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Probability Theory and Related Fields

Multiparameter Martingale Differential Forms

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1. Introduction

A substantial body of results on stochastic integration with respect to multiparameter martingales now exists. Yet, as it stands, the theory is not entirely satisfactory in a number of ways. In particular, the *calculus* for stochastic integration, already complicated in two dimension, becomes prohibitively so in higher dimensions. In retrospect, the source of the difficulty seems to be that integration over n-dimensional volumes in n-space is only a very small part of a complete theory of integration in n-space. What seems to be needed is a theory of differential forms involving martingales and integration of such forms on sets of appropriate dimensionality. To embark on a course to develop such a theory is the objective of the work reported here.

The starting point of our approach to stochastic differential forms is similar to that of Whitney [4] and forms are defined as function on chains or functions parametrized by chains satisfying certain continuity conditions. While the flat cochains defined by Whitney ([4], Ch. IX) have the representation

$$X(\sigma)=\int_{\sigma} x(t)\, dt_{i_1} \wedge \ldots \wedge dt_{i_r} \tag{1.1}$$

we cannot expect such a representation to hold for any class of martingale forms that includes the Wiener process and a different approach becomes necessary. We intend to show in this paper that an exterior calculus for martingale forms can be constructed without such a representation. In the nonrandom case the exterior calculus is coordinate independent. However, in the stochastic case there is an underlying information pattern, namely, the subsigma fields, and as a result the stochastic calculus presented here is not coordinate free. The situation is similar to those cases where boundary conditions for physical systems yield a coordinate dependent formulation.

In the next section we define the subsigma fields involved, stochastic cochains and two norms for chains: namely, the mass norm and the flat norm of Whitney, with a different and more appropriate norm being introduced later.

Stochastic differential forms are introduced in Sect. 3, these are cochains satisfying certain continuity properties. The exterior derivative of a cochain is introduced by relating the value for the derivative on a rectangle to the value of the original cochain on the boundary of the rectangle. Different classes of martingale cochains and forms are introduced in Sect. 4. In Sect. 5 it is shown that with a wide class of martingale cochains we can associate with each martingale cochain a positive cochain which plays a role analogous to that of the increasing function of a one parameter process. The notions of martingales of path independent variations and martingales of orthogonal increments are easily generalized to the multiparameter case via the positive cochain associated with martingales. The exterior product is considered in Sects. 6, 7, 8. First, in Sect. 6, we deal with the exterior product $\phi \wedge M$ where ϕ is a zero cochain (i.e., a predictable integrand) and M is a martingale cochain which plays the role of an integrator. The exterior product $X \wedge Y$ of nonrandom forms is discussed in Sect. 7. Since our assumptions on the forms X and Y are not enough to have a representation of the form (1.1) for X and Y, we cannot define $X \wedge Y$ through

$$(X \wedge Y)(\sigma)=\int_\sigma x(t)\, y(t)\, dt_{i_1} \wedge \ldots \wedge dt_{i_{r_1}} \wedge dt_{j_1} \wedge \ldots \wedge dt_{j_{r_2}}.$$

We introduce the exterior product by an approximation procedure that avoids the local representations $x(t)$ and $y(t)$. The exterior product not only extends the stochastic integrals of the second type that were introduced in [6], but is of new and independent interest in the nonrandom case. This approach is followed in Sect. 8 in introducing the exterior product of martingale forms. A formula for the exterior derivative of the exterior product $X \wedge Y$ is discussed in Sect. 9 and its relation to the Green formula of Cairoli and Walsh is pointed out.

2. Preliminaries

Notation. Let $\mathbb{R}^n_+$ denote the positive quadrant of $\mathbb{R}^n$. We associate with $\mathbb{R}^n_+$ the usual partial order

$$(t_1, t_2, \ldots, t_n) \geqq (s_1, \ldots, s_n) \quad \text{if } t_i \geqq s_i \text{ for all } i=1, 2, \ldots, n$$
$$(t_1, \ldots, t_n) > (s_1, \ldots, s_n) \quad \text{if } t_i > s_i \text{ for all } i=1, 2, \ldots, n$$

and define

$$t \wedge s=(\min(t_1, s_1), \ldots, \min(t_n, s_n))$$
$$t \vee s=(\max(t_1, s_1), \ldots, \max(t_n, s_n)).$$

For a rectangle σ, define $\underline{t}(\sigma)$ and $\bar{t}(\sigma)$ as the infimum and supremum of points in σ respectively. $\mathbf{i}=(i_1, i_2, \ldots, i_r)$ is a subset of the integers from 1 to n then $\mathbf{i}^*$ will denote the collection of all remaining integers between 1 and n, and $[\mathbf{i}]$ will denote $\mathbf{i}$ put in increasing order. Similarly, $t_{\mathbf{i}}$ denotes $(t_{i_1}, \ldots, t_{i_r})$.

Let $(\Omega, \mathbf{F}, P)$ be a complete probability space and let $\{\mathbf{F}_t, t\in\mathbb{R}^n_+\}$ be a family of sub-σ fields. Define

$$\mathbf{F}_t^{\mathbf{i}} = \bigvee_{s:\, s_{\mathbf{i}}=t_{\mathbf{i}}} \mathbf{F}_s$$

e.g., if $n=3$ and $\mathbf{i}=(1,3)$ then $\mathbf{F}_t^{\mathbf{i}} = \bigvee_{\theta>0} \mathbf{F}_{(t_1,\theta,t_3)} = \mathbf{F}_{(t_1,\infty,t_3)}$. We assume that $\mathbf{F}_t$ satisfies the following assumptions (cf. [1]):

(F_1) $t>s \Rightarrow \mathbf{F}_t \supseteq \mathbf{F}_s$

(F_2) $\mathbf{F}_t$ contains all the null sets of $\mathbf{F}$

(F_3) $\mathbf{F}_t = \bigcap_{t<s} \mathbf{F}_s$

(F_4) $\forall\, t, \mathbf{i}$, $\mathbf{F}_t^{\mathbf{i}}$ and $\mathbf{F}_t^{\mathbf{i}^*}$ are conditionally independent given $\mathbf{F}_t$.

Condition (F_4) is a generalization of the corresponding condition of Cairoli and Walsh [1].

Let a_j denote a finite interval open to the left and closed to the right on the t_j axis. For $i<j$, $a_i \wedge a_j$ will denote a possibly oriented 2-dimensional rectangle with sides a_i and a_j and $a_j \wedge a_i = -a_i \wedge a_j$ will denote the same rectangle with a negative orientation. In general, let $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ denote intervals as above. Then $a_{i_1} \wedge a_{i_2} \wedge \ldots \wedge a_{i_r}$ will denote an r dimensional rectangle with sides $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$. The orientation is positive if an even permutation of $(i_1, i_2, \ldots, i_r)$ puts it into increasing order, and the orientation is negative otherwise. We call such rectangles *oriented r-rectangles* and refer to $[\mathbf{i}]$ as the direction of $a_{i_1} \wedge a_{i_2} \wedge \ldots \wedge a_{i_r}$.

We note that the boundary $\partial\sigma$ of an oriented $(r+1)$ rectangle σ is a collection of oriented r-rectangles that overlap at most on boundaries. Subdivision of an r-rectangle produces a collection of r-rectangles. It is useful to denote such a collection by a sum $\sigma_1+\sigma_2+\ldots+\sigma_m$. Furthermore if σ is an oriented r-rectangle it is useful to denote by $-\sigma$ the same rectangle with the opposite orientation. It is therefore useful to introduce linear combinations

$$A = \sum_{k=1}^{m} \alpha_k \sigma_k \tag{2.1}$$

where α_k are real numbers taking values in $\{-1, 1\}$ and σ_k are oriented r-rectangles. We shall call any sum of the form (2.1) an *r-chain.*

Let $X(\sigma)$ be a real-valued random function defined on $(\Omega, \mathbf{F}, P)$ and parametrized by oriented r-rectangles such that

(a) $X(\sigma)$ is defined for every oriented r-rectangle σ

(b) $X(\sigma) = -X(-\sigma)$ and for disjoint rectangles $X\left(\sum_{k=1}^{m} \sigma_k\right) = \sum_{k=1}^{m} X(\sigma_k)$

(c) $X(\sigma)$ is $\mathbf{F}_{\bar{t}(\sigma)}$ adapted.

We can extend X to all rectangular r-chains by linearity and X so extended is appropriately termed a *random r-cochain.*

In the next section a further extension of X that depends on whether it satisfies some continuity conditions will be considered. For this purpose the notion of convergence of chains is necessary. Let $|\sigma|$ denote the r-dimensional volume of the oriented rectangle σ with $|\sigma|=1$ for $r=0$. For A defined by (2.1) with disjoint σ_k, $k=1, \ldots, n$, the *mass* of a chain A is defined as

$$|A|=\sum_k |\alpha_k| \cdot |\sigma_k|.$$

Turning to another norm, let $\{A_m,\ m=1, 2, \ldots\}$ be a sequence of r chaims, we shall say that the sequence is a *Cauchy sequence* if either

$$|A_m - A_k| \underset{m,\,k\to\infty}{\longrightarrow} 0$$

or, if for every m, k there is an $r+1$ chain $B_{m,k}$ such that $\partial B_{m,k}=A_m-A_k$ and

$$|B_{m,k}| \underset{m,\,k\to\infty}{\longrightarrow} 0.$$

Note that for the convergence of an n-chain in $\mathbb{R}^n$, only the first type of convergence makes sense, while for the convergence of a 1-chain in $\mathbb{R}^2$ to a curve the second type of convergence is necessary. Therefore, it is useful to define the *flat norm* $|A|^{\sim}$ for an r-chain in $\mathbb{R}^n$ by ([4], p. 154)

$$|A|^{\sim}=\inf\{|A-\partial B|+|B|\} \tag{2.2}$$

where the infimum is over all $r+1$ chains B. It is shown in [4] that $|A+B|^{\sim} \leqq |A|^{\sim}+|B|^{\sim}$ and $|A|^{\sim}=0$ if and only if $A=\phi$. Hence, $|\cdot|^{\sim}$ is a norm. Furthermore, $|\cdot|^{\sim}$ satisfies: (see [4])

$$|\partial A|^{\sim} \leqq |A|^{\sim} \leqq |A|. \tag{2.3}$$

Note that for $r=n$, $|A|^{\sim}=|A|$. For $r=0$ and A a point in $\mathbb{R}^n$, $|A|^{\sim}=1$. For the case where A is the difference of two points, s and t, $|A|^{\sim}=\min(2, |(s,t)|)$.

