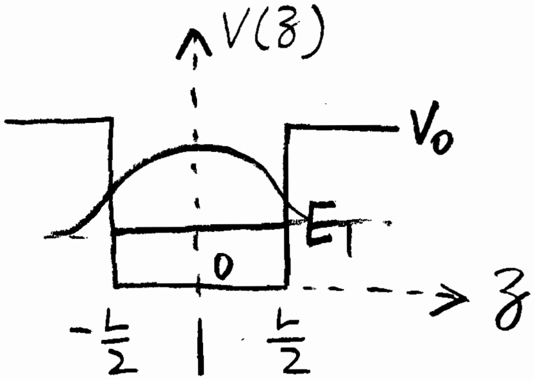
Potential Well with Finite Barrier



$$V(z) = \begin{cases} V_0 & |z| \geq \frac{L}{2} \\ 0 & |z| < \frac{L}{2} \end{cases}$$

$$-\frac{\hbar^2}{2m}\nabla^2\phi(z) + V(z)\phi(z) = E\phi(z)$$

For $|z| < \frac{L}{2}$

$$\frac{d^2\phi(z)}{dz^2} + \frac{2m_wE}{\hbar^2}\phi(z) = 0$$

$$\phi(z) = A_1 \underset{\text{odd}}{\underset{\uparrow}{\sin kz}} + A_2 \underset{\text{even function}}{\underset{\uparrow}{\cos kz}}$$

$$k = \frac{\sqrt{2m_wE}}{\hbar}$$

Since potential $V(z) = V(-z)$: symmetric

$\Rightarrow \phi(z)$ is either symmetric or anti-symmetric.

Lowest energy level usually corresponds to even function.

$$\Rightarrow \phi(z) = A_2 \cos kz$$

For $z \geq \frac{L}{2}$

$$\frac{d^2\phi(z)}{dz^2} - \frac{2m_b}{\hbar^2}(V_0 - E)\cdot\phi(z) = 0$$

$$\phi(z) = B_1 \cdot e^{-\alpha z}$$

$$\alpha = \frac{\sqrt{2m_b(V_0 - E)}}{\hbar}$$

Boundary condition

$\phi(z)$ continuous at $z = \frac{L}{2}$

Velocity continuous $\frac{\vec{P}}{m} = \frac{-i\hbar}{m}\nabla$

$\Rightarrow \frac{1}{m}\frac{d\phi}{dz}$ continuous

↑ note m may be different in well and barrier

$$\left.\begin{cases} A_2 \cos k\frac{L}{2} = B_1 \cdot e^{-\alpha L/2} & \text{— ①} \\ -\frac{k A_2}{m_w}\cdot \sin k\frac{L}{2} = -\frac{\alpha B_1}{m_b} e^{-\alpha L/2} & \text{— ②} \end{cases}\right\}$$

