (full rank)}}$$

#.

★ Canonical decomposition.

Since $F \mapsto (\Pi_{1p}, \Pi_{2p})$ is a one-to-many relation, each map gives a different reconstruction of X_p.

Define (Canonical) $F \mapsto \Pi_{1p} = [I, 0]$, $\Pi_{2p} = [(\widehat{T'})^T F, T']$

Question: how to get T'? (unique?) $\underbrace{(T')^T}_{e_2} F = 0$

☆ Reconstruction :

$$\begin{cases} \lambda_1 x_1' = [I, 0]\, X_p \\ \lambda_2 x_2' = [(\hat{T}')^T F,\ T']\, X_p . \end{cases}$$

This constraint is linear in X_p.

Sovle by SVD on $D X_p = 0$ for a data matrix D.

③ Affine reconstruction

Goal: upgrade X_p to X_a by

$$X_a = H_p^{-1} X_p = \left(\begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \in \mathbb{R}^{4\times4} \right) \cdot X_p .$$

☆ Geometric interpretation of $[v^T, v_4]$.

Notice that when $[v^T, v_4]\, X_p = 0$.

$$\Rightarrow X_a = \begin{pmatrix} X \\ Y \\ Z \\ 0 \end{pmatrix}_{\text{affine}} \Rightarrow X_a \text{ is at the infinity}$$

$\therefore \{X_p\}$ is called the plane at infinity, P_∞.

and denote $\pi_\infty^T \doteq [v^T, v_4] = [v_1, v_2, v_3, v_4] \in \mathbb{R}^4$.

$\therefore\ \pi_\infty^T X_p = 0 \Leftrightarrow X_p \in P_\infty$

Solution I: Using the vanishing points.

Under perspective projection, X_p projects onto the image as vanishing points !!

Now, given 3 reconstructed vanishing points X_p^1, X_p^2, X_p^3,

$$[v_1, v_2, v_3, v_4]\, X_p^j = 0 .$$

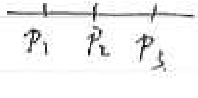
$$\Rightarrow X_a = \begin{bmatrix} I & 0 \\ v^T/v_4 & 1 \end{bmatrix} X_p$$

now, enforcing that $\left((\hat{T}')^T F - T' v^T v_4^{-1}\right) \sim KRK^{-1}$

Solution II: Equal modulus constraint.

sketch: $\lambda_2 x_2' = [(\hat{T}')^T F - T' v^T v_4^{-1},\ T' v_4^{-1}]\, X_a$.

Solution III: Can we use mid-point constraints?



$$\xRightarrow{\text{affine}} X_a^2 - X_a^1 = X_a^3 - X_a^2 \quad ?$$

④ Euclidean reconstruction (Assume $K_1 = K_2 = K$).

$$\begin{cases} \lambda_1 x_1' = [I, 0] X_a \\ \lambda_2 x_2' = [KRK^{-1}, KT] X_a \end{cases}$$

Goal: recover the distortion due to the metric $S = K^{-T}K^{-1}$.

If we denote KRK^{-1} as $\tilde{R}$.

$$S^{-1} - \tilde{R} S^{-1} \tilde{R}^T$$
$$= KK^T - \underbrace{KRK^{-1}}\underbrace{KK^T}\underbrace{K^{-T}R^TK^T} = 0.$$

This is called a Lyapunov equation

$$L : \mathbb{C}^{3\times3} \to \mathbb{C}^{3\times3} \; ; \; X \longmapsto X - CXC^T$$

∴ We are looking for the kernel of L., In fact, symmetric real kernel!

Theorem 6.9. Given two matrices $C_i = KR_iK^{-1}$, $i = 1, 2$, where $R_i = e^{\hat{u}_i \theta_i}$ with $\|u_i\| = 1$, $\theta_i \neq k\pi$, then

$$SRker(L_1) \cap SRker(L_2) \text{ is one-dimensional}$$

iff u_1 and u_2 are linearly independent

⇒ ① for a pair of affine projection matrices Π_{1a}, Π_{2a}, we can only recover K up to a one-parameter family.

② for two pairs of $\{\Pi_a\}$, and rotations are along different axes, K can be fully recovered.

Example 6.10 (pure rotation)

$$\lambda_2 x_2' = KRK^{-1} \lambda_1 x_1' \quad \text{(uncalibrated Homography)}$$
$$\Leftrightarrow \quad \widehat{x_2'} KRK^{-1} x_1' = 0$$

Given four points, $(KR_1K^{-1}) \doteq \tilde{H}_1$ can be recovered.

rotate along another direction ⇒ $(KR_2K^{-1}) \doteq \tilde{H}_2$

Then, K can be uniquely recovered by $\tilde{H}_1$, $\tilde{H}_2$

4. Calibration with scene knowledge.

① Suppose the image is a picture of a man-made building

$\therefore \exists$ three principal directions in space.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the vanishing points correspondingly are

$$v_1 = KRe_1, \quad v_2 = KRe_2, \quad v_3 = KRe_3.$$

denote

$$S = K^{-T}K^{-1} \in \mathbb{R}^{3\times3}, \text{ then}$$

$$v_i^T S v_j = v_i^T K^{-T} K^{-1} v_j = e_i^T e_j = \delta_{ij}.$$

where, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow$ three independent constraints $\begin{cases} v_1^T S v_2 = 0 \\ v_1^T S v_3 = 0 \\ v_2^T S v_3 = 0. \end{cases}$

But K has five parameters: $\begin{bmatrix} fS_x & S_\theta & O_x \\ 0 & fS_y & O_y \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ K can be recovered up to a 2-parameter family.

To simplify, set $S_\theta = 0$, and $S_y = S_y$.

$\therefore$ $K = \begin{bmatrix} S & 0 & O_x \\ 0 & S & O_y \\ 0 & 0 & 1 \end{bmatrix}$ can be fully recovered.

② Calibration with a planar pattern (checker board)

Given: ✓ multiple images of a checker board.

✓ measurement of the corner points on the board.

$\Rightarrow$ if we set the world coord. system on the board with $Z \equiv 0$, then we know all points $X_i = \begin{bmatrix} x_i \\ y_i \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4$.

Hence: $\lambda \begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} = K[r_1, r_2, T] \begin{bmatrix} X_i \\ Y_i \\ 1 \end{bmatrix}$.

where $r_1, r_2 \in \mathbb{R}^3$ are the first two columns of R.

$\Rightarrow \hat{x}'_i H [X_i, Y_i, 1]^T = 0$, $H \doteq K[r_1, r_2, T] \in \mathbb{R}^{3\times3}$

☆ This is a homography relation between X in space and its image x', different from $\hat{x}_2 H x_1 = 0$, where both x_1 and x_2 are images.

With more than 4 points, $K[r_1, r_2, T]$ can be fully recovered up to a scale factor.

Next, recover K, r_1, r_2, and T

let $[h_1, h_2] \sim [K r_1, K r_2]$. then

$\because r_1 \perp r_2 \quad \therefore \begin{cases} h_1^T K^{-T} K^{-1} h_2 = 0 \\ h_1^T K^{-T} K^{-1} h_1 = h_2^T K^{-T} K^{-1} h_2 \end{cases}$

= Two linear constraints in terms of $S = K^{-T}K^{-1}$.

Since S has 5 parameters, we need 3 images to fully recover $S = K^{-T}K^{-1} \in \mathbb{R}^{3\times3}$.

from S to K? Cholesky factorization.

$S =$

After K is recovered, we can get hold of r_1, r_2, and T.