3. Stochastic Differential Forms

Intuitively we would like to write a random r-cochain $X(\sigma)$ as an integral over σ

$$X(\sigma)=\int_\sigma X$$

where the integrand X is a "stochastic differential r-form". If we are to include such processes as the Wiener process and white noise in the theory, then the random differential forms are necessarily generalized processes (i.e., random currents). Ito defined random currents a long time ago [2], however his approach is incomplete for our purposes because it is limited to linear operations. Exterior products $X \wedge Y$, where X, Y are random currents have not been defined in [2]. As will be seen in later sections, to define such exterior products is to define stochastic integrals (of different varieties) on $\mathbb{R}^n_+$.

One possibility is to define a stochastic differential r-form as the formal integrand of a stochastic r-cochain that is continuous in probability with respect to the flat norm defined in the previous section, i.e.,

$$X(A_m) \xrightarrow{p} 0 \quad \text{whenever } |A_m|^{\sim} \to 0 \quad \text{as} \quad m \to \infty. \tag{3.1}$$

Similarly a random differential form is said to be an L_q form or a q-integrable form if

$$E|X(A)|^q < \infty \tag{3.2$'$}$$

and

$$E|X(A_m)|^q \to 0 \tag{3.2$''$}$$

whenever $|A_m|^{\sim} \to 0$. In (3.1) and (3.2) we extend the definition of X to limits of chains under the flat norm by adjoining $X(A_\infty)$.

As an example let η be "Gaussian white noise" on $\mathbb{R}_+^2$ defined as follows:

(a) $\eta(\sigma)$ is a Gaussian random function parametrized by oriented 2-rectangles σ on $\mathbb{R}_+^2$

(b) $E\eta(\sigma)=0$

(c) $E\eta(\sigma)\eta(\sigma')=\mu(\bar{\sigma}\cap\bar{\sigma}')$ if σ and σ' are similarly oriented
$= -\mu(\bar{\sigma}\cap\bar{\sigma}')$ otherwise

where $\bar{\sigma}$ denotes σ without orientation and μ denotes the Lebesgue measure. The white noise η is a random rectangular 2-cochain. Since $E\eta^2(\sigma)=|\sigma|$, (3.2) is satisfied.

A Wiener process $\{W_t, t\in\mathbb{R}_+^2\}$ is defined by

$$W_t = \eta(A_t)$$

where A_t is the rectangle $\{s: 0 \leqq s \leqq t\}$. Wiener process is a 0-cochain satisfying (3.2). Furthermore, suppose that σ is an oriented 1-rectangle $[(\alpha_1, \alpha_2), (\beta_1, \alpha_2)]$, then

$$dW(\partial\sigma) = W_{\beta_1, \alpha_2} - W_{\alpha_1, \alpha_2}$$
$$E(dW(\partial\sigma))^2 = \alpha_2(\beta_1 - \alpha_1)$$

and $|\sigma| = \beta_1 - \alpha_1$. Hence (3.2)$''$ is verified for horizontal 1-rectangles.

Continuing with our example, suppose that we define an oriented 1-cochain as follows:

$$\text{for } \sigma = ((\alpha_1, t_2), (\beta_1, t_2)] \text{ set } X_1(\sigma) = W_{\beta_1, t_2} - W_{\alpha_1, t_2}$$
$$\text{for } \sigma = ((t_1, \alpha_2), (t_1, \beta_2)] \text{ set } X_1(\sigma) = 0.$$

Since the only 1-rectangles in $\mathbb{R}_+^2$ are horizontal and vertical line segments, X_1 is well defined as a random cochain. For a 2-rectangle σ defined by the vertices (α_1, α_2), (β_1, α_2), (β_1, β_2), (α_1, β_2), $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, and anticlockwise orientation

$$X_1(\partial\sigma) = (W_{\beta_1, \alpha_2} - W_{\alpha_1, \alpha_2}) - (W_{\beta_1, \beta_2} - W_{\alpha_1, \beta_2}) \tag{3.3}$$

and $EX_1^2(\partial\sigma) = |\sigma|$ so that X_1 satisfies (3.2). We can similarly define a vertical cochain X_2 and we shall see that $X_1 + X_2$ can be viewed as dW i.e., the exterior derivative of the 0-cochain W_t.

It turns out that convergence in the flat norm is not convenient for martingale forms and exterior products. For this reason we introduce the following type of convergence: Let A be a r-chain and let $A=\sum_1^K \alpha_k \sigma_k$ where $\alpha_k \in \{-1, 0, 1\}$ and the σ_k are disjoint rectangles. Set

$$\|X(A)\|_{(1)}=\sup \sum_{k=1}^K |\alpha_k| \cdot E|X(\sigma_k)|$$

where the supremum is over all representations of A as $\sum_1^K \alpha_k \sigma_k$ with σ_k disjoint (and over all K). Similarly,

$$\|X(A)\|_{(2)}=\sup \left(\sum_{k=1}^K \alpha_k^2 E(X(\sigma_k))^2\right)^{1/2}$$

where the supremum is the same as for the definition of $\|\cdot\|_{(1)}$. Obviously, $\|\cdot\|_{(1)}$ satisfies the triangle inequality:

$$\|X(A)+Y(A)\|_{(1)} \leqq \|X(A)\|_{(1)}+\|Y(A)\|_{(1)}.$$

So does $\|\cdot\|_{(2)}$, and the proof of this is as follows

$$\begin{aligned}\|X(A)+Y(A)\|_{(2)} &= \sup (E \sum (\alpha_k X(\sigma_k)+\alpha_k Y(\sigma_k))^2)^{1/2} \\ &\leqq \sup (E((\sum \alpha_k^2 X^2(\sigma_k))^{1/2}+(\sum \alpha_k^2 Y^2(\sigma_k))^{1/2})^2]^{1/2} \\ &\leqq \sup \{E^{1/2} \sum \alpha_k^2 X^2(\sigma_k)+E^{1/2} \sum \alpha_k^2 Y^2(\sigma_k)\}\end{aligned}$$

where the last inequality follows by the Minkowski inequality. Consequently, for A fixed, $\|X(A)\|_{(q)}$ with $q=1$ or 2 is a norm.

Definition. *X will be said to be a Σ_q cochain, $q=1$ or 2, if*

$$\|X(A_m)\|_{(q)} \to 0$$

whenever $|A_m| \to 0$ as $m \to \infty$.

For example, if $EX^2(\sigma) \leqq C \cdot |\sigma|$ for every r-rectangle σ then X is a Σ_2 cochain. The notion of Σ_q cochains is not sufficient to extend a cochain X to manifolds, this will be done later after dX, the exterior derivative of X, is defined.

As in the non-random case, an advantage of working with differential forms rather than with linear functionals or generalized functions is a conceptual one. The non-random differential forms are defined locally so that exterior differentials and exterior products of forms make sense. Exterior products, in particular, lead to a nonlinear analysis of cochains. We intend to present a similar stochastic-calculus approach in the stochastic "generalized" case by defining the operations on the corresponding random cochains.

Remark. As we have defined them, random differential r-forms are random currents of Ito [2], but not every Ito current is an r-form in our sense. While linear operations are definable on all random currents, nonlinear operations (e.g., exterior products) are not. The r-forms that we have defined have the right degree of localization to allow exterior products to be defined.

If X is a regular (nongeneralized) differential form, then X can be represented as

$$X_t = \sum_{[\mathbf{i}]} \alpha_{[\mathbf{i}]}(t)\, dt_{[\mathbf{i}]} \tag{3.4}$$

where the differentials $dt_{[\mathbf{i}]} = dt_{i_1} \wedge dt_{i_2} \wedge \ldots \wedge dt_{i_r}$ provide a local coordinate system. For a random current such a representation is in general not possible, but a useful representation similar to this one still exists. For a random cochain X, define $X_{[\mathbf{i}]}$ as the cochain such that for every rectangle σ

$$\begin{aligned} X_{[\mathbf{i}]}(\sigma) &= X(\sigma) && \text{if } \sigma \text{ has the direction } [\mathbf{i}] \\ &= 0 && \text{otherwise} \end{aligned} \tag{3.5}$$

Then for any rectangular chain A

$$X(A) = \sum_{[\mathbf{i}]} X_{[\mathbf{i}]}(A) \tag{3.6}$$

and if X is a random differential form so is $X_{[\mathbf{i}]}$. Hence we can write

$$X = \sum_{[\mathbf{i}]} X_{[\mathbf{i}]} \tag{3.7}$$

and this is the equivalent of (3.4) for random differential forms.

Next, we define the exterior derivative dX of a random r-cochain X (via the Stokes theorem) as follows. Set

$$dX(A) = X(\partial A) \tag{3.8}$$

for all oriented $(r+1)$ chains A. An equivalent definition for the exterior derivative is the following. Define $d_k X_{[\mathbf{i}]}$ for rectangles σ as

$$\begin{aligned} d_k X_{[\mathbf{i}]}(\sigma) &= dX_{[\mathbf{i}]}(\sigma) && \text{if } k \text{ is not in } [\mathbf{i}] \text{ and } \sigma \text{ has direction } [k, [\mathbf{i}]] \\ &= 0 && \text{otherwise} \end{aligned} \tag{3.9}$$

Then we can write

$$d_k X = \sum_{[\mathbf{i}]} d_k X_{[\mathbf{i}]} \tag{3.10}$$

and

$$dX = \sum_k d_k X. \tag{3.8$'$}$$

Let $\partial_k^+(\sigma)$, $\partial_k^-(\sigma)$ denote the upper and lower boundaries of σ in the k direction ($\partial_k^+(\sigma) = \partial_k^-(\sigma)$ if k is not in the direction of σ)

$$(d_k X)(\sigma) = X(\partial_k^+(\sigma)) - X(\partial_k^-(\sigma)). \tag{3.9$'$}$$

The exterior derivative of a stochastic differential form as defined by (3.1) and (3.2) is also a stochastic differential form of the same type, this follows directly from the definition of dX and the fact that $|\partial A|^{\sim} \leq |A|^{\sim}$.