2 Eqs. $\Rightarrow$ 2 variable $\begin{cases} E \\ \left(\frac{B_1}{A_2}\right) \end{cases}$

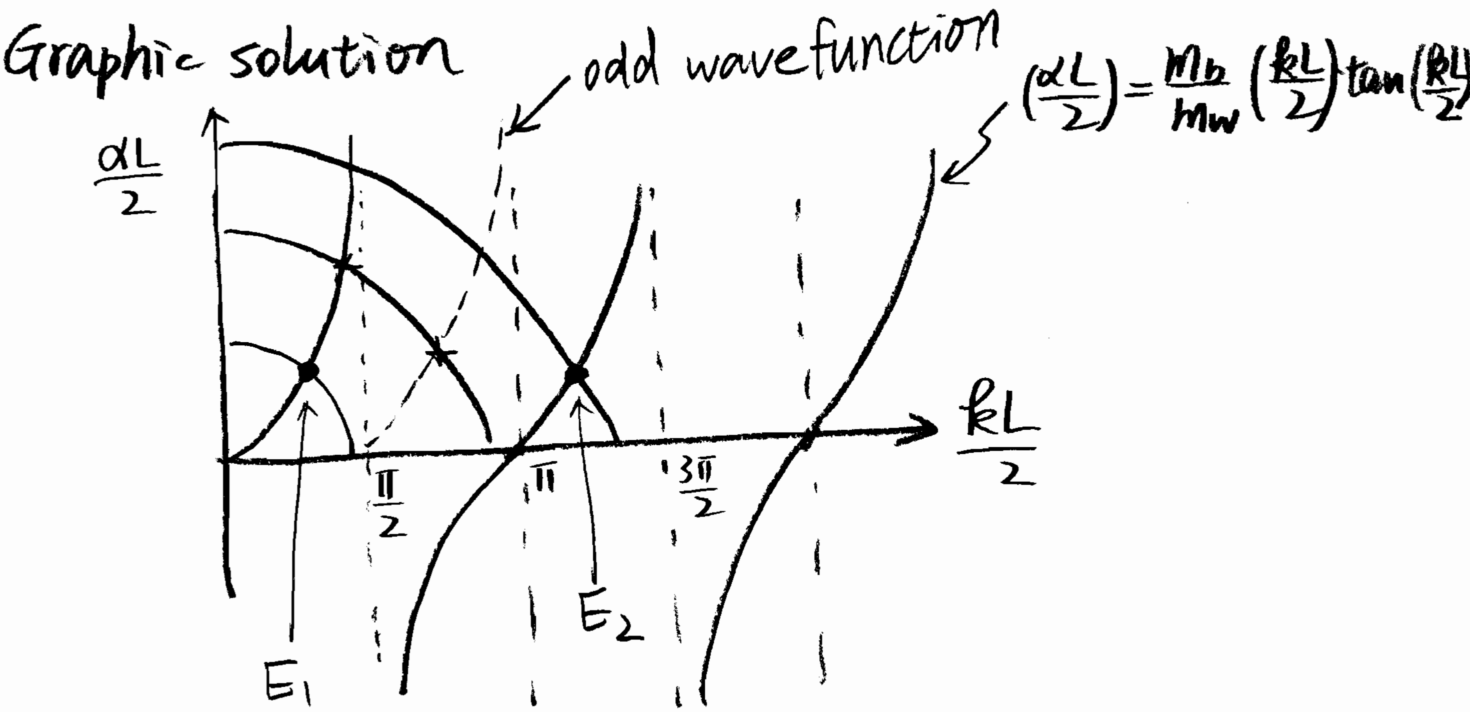
$$\frac{②}{①}: \quad \frac{k}{m_w}\tan k\frac{L}{2} = +\frac{\alpha}{m_b}$$

$$\alpha = \frac{m_b}{m_w}\cdot k \cdot \tan\frac{kL}{2}$$ ← Eigenequation for E

both k and α are functions of E

$$\frac{k^2}{m_W} + \frac{\alpha^2}{m_b} = \frac{2V_0}{\hbar^2} \rightarrow \text{ellipse in } \left(\frac{\alpha L}{2}\right)\text{-vs-}\left(\frac{kL}{2}\right) \text{ plane.}$$

Graphic solution



A_2, B_1 can be solved by substituting solution E_1 (or E_2) in ① (or ②)

$$\Rightarrow B_1 = A_2 \cos\frac{kL}{2} \cdot e^{\frac{\alpha L}{2}}$$

$$\Rightarrow \phi(z) = \begin{cases} A_2 \cos kz & |z| < \frac{L}{2} \\ A_2 \cdot \cos\left(\frac{kL}{2}\right) e^{-\alpha\left(|z| - \frac{L}{2}\right)} & \end{cases}$$

A_2: normalization constant.

From the graphical solution. we can conclude

* There are at least one bound state for symmetric potential well

* The number of bound states depends on barrier height = V_0

In graphic solution. set $\alpha = 0$ in elliptical eq.

$$k = \frac{\sqrt{2 V_0 m_w}}{\hbar}$$

$$(N-1)\frac{\pi}{2} \le \sqrt{2 V_0 m_w} \cdot \frac{L}{2\hbar} < N\frac{\pi}{2}$$