We turn now to the definition of Σ_q forms, $q = 1, 2$.

Definition. *A cochain X will be said to be a Σ_q form if both X and dX are Σ_q cochains. Note that in general $\|X(\partial B)\|_{(q)} \neq \|dX(B)\|_{(q)}$. Also note that if X is a Σ_q form so is dX since $ddX=0$, and that every Σ_q n-cochain is also a Σ_q n-form.*

For example, if $EX^2(\sigma) \leqq c|\sigma|$ for every r-rectangle σ and $EX^2(\partial\tau) \leqq c|\tau|$ for every $(r+1)$ rectangle τ then X is a Σ_2 form. If for every r-chain A

$$EX^2(A) \leqq c|A|^{\sim}$$

then

$$EX^2(A) \leqq c|A|,$$

since $|A|^{\sim} \leqq |A|$ and $E(dX(B))^2 = E(X(\partial B))^2 \leqq c|\partial B|^{\sim} \leqq c|B|^{\sim} \leqq c|B|$, X is a Σ_2 form.

Consider a Wiener process W_t, $t \in \mathbb{R}_+^2$. Take σ_1 and σ_2 to be the horizontal and vertical 1-rectangles

$$\sigma_1 = ((t_1, t_2), (t_1 + a, t_2)], \qquad \sigma_2 = ((t_1, t_2), (t_1, t_2 + b)]$$

oriented from the left (from below) to the right (to above). We have

$$\begin{aligned} d_1 W(\sigma_1) &= W_{t_1+a,\, t_2} - W_{t_1,\, t_2} \\ d_2 W(\sigma_2) &= W_{t_1,\, t_2+b} - W_{t_1,\, t_2} \\ d_1 W(\sigma_2) &= d_2 W(\sigma_1) = 0. \end{aligned}$$

Now, take a positively oriented 2-rectangle σ with $\underline{t} = (t_1, t_2)$ and $\bar{t} = (t_1 + a, t_2 + b)$. Its boundary $\partial\sigma$ is given by:

$$\partial\sigma = \{\sigma_1, -\sigma_2, ((t_1+a, t_2), (t_1+a, t_2+b)], \quad -((t_1, t_2+b), (t_1+a, t_2+b)].$$

Hence

$$\begin{aligned} d(d_1 W)(\sigma) &= (W_{t_1+a,\, t_2} - W_{t_1,\, t_2}) - (W_{t_1+a_1,\, t_2+b} - W_{t_1,\, t_2+b}) \\ &= -\eta(\sigma) \end{aligned} \tag{3.11}$$

and

$$d(d_2 W)(\sigma) = \eta(\sigma). \tag{3.12}$$

We can interpret (3.11) and (3.12) as follows

$$\begin{aligned} d(d_1 W) &= d_1 d_1 W + d_2 d_1 W \\ d(d_2 W) &= d_1 d_1 W + d_1 d_2 W \end{aligned}$$

with $d_1 d_1 W = d_2 d_2 W = 0$, $d_2 d_1 W = -d_1 d_2 W$ and $d_1 d_2 W = d_{12} W = \eta$. Observe that

$$ddW = d(d_1 W + d_2 W) = d_2 d_1 W + d_1 d_2 W = 0$$

as it should be.

Remark. Note that the Hodge star operator $*$ is a linear operator defined on all Ito random currents. Hence $*X$ is well defined as an Ito random current for any r-cochain X considered as an Ito random current. However, $*X$ is not necessarily a cochain (equivalently a differential form) and for many interesting cases it is not. For example, let η be an n-cochain representing Gaussian white noise, for $*\eta$ to be a 0-cochain it must be a continuous random function. However

$$\eta(\sigma)=\int_\sigma *\eta\, dt_1 \wedge dt_2 \wedge \ldots \wedge dt_n$$

so that $*\eta$ can not be a continuous random function and hence is not a 0-cochain.

4. Martingale Cochains and Forms

For an oriented r-rectangle σ, denote by $\underline{t}(\sigma)$ its infimum point and $\bar{t}(\sigma)$ its maximum point. Recall that a random r-cochain was defined to be adapted i.e., $X(\sigma)$ is $\mathbf{F}_{\bar{t}(\sigma)}$ adapted for every r-rectangle σ; also, recall the notation

$$\mathbf{F}_t^{\mathbf{i}}=\bigvee_{s:\, s_{\mathbf{i}}=t_{\mathbf{i}}} \mathbf{F}_s.$$

For $n \geqq r \geqq 1$ and a fixed integer k, $1 \leqq k \leqq n$, a random r-cochain M is said to be a k-martingale r-cochain if for every r rectangle σ with direction $[\sigma]$ containing k, $E|M(\sigma)|<\infty$ and

$$E[M(\sigma)\,|\,\mathbf{F}^k_{\underline{t}(\sigma)}]=0. \tag{4.1}$$

Note that if $[\sigma]$ does not contain k, then σ lies in the $(n-1)$ hyperplane $\{s\colon s_k = t_k\}$ and

$$E[M(\sigma)\,|\,\mathbf{F}^k_{\underline{t}(\sigma)}]=M(\sigma) \quad \text{a.s.}$$

It follows immediately from the definition that if M is a k-martingale then $M_{[\mathbf{i}]}$ is also a k-martingale for all $[\mathbf{i}]$ (cf. (3.5)). M is said to be a martingale if it is a k-martingale for every k, $1 \leqq k \leqq n$. (Note that if M is a martingale then $M_{[\mathbf{i}]}$ is also a martingale.) M is said to be a weak martingale r-cochain if $E|M(\sigma)|<\infty$ and

$$E[M(\sigma)\,|\,\mathbf{F}_{\underline{t}(\sigma)}]=0. \tag{4.2}$$

Since $\mathbf{F}^k_t \supset \mathbf{F}_t$ for every k, any k-martingale is a weak martingale.

For zero cochains we define the martingale property in terms of its exterior derivatives. A zero cochain M is said to be an i-martingale if it is bounded in L_1 and $d_i M$ is an i-martingale cochain. (Note that by definition $d_i M$ is zero except on 1-rectangles with direction i. Hence $d_i M$ is a martingale if and only if it is an i-martingale.) A zero cochain M is said to be a martingale cochain if $d_i M$ is a martingale for all $i \leqq n$.

If M is a martingale zero cochain then $d_{i_r} \ldots d_{i_1} M$ is a martingale r-cochain. If M is a k-martingale then $dM_{[\mathbf{i}]}$ is a k-martingale for any $[\mathbf{i}]$ for which $k \in [\mathbf{i}]$ but dM need not be k-martingale. The relationship between the martingale properties of M and those of dM are as follows:

(a) M is a k-martingale $\Rightarrow dM_{[\mathbf{i}]}$ is a k-martingale for every $[\mathbf{i}]$ such that $k \in [\mathbf{i}]$.

(b) M is a k-martingale and $d_k M$ is a k martingale $\Rightarrow dM$ is a k-martingale.

(b′) M is a martingale and $d_k M$ is a k martingale for every $k \Rightarrow dM$ is a martingale.

Note that by itself the condition "$d_k M = k$-martingale for every k" implies that dM is a weak martingale but not necessarily a martingale, and if M is a martingale then dM is a weak martingale.

In both [1] and [6], a 2-parameter martingale is defined as a random function $\{M_t, t \in \mathbb{R}_+^2\}$ such that

$$t > s \Rightarrow E(M_t \mid \mathbf{F}_s) = M_s \quad \text{a.s.}$$

Such a process is a 0-form in our sense characterized by the property that dM is a martingale 1-form.

Cairoli and Walsh have defined 1 and 2 martingales for $\mathbb{R}_+^2$ in [1] and it is interesting to compare their definition with ours. Let M be a 0-form and let σ be a 2-rectangle $\{a_1 < t \leqq b_1,\ a_2 < t_2 \leqq b_2\}$. Denoting

$$\Delta M(\sigma) = M_{b_1, b_2} + M_{a_1, a_2} - M_{a_1, b_2} - M_{b_1, a_2}.$$

Cairoli and Walsh define M to be a 1-martingale if

$$E(\Delta M(\sigma) \mid \mathbf{F}_{a_1, \infty}) = 0$$

for all 2-rectangles σ and a 2-martingale if

$$E[\Delta M(\sigma) \mid \mathbf{F}_{\infty 1 a_2}] = 0$$

for all 2-rectangles σ. Now, ΔM can be viewed as a 2-form derived from M as follows:

$$\Delta M = d(d_2 M) = -d(d_1 M) = d_1 d_2 M.$$

Thus, M is a k-martingale, ($k=1$ or $k=2$), in the sense of Cairoli and Walsh if and only if $d_1 d_2 M$ is a k martingale 2-form in the sense of this paper.

Assume now that $d_2 M$ is a 2-martingale 1-form in our sense. Then since $\Delta M = d(d_2 M)$, we have by (3.6)

$$\Delta M(\sigma) = (d_2 M)(\partial \sigma).$$

Note that $d_2 M$ is zero on horizontal 1-rectangles so that

$$\Delta M(\sigma) = d_2 M((a_2, b_1) \uparrow (a_2, b_2) + (a_1, b_2) \downarrow (a_2, b_1))$$

and $d_2 M$ being 2-martingale 1-form implies

$$E(\Delta M(\sigma) \mid \mathbf{F}_{\infty, a_2}) = 0$$

so that M is a 2-martingale in the sense of Cairoli and Walsh. Similarly $d_1 M$ being a 1-martingale in our sense implies that M is a 1-martingale in the sense of Cairoli and Walsh. Since $dM = d_i M$ on 1 rectangles in the i-direction, we may summarize the above: M is an i-martingale in the sense of Cairoli and Walsh if dM is an i-martingale in the sense of this paper. The relationship among the various definitions can be displayed as follows:

$$
\begin{array}{ccc}
\text{This Paper} & & \text{C-W [1]} \\
\hline
dM = \text{martingale 1-cochain} & \Longleftrightarrow & M = L_2 \text{ martingale} \\
\Downarrow & & \\
dM = k \text{ martingale 1-cochain} & & \\
\Downarrow & & \\
d_1 d_2 M = k \text{ martingale 2-cochain} & \Longleftrightarrow & M = L_2 k \text{ martingale}
\end{array}
$$

We conclude the definitions of the martingale cochains with a definition of strong martingale r-cochains. Let σ be a rectangle and let $[\sigma]$ denote the orientation of σ. An r-cochain X will be said to be a strong martingale form if for all r-rectangles σ, $E|X(\sigma)|<\infty$ and

$$E(X(\sigma) \mid \bigvee_{k\in[\sigma]} \mathbf{F}^k_{\underline{t}(\sigma)})=0.$$

Note that for $r=1$ every martingale cochain is strong and for $n=r=2$ this reduces to the definition of [1].

Finally, every martingale cochain of the different types defined here will be said to be a Σ_2 martingale cochain and a Σ_2 martingale form of the same type if in addition it is a Σ_2 cochain or form respectively.

5. Positive Cochains and Forms Associated with Martingale Forms

A differential form or cochain X will be said to be positive if $X(\sigma)$ is non-negative for every positively oriented rectangle σ.

Proposition 5.1. *Let M be a Σ_2 martingale r-cochain, $1\leqq r\leqq n$. Then there exists a Σ_1 r-cochain $\langle M\rangle$ which is positive and satisfies*

$$E(M^2(\sigma)-\langle M\rangle(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)})=0 \tag{5.1}$$

for every positively oriented rectangle σ.

Remarks. (a) Note that relation (5.1) is for rectangles only and for the case where A is a chain and $r<n$ we may well have

$$EM^2(A)\neq E\langle M\rangle(A).$$

However (5.1) implies that

$$\|M(A)\|^2_{(2)}=\|\langle M\rangle(A)\|_{(1)}.$$

(b) For the case $r=n$ it will follow from the proof that $EM^2(\sigma)=E\langle M\rangle(\sigma)$ for rectangles σ. Since M is a martingale form, it follows that $EM^2(A)=E\langle M\rangle(A)$ for any chain A.

(c) Note that $\|X(A)\|_{(2)}$ is a convenient norm for square integrable martingale cochains since for $A \cap B = \phi$ we have

$$\|X(A \cup B)\|_{(2)}^2 = \|X(A)\|_{(2)}^2 + \|X(B)\|_{(2)}^2.$$

Before turning to the proof consider the following example. Let η and ξ be two independent white Gaussian noises on $\mathbb{R}_+^2$, and let Y and Z be the two zero forms induced by η and ξ respectively: $d_2 d_1 Y = \eta$, $d_2 d_1 Z = \xi$. Let U be the 1-form $U = dY = d_1 Y + d_2 Y$ then the 1-form $\langle U \rangle = t_2 dt_1 + t_1 dt_2$ satisfies (5.1). Let $V = d_1 Y + d_2 Z$ then $\langle V \rangle = t_2 dt_1 + t_1 dt_2 = \langle U \rangle$, note however that U and V are 1-forms with different probabilistic properties since

$$dU = d(dY) = 0 \quad \text{while} \quad dV = d_2 d_1 Y + d_1 d_2 Z \neq 0.$$

In one dimension, Gaussian martingales are characterized by their quadratic variations. This example shows that this is not the case for Gaussian martingale one forms in $\mathbb{R}^n$.

Proof. The proof will be divided into several parts. Parts (b) and (c) of the proof follow Cairoli and Walsh [1] (cf. also pp. 21, 22 of [3]).

(a) Note that it suffices to prove the existence of a positive cochain $\langle M \rangle$ satisfying (5.1) since the assumption that M is a martingale Σ_2 cochain implies that the cochain $\langle M \rangle$ is a Σ_1 cochain.

(b) Let M be Σ_2 martingale n-cochain in $\mathbb{R}_+^n$. Let R_t denote the rectangle generated by 0 and t, $t \in \mathbb{R}_+^n$, (i.e., $R_t = \{y : 0 \leqq y \leqq t\}$) and let m_t denote the zero cochain $m_t = M(R_t)$. We shall also use $m(\sigma)$ to denote the n-cochain $M(\sigma)$. Let $\phi_t = m_t^2$ and let $\phi(\sigma)$ denote $\phi(\sigma) = (d_1 d_2 \dots d_n \phi)(\sigma)$.

Lemma 5.2. (i) *m_t is a one parameter martingale on every increasing path in $\mathbb{R}_+^n$. Consequently ϕ_t is a (one parameter) submartingale on every increasing path in $\mathbb{R}^n$.* (ii) *For every rectangle σ*

$$E(\phi(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) = E((M(\sigma))^2 \mid \mathbf{F}_{\underline{t}(\sigma)}). \tag{5.2}$$

Proof of Lemma. (i) Note that $(m_{t_1+\alpha, t_2, \dots, t_n}, \mathbf{F}_{t_1+\alpha, t_2, \dots, t_n})$, $\alpha > 0$, is a one parameter martingale in the parameter α. Consequently m_t is a one parameter martingale on every increasing path since $t > s$ implies that t can be reached from s along a stepped path i.e., one that is a chain. Hence, m_t is a martingale along such stepped paths which proves (i). Turning to (ii), we note that for a given σ

$$M(\sigma) = \sum_i \delta_i m_{t_i}$$

where t_i denotes the vertices of σ and δ_i is $+1$ or -1. Therefore

$$\begin{aligned}
E(\phi(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) &= E \sum_i (\delta_i m_{t_i}^2 \mid \mathbf{F}_{\underline{t}(\sigma)}) \\
&= E \sum_i (\delta_i m_{\bar{t}(\sigma)} m_{t_i} \mid \mathbf{F}_{\underline{t}(\sigma)}) \\
&= E(m_{\bar{t}(\sigma)} M(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) \\
&= E(M^2(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) + E((m_{\bar{t}(\sigma)} - M(\sigma)) M(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) \\
&= E(M^2(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)})
\end{aligned}$$

which proves (i).

(c) As in part (b), let M be an L_2 martingale n-form and $\phi_t = m_t^2$. Let $t = (t_1, t_2, \ldots, t_n)$ and consider $\phi_{(\theta, t_2, \ldots, t_n)}$ as a function of θ, $0 \leqq \theta \leqq t_1$. By Lemma 5.2 $\phi_{\theta, t_2, \ldots, t_n}$ is a 1-parameter submartingale and therefore by the Doob-Meyer decomposition $\phi_t = \phi_t^{\pi_1} + \psi_t^1$, where ψ_t^1 is a 1-parameter martingale in the t_1 direction relative to $(\mathbf{F}_t)_{t_1}$ (or by the F-4 property, relative to $(\mathbf{F}_t^1)_{t_1}$), and $\phi_t^{\pi_1}$ is the predictable function of bounded variation in the t_1 direction relative to $(\mathbf{F}_t)_{t_1}$ or $(\mathbf{F}_t^1)_{t_1}$. Therefore, for $\alpha > 0$,

$$\phi^{\pi_1}_{t_1+\alpha, t_2, \ldots, t_n} - \phi^{\pi_1}_{t_1, t_2, \ldots, t_n} = \lim \sum_i E(\phi_{\theta_{i+1}, t_2, \ldots, t_n} - \phi_{\theta_i, t_2, \ldots, t_n} \mid \mathbf{F}_{\theta_i, \infty, \ldots, \infty}) \tag{5.3}$$

where θ_i denote the points of a partition of the t_1 axis and the limit is as the partition is refined. Consider now the behavior of $\phi_t^{\pi_1}$ in the t_2 direction ($t = t_1, \ldots, t_n$):

$$\begin{aligned} &E(\phi^{\pi_1}_{t_1, t_2+\alpha, \ldots, t_n} - \phi^{\pi_1}_{t_1, t_2, \ldots, t_n} \mid \mathbf{F}_t) \\ &\quad = E\{\lim \sum_i E(\phi_{\theta_{i+1}, t_2+\alpha, \ldots, t_n} - \phi_{\theta_i, t_2+\alpha, \ldots, t_n} \\ &\qquad - \phi_{\theta_{i+1}, t_2, \ldots, t_n} + \phi_{\theta_i, t_2, \ldots, t_n} \mid \mathbf{F}_{\theta_i, t_2, \ldots, t_n}) \mid \mathbf{F}_t\} \end{aligned} \tag{5.4}$$

where θ_i and the limit is as in (5.3). It follows from (5.4) and part (ii) of Lemma 5.2 that $\phi_t^{\pi_1}$ is a submartingale in the t_2 direction. Let $\phi_t^{\pi_i}$ denote the dual predictable function of bounded variation appearing in the Doob-Meyer decomposition of ϕ_t in the t_i direction and consider $(\phi^{\pi_1})_t^{\pi_2}$, then

$$(\phi^{\pi_1})^{\pi_2} - \phi^{\pi_1}$$

is a one parameter martingale in the t_2 direction. Furthermore $(\phi^{\pi_1})^{\pi_2}$ is a submartingale in the t_3 direction. Repeating, we construct

$$\begin{aligned} A_t^0 &= \phi_t \\ A_t^1 &= \phi_t^{\pi_1} \\ A_t^2 &= (\phi_t^{\pi_1})^{\pi_2} \\ A_t^k &= ((\phi_t^{\pi_1})^{\pi_2} \cdots)^{\pi_k} \end{aligned}$$

where, as before, A_t^k is a submartingale in the $k+1$ direction. Then $A_t^k - A_t^{k-1}$ is a one parameter martingale in the k direction. Let Δ_q be a partition of $\mathbb{R}_+^n$ then

$$A^n = \lim \sum_q E(\phi(\Delta_q) \mid \mathbf{F}_{\underline{t}(\Delta_q)}) \tag{5.5}$$

where the limit denotes a proper sequence of refinements of the partitions (first in the t_1 direction, then in the t_2 direction etc.).

Let B^n denote the cochain $B^n(\sigma) = (d_1 d_2 \ldots d_n A^n)(\sigma)$ or

$$B^n(\sigma) = \lim \sum_q E(\phi(\Delta_q \cup \sigma) \mid \mathbf{F}_{\underline{t}(\Delta_q)}).$$

Then B^n is a positive cochain and since $A^{k+1} - A^k$ is a one parameter martingale in the $k+1$ direction

$$E(M^2(\sigma) - B^n(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) = 0$$

therefore we may set $B^n = \langle M \rangle$ which proves (5.1) for the case $r=n$. Note that as in the two parameter case, uniqueness is not assured by this argument since (5.5) may depend on the order in which the limit in (5.5) is taken.

(d) Let $M = M_{[\mathbf{i}]}$ be a Σ_2 r-martingale cochain in $\mathbb{R}^n$. Consider $[\mathbf{i}]^*$, fix $t_j = \alpha_j$ for all $j \in [\mathbf{i}]^*$, $\alpha_j \geqq 0$ and denote $S_\alpha = \{t: t_j = \alpha_j,\ j \in [\mathbf{i}]^*\}$. Then, since $M(\sigma) = M_{[\mathbf{i}]}(\sigma)$ is zero whenever the direction of σ is different from $[\mathbf{i}]$, we can map $(M_{[\mathbf{i}]}(\sigma),\ \underline{t}(\sigma) \in S_\alpha)$ on $\mathbb{R}^r_+$. Applying part (c) to $\mathbb{R}^r_+$, yields $\langle M_{[\mathbf{i}]} \rangle = \langle M_{[\mathbf{i}]} \rangle_{[\mathbf{i}]}$ which satisfies (5.1).

Consider now $M_{[\mathbf{i}]}$ and $M_{[\mathbf{j}]}$, $[\mathbf{i}] \neq [\mathbf{j}]$ and σ_1, σ_2 r-rectangles. Let k be a direction included in $[\mathbf{i}]$ but not in $[\mathbf{j}]$. Then, by F-4

$$E(M_{[\mathbf{i}]}(\sigma_1) M_{[\mathbf{j}]}(\sigma_2) \mid \mathbf{F}^k_{\underline{t}(\sigma_1)}) = 0. \tag{5.6}$$

Consequently, setting

$$\langle M \rangle = \sum_{[\mathbf{i}]} \langle M_{[\mathbf{i}]} \rangle \tag{5.7}$$

yields (5.1).

Remark. Note that for $r<n$, because of the F-4 property as defined in Sect. 2

$$E(M^2(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) = E(M^2(\sigma) \mid \mathbf{F}^{[\sigma]}_{\underline{t}(\sigma)}) \tag{5.8}$$

where $[\sigma]$ is the direction of σ. Consequently, $\langle M \rangle$ can be constructed by conditioning with respect to $\mathbf{F}^{[\sigma_{mk}]}_{\underline{t}(\sigma_{mk})}$ instead of conditioning with respect to $\mathbf{F}_{\underline{t}(\sigma_{mk})}$ where σ_{mk} are elements of the partitions.

For a pair of Σ_2 martingale r-cochains M and N, $\langle M, N \rangle$ is defined by polarization

$$\langle M, N \rangle = \tfrac{1}{4}(\langle M+N \rangle - \langle M-N \rangle).$$

Lemma 5.3. $\langle M, N \rangle = 0$ *iff:*

$$E(M(\sigma) N(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) = 0 \tag{5.9}$$

for all r-rectangles σ.

The proof follows directly from the construction of $\langle M \rangle$.

Lemma 5.4. *If M is a Σ_2 martingale r-form $(1 \leqq r \leqq n-1)$ and $d_k M$ is a martingale cochain for all k, $1 \leqq k \leqq n$ then whenever $k_1 \neq k_2$*

$$\langle d_{k_1} M, d_{k_2} M \rangle = 0 \tag{5.10}$$

and consequently

$$\langle dM \rangle = \sum_{k=1}^{n} \langle d_k M \rangle. \tag{5.11}$$

Proof. Let $\underline{t}(\sigma_1) = \underline{t}(\sigma_2)$, $[\sigma_1] \neq [\sigma_2]$ where both σ_1 and σ_2 are r-rectangles then as in (5.8)

$$E(M(\sigma_1) M(\sigma_2) \mid \mathbf{F}_{\underline{t}(\sigma_1)}) = 0. \tag{5.12}$$

Now, $d_k M(\sigma) = 0$, if $k \notin [\sigma]$; for $k \in [\sigma]$

$$d_k M(\sigma) = M(\partial_k^+(\sigma) - \partial_k^-(\sigma)).$$

It follows from (5.12), since $d_k M$ is a martingale cochain that

$$E(d_{k_1} M(\sigma)\, d_{k_2} M(\sigma) \mid \mathbf{F}_{\underline{t}(\sigma)}) = 0$$

and (5.10) follows from the previous lemma.

We conclude this section with a discussion of martingales of path independent variation and martingales of orthogonal increments [10].

Definitions. (a) *A Σ_2 martingale r-form $1 \leqq r \leqq n-1$ will be said to be of closed variation if $\langle M \rangle$ is a closed cochain i.e., $d\langle M \rangle = 0$.*

(b) *A Σ_2 martingale r-form M, $1 \leqq r \leqq n-1$ will be said to be of orthogonal increments if dM is a martingale form and*

$$\langle dM \rangle = d\langle M \rangle.$$

Lemma 5.5. *If M is of orthogonal increments and M is of the form dm where m is an $(r-1)$ martingale cochain then M is of closed variation.*

Proof. $d\langle M \rangle = \langle dM \rangle = \langle ddm \rangle = 0$.

Lemma 5.6. *If M is a Σ_2 martingale r-form $1 \leqq r \leqq n-1$ and $d_k M$ is a strong martingale Σ_2 $(r+1)$ form for $1 \leqq k \leqq n$, then M is of orthogonal increments.*

Proof. It follows directly from the definition of strong martingales given in the previous section and the proof of Proposition 5.1 that $\langle d_k M \rangle = d_k \langle M \rangle$ and the rest follows from Lemma 5.4.

6. Exterior Products I

If X and Y are ordinary r_1 and r_2 ordinary differential forms then their exterior product is well defined as an $(r_1 + r_2)$ form $X \wedge Y$. Our goal is to extend this definition to Σ_q cochains and forms. We begin by observing that if X is a Σ_1 n-cochain (which in this case $(r=n)$ is the same as being a Σ_1 n-form) and ϕ is a bounded function ϕ_t, $t \in \mathbb{R}^n_+$ then

$$(\phi \wedge X)(\sigma) = \int_\sigma \phi X$$

is an $n - \Sigma_1$ cochain defined by an ordinary Lebesque integral that can be expressed in a more convenient form as

$$\int_\sigma \phi_t X(dt).$$

Similarly if M is a Σ_2 martingale n-cochain and ϕ is predictable then

$$(\phi \wedge M)(\sigma) = \int_\sigma \phi_t M(dt)$$

is nothing but a stochastic integral as has been defined in the literature [6, 1, 9]. Therefore if ϕ_t, $t \in \mathbb{R}^n_+$ is predictable and

$$E \int_{\mathbb{R}^2_+} \phi_t^2 \langle M \rangle (dt) < \infty$$

then $\phi \wedge M$ is the unique Σ_2 martingale n-cochain such that

$$\langle \phi \wedge M, M \rangle = \phi \wedge \langle M \rangle$$

and for every rectangle σ

$$E((\phi \wedge M)(\sigma))^2 = E(\phi^2 \wedge \langle M \rangle)(\sigma).$$

For the case where X is an r-cochain we set, first

$$\phi \wedge X = \sum_{[\mathbf{i}]} \phi \wedge X_{[\mathbf{i}]}$$

and proceed to define $\phi \wedge X_{[\mathbf{i}]}$. Let σ be a rectangle with direction $[\mathbf{i}]$. By mapping the r-hyperplane which includes σ onto $\mathbb{R}^r$, we reduce the case of $\phi \wedge X_{[\mathbf{i}]}$ where X is an r-cochain in $\mathbb{R}^n_+$ to the case of integrating ϕ with respect to an r-cochain in $\mathbb{R}^r_+$. Therefore by the previous result, if ϕ_t, $t \in \mathbb{R}^n_+$ is predictable and M is a Σ_2 martingale r-form satisfying

$$E\phi^2 \wedge \langle M \rangle(\sigma) < \infty$$

for all rectangles σ, then $\phi \wedge M$ is a well defined Σ_2 martingale r-cochain such that

$$\langle \phi \wedge M, M \rangle = \phi \wedge \langle M \rangle$$

and for every rectangle σ

$$E\{(\phi^2 \wedge \langle M \rangle)(\sigma)\} = E((\phi \wedge M)(\sigma))^2.$$

Note that the result states that $\phi \wedge M$ is a Σ_2 martingale r-cochain, not r-form, and further assumptions on ϕ and M are necessary to assure that $\phi \wedge M$ is an r-form. This will be discussed in Sect. 9.

To further motivate the exterior product, first consider a Wiener process W and a Gaussian white noise η on $\mathbb{R}^2_+$. For a 0-form ϕ, $\phi \wedge \eta$ is just the stochastic integral

$$(\phi \wedge \eta)(\sigma) = \int_\sigma \phi_t \eta(dt).$$

For the 1-forms $d_1 W$ and $d_2 W$ we should have

$$d_1 W \wedge d_1 W = d_2 W \wedge d_2 W = 0$$

and

$$(d_1 W \wedge d_2 W)(\sigma) = -(d_2 W \wedge d_1 W)(\sigma) = \int_{t \vee t' \in \sigma} \eta(dt)\, \eta(dt')$$

when the last integral is a stochastic integral of second type as introduced in [6]. If X is a ordinary 1-form

$$X = \phi_t \, dt_1$$

then we should have

$$X \wedge d_1 W = 0$$

$$(X \wedge d_2 W)(\sigma) = \int_{(t \vee t') \in \sigma} \phi_t \, dt \, \eta(dt')$$

where the last integral is a mixed integral as defined in [1] and [7]. If ϕ and ψ are 0-forms then $\phi \wedge d_1 W + \psi \wedge d_2 W$ is a 1-form and for suitable paths Γ in $\mathbb{R}_+^2$

$$(\phi \wedge d_1 W + \psi \wedge d_2 W)(\Gamma) = \int_\Gamma (\phi \partial_1 W + \psi \partial_2 W)$$

where the last quantity is the path integral introduced in [1].

7. Exterior Product II

In this section we consider the exterior product $X \wedge Y$ of nonrandom r_1 and r_2 cochains, which are continuous in the sense that they are Σ_1 cochains. The case of stochastic integration will be considered in the next section.

Let X and Y denote nonrandom Σ_1 r_1 and r_2 cochains respectively and assume that $1 \leqq r_1, r_2 < n, r_1 + r_2 \leqq n$. Let

$$X = \sum_{[\mathbf{i}]} X_{[\mathbf{i}]}, \qquad Y = \sum_{[\mathbf{j}]} Y_{[\mathbf{j}]}.$$

We require $X \wedge Y$ to be an $r_1 + r_2$ form with representation

$$X \wedge Y = \sum \delta(\mathbf{i}; \mathbf{j}, \mathbf{k}) X_{[\mathbf{j}]} \wedge Y_{[\mathbf{k}]} \tag{7.1}$$

where $\delta(\mathbf{i}; \mathbf{j}, \mathbf{k}) = 1$ if $(\mathbf{j}, \mathbf{k})$ is a permutation of $[\mathbf{i}]$ and zero otherwise. In short,

$$(X \wedge Y)_{[\mathbf{i}]} = \sum_{[\mathbf{j}], [\mathbf{k}]} \delta(\mathbf{i}; \mathbf{j}, \mathbf{k}) X_{[\mathbf{j}]} \wedge Y_{[\mathbf{k}]}. \tag{7.2}$$

Therefore, in order to define $X \wedge Y$ we only need to consider $(X \wedge Y)(\sigma)$ where $X = X_{[\mathbf{j}]}$, $Y = Y_{[\mathbf{k}]}$ for some fixed $[\mathbf{j}]$ and $[\mathbf{k}]$ such that $[\mathbf{j}]$ and $[\mathbf{k}]$ are disjoint, and σ is an $r_1 + r_2$ rectangle with direction $[\mathbf{i}] = [[\mathbf{j}], [\mathbf{k}]]$. Suppose $X = X_{[\mathbf{j}]}$, $Y_{[\mathbf{k}]}$ were ordinary or flat differential forms satisfying

$$X(dt) = \alpha(t)\, dt_{j_1} \wedge dt_{j_2} \wedge \ldots \wedge dt_{j_r},$$
$$Y(dt) = \beta(t)\, dt_{k_1} \wedge \ldots \wedge dt_{k_p}.$$

Then obviously

$$(X \wedge Y)(\sigma) = \varepsilon \int_\sigma \alpha(t) \beta(t)\, dt_{j_1} \wedge \ldots \wedge dt_{j_r} \wedge dt_{k_1} \wedge \ldots \wedge dt_{k_p} \tag{7.3}$$

where $\varepsilon = +1$ or -1 according to whether $([\mathbf{j}], [\mathbf{k}])$ is an even or odd permutation of $[\mathbf{i}]$.

Let $[\sigma]$ denote the direction of the rectangle σ. Let $X_{[\mathbf{i}]}$, $Y_{[\mathbf{j}]}$ be r_1 and r_2 forms respectively. Given an $(r_1 + r_2)$ rectangle σ, if σ can be factored into the product of one rectangle of direction $[\mathbf{i}]$ and another rectangle of direction $[\mathbf{j}]$ then we denote the first factor by $\sigma^{(1)}$ and the second factor by $\sigma^{(2)}$ (so that $\sigma = \sigma^{(1)} \wedge \sigma^{(2)}$). If σ cannot be factored in the $[\mathbf{i}]$ and $[\mathbf{j}]$ directions, set $\sigma^{(1)} = \emptyset$, $\sigma^{(2)} = \emptyset$. Now, let

$$\bigcup_q \theta_{m,q} = \mathbb{R}_+^n \tag{7.4}$$

denote the dyadic partition of $\mathbb{R}_+^n$ into n-cubes of volume $(2^{-m})^n$ each. Set

$$(X \wedge Y)_m(\sigma) = \sum_q X((\sigma \cap \theta_{m,q})^{(1)})\, Y((\sigma \cap \theta_{m,q})^{(2)}). \tag{7.5}$$

Define, now

$$(X \wedge Y)(\sigma) = \lim_{m \to \infty} ((X \wedge Y)(\sigma))_m \tag{7.6}$$

provided the limit exists for all finite σ. Note that in the case where X, Y are ordinary forms, $(X \wedge Y)(\sigma)$ of (7.6) is related to (7.3) via approximating the integrands $\alpha(t)$, $\beta(t)$ by piecewise constant integrands.

Lemma 7.1. *If ϕ is piecewise constant and $X \wedge Y$ exists then*

$$\phi \wedge (X \wedge Y) = (\phi \wedge X) \wedge Y = X \wedge (\phi \wedge Y).$$

The proof follows directly from (7.5).

We conclude this section with a general condition for the existence of the exterior product. Let $X = X_{[\mathbf{i}]}$, $Y = Y_{[\mathbf{j}]}$ be as before, define the cartesian product of X and Y on $\mathbb{R}^{2n}$ as follows, for any rectangle $\tilde{\rho}$ in $\mathbb{R}^{2n}$ set $\tilde{\rho} = \tilde{\rho}^a \times \tilde{\rho}^b$ where $\tilde{\rho}^a$ is the rectangle obtained by projecting $\tilde{\rho}$ on the first n coordinates and $\tilde{\rho}^b$ is obtained by projecting $\tilde{\rho}$ on the last n coordinates. From $\tilde{\rho}^a$, $\tilde{\rho}^b$ we derive rectangles ρ_a and ρ_b in $\mathbb{R}^n$ as follows, $\rho_a(\rho_b)$ is the rectangle obtained by deleting the last (first) n-coordinates of the points of $\tilde{\rho}^a(\tilde{\rho}^b)$. Now define

$$(X \times Y)(\tilde{\rho}) = X(\rho_a)\, Y(\rho_b). \tag{7.7}$$

Having defined the "lifting" of X, Y (in $\mathbb{R}^n$) to $X \times Y$ in $\mathbb{R}^{2n}$, consider now the "contraction" of an r-cochain in $\mathbb{R}^{2n}$ into cochain in $\mathbb{R}^n$ as follows. Let σ be an r-rectangle in $\mathbb{R}^n$ defined by $(t, t+c)$ where t and c are n-vectors and only r components of c are strictly positive, the others being zero. Let $T_a(\sigma)$ denote the r-rectangle in $\mathbb{R}^{2n}$ defined by $((t,t),(t+c,t))$ where (t_α, t_β) denotes the concatenation of the n-tuples t_α and t_β. Similarly let $T_b(\sigma)$ denote the r-rectangle in $\mathbb{R}^{2n}$ defined by $((t,t),(t,t+c))$. Now, let $\sigma_{[\mathbf{j}]}$ and $\sigma_{[\mathbf{k}]}$ be r_1 and r_2 rectangles in $\mathbb{R}^n$, set

$$T(\sigma_{[\mathbf{j}]} \times \sigma_{[\mathbf{k}]}) = T_a(\sigma_{[\mathbf{j}]}) \times T_b(\sigma_{[\mathbf{k}]}). \tag{7.8}$$

If Z is an $(r_1 + r_2)$ cochain in $\mathbb{R}^{2n}$, then define $\underset{\sim}{Z}$, the "contraction" of Z, as the $(r_1 + r_2)$ cochain in $\mathbb{R}^n$ obtained by the pullback

$$\underset{\sim}{Z}(\sigma_{[\mathbf{j}]} \times \sigma_{[\mathbf{k}]}) = Z(T_a(\sigma_{[\mathbf{j}]}) \times T_b(\sigma_{[\mathbf{k}]})).$$

Proposition 7.2.

$$\tau_m = \bigcup_q T((\sigma_{[\mathbf{j}]} \times \sigma_{[\mathbf{k}]}) \cap \theta_{m,q}) \tag{7.9}$$

is a Cauchy sequence in the flat norm in $\mathbb{R}^{2n}$ and there exists a sequence of $(r_1 + r_2 + 1)$ cochains B_m in $\mathbb{R}^{2n}$ such that $(\tau_{m+1} - \tau_m) \subseteq \partial B_m$ and $\sum_{\mu=m}^{\infty} |B_\mu| \to 0$ as $m \to \infty$.

Proof. Consider an (r_1+r_2) rectangle σ in $\mathbb{R}^n$ with sides of length 2^{-m} and starting at the origin. Let $\sigma=\sigma_{[\mathbf{j}]}\times\sigma_{[\mathbf{k}]}$, then $\sigma\cap\theta_{m,0}=\sigma$. Set

$$\tau(\sigma)=T_a(\sigma_{[\mathbf{j}]})\times T_b(\sigma_{[\mathbf{k}]})$$

and

$$\mathbf{Q}(\tau(\sigma))=\bigcup_q \tau(\sigma\cap\theta_{m+1,q}).$$

We want to evaluate $|\tau(\sigma)=Q(\tau(\sigma))|^{\sim}$. It will, however, be convenient to augment $\tau(\sigma)$ as follows. Let τ be a q-rectangle in $\mathbb{R}^{2n}$, $r<2n$, let $[\underline{u}]$ be the direction of this rectangle and let α be a coordinate direction not in $[\underline{u}]$. Let τ^+ denote the $(q+1)$ rectangle generated by decreasing τ in the α direction, i.e., the shadow of τ in the α direction or:

$$\tau^+=\{(t_1,t_2,\dots,t_{\alpha-1},\lambda t_\alpha,t_{\alpha+1},\dots,t_{2n})\colon 0\leqq\lambda\leqq 1 \text{ and } (t_1,t_2,\dots,t_{2n})\in\tau\}.$$

Note that $\tau\subseteq\partial\tau^+$, comparing each rectangle part of $Q^+(\tau(\sigma))$ with a corresponding part of τ^+ yields

$$|\tau^+(\sigma)-Q^+(\tau(\sigma))|\leqq|\tau(\sigma)|\cdot 2^{-m}. \tag{7.10}$$

Therefore

$$\begin{aligned}|\tau(\sigma)-Q(\tau(\sigma))|^{\sim}&\leqq|\partial(\tau^+(\sigma))-\partial Q^+(\tau(\sigma))|^{\sim}\\&\leqq|\tau^+(\sigma)-Q^+(\tau(\sigma))|^{\sim}\\&\leqq|\tau^+(\sigma)-Q^+(\tau(\sigma))|\\&\leqq|\tau(\sigma)|\cdot 2^{-m}\end{aligned} \tag{7.11}$$

where for a k-rectangle A, $|A|$ denotes the k dimensional volume of A. The first inequality in (7.11) follows from the triangle inequality $(\tau\subseteq\partial\tau^+)$, the next two inequalities follow from the properties of the flat norm and the last inequality follows from (7.10). Consequently, since $|\tau(\sigma)|=|Q(\tau(\sigma))|$, τ_m is a Cauchy sequence in the flat norm. Setting $B_m=\tau^+_{m+1}-\tau^+_m$, it follows from (7.10) that $\sum_m^\infty B_m\to 0$ as $m\to\infty$.

Proposition 7.3. *Let X and Y be r_1 and r_2 cochains in $\mathbb{R}^n$, $r_1+r_2\leqq n$. If $X\times Y$ is continuous in the flat norm in $\mathbb{R}^{2n}$ then $X\wedge Y$ exists and is also continuous in the flat norm.*

Proof. Note that $((X\wedge Y)(\sigma))_m$ as defined by (7.5) can be written as

$$((X\wedge Y)(\sigma))_m=(X\times Y)(\tau_m) \tag{7.12}$$

where τ_m is as defined by (7.9) and the existence and continuity of the limit as $m\to\infty$ now follows directly from the assumptions $X\times Y$ is a form and from Proposition (7.2).

Remark. The construction of $X\wedge Y$ via $X\times Y$ and (7.12) can be generalized in different directions e.g.,

(a) In order to construct $X\times Y$ it is not necessary to require that $r_1+r_2\leqq n$ all that is necessary is that $r_1, r_2\leqq n$.

(b) Let ϕ be a zero form on $\mathbb{R}^{2n}$. Then we can construct the (r_1+r_2) form $Z=\phi \wedge (X \times Y)$ in $\mathbb{R}^{2n}$, and from Z we can construct a (r_1+r_2) form in $\mathbb{R}^n$ as was done for $Z=X \times Y$. This will be a natural extension of the integral of the second kind of [6].

8. Exterior Product III

Let X, Y be L_q r_1 and r_2 stochastic cochains respectively, $r_1+r_2 \leqq n$. We define $X \wedge Y$ to be the L_q limit of (7.5) provided that the limit exists for all chains. The exterior product $X \wedge Y$ thus defined is a cochain. We shall be particularly interested in the case where X and Y are martingale forms and $X \wedge Y$ is a form.

Proposition 8.1. *Let X, Y be martingale r_1 and r_2 forms respectively, $(r_1+r_2) \leqq n$, satisfying for every rectangle σ*

$$\begin{aligned} &E^{1/2}(X(\sigma))^4 \leqq K|\sigma|, \qquad E^{1/2}(dX(\sigma))^4 \leqq K|\sigma|, \\ &E^{1/2}(Y(\sigma))^4 \leqq K|\sigma|, \qquad E^{1/2}(dY(\sigma))^4 \leqq K|\sigma|. \end{aligned} \tag{8.1}$$

Then the L_2 limit of (7.5) *exists and $X \wedge Y$ is a Σ_2 martingale (r_1+r_2) form.*

Proof. Note that without loss of generality we may assume that all rectangles are included in the unit cube and $X=X_{[\mathbf{i}]}$, $Y=Y_{[\mathbf{j}]}$. We first prove the results of the proposition by an approach similar to the one given in the previous section. The existence of the L_2 limit of (7.6) will also be proved by a direct calculation.

We construct now an (r_1+r_2) cochain Z in $\mathbb{R}_+^{2n}$ as follows. Recall that $X \times Y$ was constructed by defining for rectangles

$$(X \times Y)(\tilde{\rho})=X(\rho_a)\, Y(\rho_b)$$

we want to construct Z to be as $X \times Y$, i.e.,

$$Z(\tilde{\rho})=X(\rho_a) \cdot Y(\rho_b)$$

only if $\underline{t}(\rho_a)=\underline{t}(\rho_b)$ "and zero otherwise" namely, if $\underline{t}(\rho_a) \neq \underline{t}(\rho_b)$ and $\tilde{\rho}$ does not include any rectangle $\tilde{\rho}'$ for which $\underline{t}(\rho'_a)=\underline{t}(\rho'_b)$ then set

$$Z(\tilde{\rho})=0.$$

Otherwise stated, let $\tilde{\theta}_{m,q}$ denote a dyadic partition of $\mathbb{R}^{2n}(\bigcap_q \tilde{\theta}_{m,q}=\mathbb{R}^{2n})$. Let $(\sigma_{[\mathbf{i}]}^{m;q})^a$, $(\sigma_{[\mathbf{j}]}^{m;q})^b$ denote the rectangles formed by the $[\mathbf{i}]$ and $[\mathbf{j}]$ intervals of length 2^{-m} starting at $\underline{t}(\tilde{\theta}_{m,q})$ and $(\ )_a$, $(\ )_b$ their projection on $\mathbb{R}_+^n$. Set

$$Z((\sigma_{[\mathbf{i}]}^{m;q})^a \times (\sigma_{[\mathbf{j}]}^{m;q}))=\begin{cases} X(\sigma_{[\mathbf{i}]}^{m;q})_a \cdot Y(\sigma_{[\mathbf{j}]}^{m;q})_b, & \text{if } \underline{t}(\sigma_{[\mathbf{i}]}^{m;q})_a=\underline{t}(\sigma_{[\mathbf{j}]}^{m;q})_b \\ 0, & \text{otherwise} \end{cases} \tag{8.2}$$

Z can be extended by linearity to be defined on chains in $\mathbb{R}^{2n}$. Note that for τ_m as defined by (7.9) $Z(\tau_m)=(X \times Y)(\tau_m)=((X \wedge Y)(\sigma))_m$. Consequently, in view of

Proposition 7.2, in order to prove the existence of $X \wedge Y$ it suffices to prove that Z is a Σ_2 r_1+r_2 form in $\mathbb{R}_+^{2n}$. Let $\sigma_{[\mathbf{i}]}^k$, $k=1,2,\ldots,K$ be disjoint r_1 rectangles in $\mathbb{R}^n$ and let $\sigma_{[\mathbf{j}]}^k$ be disjoint r_2 rectangles in $\mathbb{R}_+^n$. Let $\underline{t}(\sigma_a^k)=\underline{t}(\sigma_b^k)$ then

$$Z(\bigcup_k T_a(\sigma_{[\mathbf{i}]}^k)\times T_b(\sigma_{[\mathbf{j}]}^k))=\sum_{k=1}^{K} X(\sigma_{[\mathbf{i}]}^k)\, Y(\sigma_{[\mathbf{j}]}^k) \tag{8.3}$$

and

$$\begin{aligned} &\|Z(\bigcup_k T_a(\sigma_{[\mathbf{i}]}^k)\times T_b(\sigma_{[\mathbf{j}]}^k))\|_{(2)}^2 \\ &\qquad=\sum_k E\{X^2(\sigma_{[\mathbf{i}]}^k)\cdot Y^2(\sigma_{[\mathbf{j}]}^k)\} \\ &\qquad\leqq \sum_k E^{1/2}(X^4(\sigma_{[\mathbf{i}]}^k))\cdot E^{1/2}(Y^4(\sigma_{[\mathbf{j}]}^k)) \\ &\qquad\leqq K\sum_k |\sigma_{[\mathbf{i}]}^k|\cdot|\sigma_{[\mathbf{j}]}^k|. \end{aligned} \tag{8.4}$$

In particular, $\|Z(\sigma)\|_{(2)}^2 \leqq C|\sigma|$ for every rectangles σ and Z is a Σ_2 cochain in $\mathbb{R}_+^{2n}$.

Remark. If we introduce the filtration in $\mathbb{R}_+^{2n}$

$$\mathbf{G}_{(t_a,\,t_b)}=\mathbf{F}_{t_a\vee t_b} \tag{8.5}$$

then Z becomes an (r_1+r_2) martingale cochain in $\mathbb{R}_+^{2n}$.

Turning to dZ, it follows from (8.3) that dZ is the sum of products of X with dY and dX with Y. dX, dY (and dZ) need not be martingales but a direct modification of the arguments of (8.4) yields that for every rectangle τ

$$\|dZ(\tau)\|_{(2)}^2 \leqq c|\tau|$$

which proves that dZ is also a Σ_2 cochain. Therefore, by Proposition (7.2), $X \wedge Y$ is a Σ_2 form.

A direct of the existence of the limit (7.6) without using Proposition 7.2 and Z will now be given. Consider the q in $\theta_{m,q}$ defined by (7.4), this is the address of each n-cube in the m-th partition. Assume that for any given m, q is represented as an n-tuple of numbers $q=(q_1,q_2,\ldots,q_n)$. Each q_p is a binary fraction with m binary digits after the "decimal" point denoting the p-th coordinate of $\underline{t}(\theta_{m,q})$. Let $[q]_1$ denote the following modification of q: for each q_p for which $p\in[\mathbf{i}]$, if the last digit (to the right) is a one modify it into a zero. All other entries of q_p remain uncharged. Similarly, $[q]_2$ is the same modification of q for $p\in[\mathbf{j}]$. With the notation we can write I_m and I_{m+1} as follows:

$$I_{m+1}=\sum_q X((\sigma\cap\theta_{m+1,\,q})^{(1)})\cdot Y((\sigma\cap\theta_{m+q,\,q})^{(2)}), \tag{8.6}$$

$$I_m=\sum_q X((\sigma\cap\theta_{m+1,\,[q]_1})^{(1)})\cdot Y((\sigma\cap\theta_{m+1,\,[q]_2})^{(2)}). \tag{8.7}$$

Note that q_p has $(m+1)$ binary digits in both (8.6) and (8.7). Set

$$I_{m+1}=\sum_q a_q b_q, \qquad I_m=\sum_q \alpha_q \beta_q$$

where α_q, β_q, a_q, b_q are as defined by (8.6) and (8.7). We want to prove that I_m is a Cauchy sequence in L_2. Note first that for $q \neq u$

$$E\alpha_q \beta_q a_u b_u = 0$$

for the following reasons. Let

$$t^1 = \underline{t}(\sigma \cap \theta_{m+1,[q]_1})^{(1)}$$
$$t^2 = \underline{t}(\sigma \cap \theta_{m+1,[q]_2})^{(2)}$$
$$t^3 = \underline{t}(\sigma \cap \theta_{m+1,u})^{(1)}$$
$$t^4 = \underline{t}(\sigma \cap \theta_{m+1,u})^{(2)}$$

and

$$t^a = t^1 \vee t^2; \qquad t^b = t^3 \vee t^4.$$

Then $t^a = \underline{t}(\sigma \cap \theta_{m+1,q})$ and $t^a \neq t^b$. Let p be a direction such that $p \in [\sigma]$ and such that $(t^a)_p < (t^b)_p$ then

$$E(\alpha_q \beta_q a_u b_u) = E\{a_u b_u E(\alpha_q \beta_q \mid \mathbf{F}^p_{t^a})\} = 0.$$

If such a direction does not exist, let p be a direction for which $(t^a)_p > (t^b)_p$. Then

$$E(\alpha_q \beta_q a_u b_u) = E\{\alpha_q \beta_q E(a_u b_u \mid \mathbf{F}^p_{t^b})\} = 0.$$

Therefore,

$$\begin{aligned} E(I_{m+1} - I_m)^2 &= E \sum_q (a_q b_q - \alpha_q \beta_q)^2 \\ &= E \sum_q (a_q b_q - \alpha_q b_q + \alpha_q b_a - \alpha_q \beta_q)^2 \\ &\leqq 2E \sum_q b_q^2 (a_q - \alpha_q)^2 + 2E \sum_q \alpha_q^2 (b_q - \beta_q)^2. \end{aligned} \tag{8.8}$$

Consider now a term in the first sum of (8.8)

$$\begin{aligned} E(b_q^2 (a_q - \alpha_q)^2) = E\{(Y((\sigma \cap \theta_{m+1,q})^{(2)}))^2 \cdot [X((\sigma \cap \theta_{m+1,q})^{(1)}) \\ - X((\sigma \cap \theta_{m+1,[q]_1})^{(1)}]^2. \end{aligned} \tag{8.9}$$

Let k be a direction in $(\sigma \cap \theta_{m+1,q})^{(2)}]$ and let $\underline{t}$ denote $\underline{t}((\sigma \cap \theta_{m+1,q})^{(2)})$, then

$$E b_q^2 (a_q - \alpha_q)^2 \leqq E^{1/2} b_q^4 E^{1/2} (a_q - \alpha_q)^4. \tag{8.10}$$

Therefore, by the assumptions and Lemma 7.2

$$E b_q^2 (a_q - \alpha_q)^2 \leqq K^2 r_1 2^{-(m+1)(r_1 + r_2 + 1)}.$$

Similarly, for a term in the second sum of (8.8) we have by similar arguments

$$E \alpha_q^2 (b_q - \beta_q)^2 \leqq r_2 K^2 2^{-(m+1)(r_1 + r_2 + 1)}.$$

Substituting (8.10), (8.11) into (8.8) yields

$$E^{1/2} (I_{m+1} - I_m)^2 \leqq K_1 2^{-(m+1)/2}.$$

Consequently I_m is a Cauchy sequence.

From the proof of Proposition 8.1 it also follows that:

Proposition 8.2. *Under the assumptions of Proposition* 8.1, $\langle X\rangle \wedge \langle Y\rangle$ *exists and*

$$\langle X \wedge Y\rangle = \langle X\rangle \wedge \langle Y\rangle. \tag{8.12}$$

Proof. If M_m, $m=1,2,\ldots$ is a sequence of Σ_2 r-martingale cochains and $M_m \to M$ in L_2 then M is an L_2 martingale cochain and $\langle M_m\rangle(\sigma) \to \langle M\rangle(\sigma)$ for every rectangle σ. Applying this to $(X \wedge Y)_m$ of (7.5) yields (8.12).

In Sect. 3 of [8], it is shown that stochastic integration in the plane provides a class of measure transformations that preserves the Markovian property for a Gaussian process. The results on exterior product up to this point extend this result of [8] to parameter space of any finite dimension.

9. A Differentiation Formula

In the non random case it is well known that

$$d(X \wedge Y) = dX \wedge Y + (-1)^{r_1} X \wedge dY.$$

This will, in general, not be true for the stochastic case. Let X and Y be respectively r_1 and r_2 Σ_2 martingale forms. Assume that $dX \wedge Y$ and $X \wedge dY$ are well defined, and further assume that dX and dY are also Σ_2 martingale forms. Define the (r_1+r_2+1) form $[X, Y]$ by

$$d(X \wedge Y) = dX \wedge Y + (-1)^{r_1} X \wedge dY + [X, Y] \tag{9.1}$$

we will call $[X, Y]$ the cross variation between X and Y. (Incidentally, we have not defined the exterior product between two zero cochains ϕ and ψ but if we set $\phi \wedge \psi = \phi\psi$ then (9.1) reduces to the Ito formula). Some simple properties of $[X, Y]$ can be derived directly from (9.1) as follows. Because $X \wedge Y = (-1)^{r_1 r_2}(Y \wedge X)$, we have

$$[Y, X] = (-1)^{r_1 r_2}[X, Y] \tag{9.2}$$

and it also follows that for r_1 odd $X \wedge X = 0$. Hence, for r_1 odd

$$\begin{aligned} d(X \wedge X) &= dX \wedge X - X \wedge dX + [X, X] \\ &= 0 + [X, X] = 0 \end{aligned}$$

and $[X, X] = 0$ for r_1 odd. For r_1 even we have

$$d(X \wedge X) = 2X \wedge dX + [X, X].$$

Finally, note that $dd(X \wedge Y) = 0$ whence it follows from 9.1 that

$$d[X, Y] + [dX, Y] + (-1)^{r_1}[X, dY] = 0$$

for $X = Y$ and r_1 even it reads

$$d[X, X] + 2[X, dX] = 0.$$

We conjecture now that if either $d_k X$ (or $d_k Y$) is a strong martingale form for every k then $[X, Y]=0$. The heuristic arguments for this are as follows:

$$\begin{aligned} d(X \wedge Y)(\tau) &= (X \wedge Y)(\partial \tau) \\ &= \lim (X \wedge Y)_m(\partial \tau) \end{aligned}$$

where $(X \wedge Y)_m$ is as defined by (7.5). Therefore

$$(d_k(X \wedge Y))(\tau) = \lim (X \wedge Y)_m(\partial^+ - \partial^-)$$

where τ is an (r_1+r_2+1) rectangle $\tau=\sigma^- \times \tau_k$ and τ_k is an interval in the k direction, σ^- is therefore the lower face of τ in the k direction and σ^+ is the upper face of τ is the k direction. Now, $(X \wedge Y)_m$ is of the form of a sum of products $X(\sigma_1^{\cdot}) \cdot Y(\sigma_2^{\cdot})$ and therefore $d(X \wedge Y)_m$ will be of the form of sums of terms of the following type

$$\begin{aligned} & X(\sigma_1^+)\, Y(\sigma_2^+) - X(\sigma_1^-)\, Y(\sigma_2^-) \\ &\quad = X(\sigma_1^+)(Y(\sigma_2^+) - Y(\sigma_2^-)) + (X(\sigma_1^+) - X(\sigma_1^-))\, Y(\sigma_2^-) \\ &\quad = X(\sigma_1^-)(Y(\sigma_2^+) - Y(\sigma_2^-)) + (X(\sigma_1^+) - X(\sigma_1^-))\, Y(\sigma_2^-) \\ &\quad\quad + (X(\sigma_1^+) - X(\sigma_1^-))(Y(\sigma_2^+) - Y(\sigma_2^-)) \end{aligned}$$

with $\underline{t}(\sigma_1)=\underline{t}(\sigma_2)$ and $(X \wedge Y)_m$ will be the sum of the three types of terms of the last equation. The sum of the terms of the first type will yield $(-1)^{r_1}(X \wedge dY)$ as $m \to \infty$, the sum of the terms of the second type will yield $(dX \wedge Y)$ as $m \to \infty$ and the sum of the terms of the last type will yield $[X, Y]$ as $m \to \infty$. The $[X, Y]$ term is therefore very similar to the cross quadratic variation of continuous one-parameter martingales. In the one parameter case

$$\lim \sum_i (M_{t_{i+1}} - M_{t_i})^2 = \lim \sum_i E((M_{t_{i+1}} - M_{t_i})^2 \mid \mathbf{F}_{t_i})$$

and what we conjecture is that the same is true in the present case; namely, we assume that we may replace the terms limit of terms of the form

$$(X(\sigma_1^+) - X(\sigma_1^-))(Y(\sigma_2^+) - Y(\sigma_2^-))$$

by the limit of terms of the form

$$E\{(X(\sigma_1^+) - X(\sigma_1^-))(Y(\sigma_2^+) - Y(\sigma_2^-)) \mid \mathbf{F}^k_{\underline{t}(\sigma_1^-)}\}$$

which vanishes if either $d_k X$ or $d_k Y$ is a strong martingale.

As an application of (9.1) consider the case where ϕ is a Σ_2 martingale zero form and M is a Σ_2 martingale one form. Note that our definition of a strong martingale implies that every martingale 1-form is strong therefore $[\phi, M]=0$ and

$$d(\phi \wedge M) = d\phi \wedge M + \phi \wedge dM$$

which for $\mathbb{R}_+^2$ is the Green formula of Cairoli and Walsh [1].

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