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A LINEAR SEARCH PROBLEM*

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SUMMARY

A class of one-dimensional search problems is considered. In general, the formulation results in a functional-minimization equation of the dynamic programming type. In a special case the optimal solution for both the objective and policy have been found.

INTRODUCTION

WE CONSIDER a storage unit consisting of N cells, with information stored in tabular form. That is, the record $r(i)$ stored in cell i is the form of argument-function $[x_i, f(x_i)]$, the file being arranged in ascending order of the argument x_i. An example of such an arrangement is a dictionary.

Give a particular argument x, we find $f(x)$ by searching for the cell containing $[x, f(x)]$. The search proceeds by comparing x against the arguments in a sequence of cells i_1, i_2, etc. This sequence is to be chosen so as to minimize the average number of comparisons required for locating the correct cell. Problems of this kind occur in addressing computer storage [1].

II. EQUATION OF OPTIMIZATION

We begin with the following assumptions.

(1) In a comparison of x against x_i, only three possible outcomes exist, namely,

$$x > x_i, \qquad x < x_i, \qquad x = x_i .$$

(2) Let ξ be an integer-valued random variable denoting the location of x. We assume that the *à priori* probabilities $p_k = \text{Prob}\,[\xi = k]$ are given, with

$$\sum_{k=1}^{N} p_k = 1 \tag{1}$$

(3) Let S be the set of integers 1 through N, and let σ be a non-empty subset of S. We assume that the *à posteriori* probability distribution of ξ is unchanged except for renormalization; i.e.,

$$\begin{aligned} \text{Prob}\,[\xi = k \mid \xi \in \sigma] &= \frac{p_k}{P(\sigma)}, k \in \sigma \\ &= 0, \mathrm{k} \notin \sigma \end{aligned} \tag{2}$$

where $P(\sigma) = \sum_{i \epsilon \sigma} p_i$.

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Let $T[(p_k), N]$ formally denote the *minimum* average number of comparisons per successful search, given N cells and *à priori* distribution (p_k). It is clear that the search procedure starts with the selection of a cell for the first comparison. Suppose cell n is selected and x is compared with x_n. The following situation then results:

(1) With probability p_n, $x = x_n$ and the search terminates.

(2) With probability $P_{n-1} = \sum_{i=1}^{n-1} p_i$, $x < x_n$ and x must be contained in the first $n - 1$ cells. If we renumber the first $n - 1$ cells backwards starting with cell $n - 1$, the new distribution becomes

$$p_k' = \frac{p_{n-k}}{P_{n-1}}, \qquad k = 1, \cdots, n - 1. \tag{3}$$

(3) With probability $1 - P_n = \sum_{i=n+1}^{n} p_i$, $x > x_n$. Upon renumbering the last $N - n$ cells, we find the new distribution to be

$$p_k'' = \frac{p_{n+k}}{1 - P_n}, \qquad k = 1, \cdots, N - n. \tag{4}$$

It is clear that whichever cell is optimal for the first choice, succeeding choices choices must remain optimal for the overall sequence to be optimal. Therefore, $T[., N]$ must satisfy the following functional equation:

$$\begin{aligned} T[(p_k), N] = \min_{1 \leqq n \leqq N} \Bigg\{ & 1 + P_{n-1} T\left[\left(\frac{p_{n-k}}{P_{n-1}}\right), n - 1\right] \\ & + (1 - P_{n-1}) T\left[\left(\frac{p_{n+k}}{1 - P_n}\right), N - n\right] \Bigg\}. \end{aligned} \tag{5}$$

Equation (5) is in the formalism of dynamic programming, [2], yielding as solutions the objective $T[(p_k), N]$ and the optimal policy $n^*[(p_k), N]$. As initial conditions we set $P_0 = O$, $T(., 0) = 0$, and $T(., 1) = 0$.[1]

III. OPTIMAL SOLUTIONS FOR UNIFORM DISTRIBUTION

If $p_k = \dfrac{1}{N}$, $k = 1, 2, \cdots, N$, explicit solution of (5) can be found. In this case it is clear that $T(., .)$ and $n^*(., .)$ are functions of N only. With a slight change in notation we can rewrite (5) as

$$T(N) = 1 + \min_{1 \leqq n \leqq N} \left\{ \frac{n - 1}{N} T(n - 1) + \left(1 - \frac{n}{N}\right) T(N - n) \right\}. \tag{6}$$

The solution $T(N)$ of (6) is given by

$$\begin{aligned} &(2^{k+1} + 2m - 1) T(2^{k+1} + 2m - 1) \\ &\qquad = 2^{k+1}(k - \tfrac{1}{2}) + 2mk + 3m + 1, \quad k = 0, 1, 2, \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad m = 0, 1, \cdots, 2^k \\ &(2^{k+1} + 2m) T(2^{k+1} + 2m) \\ &\qquad = 2^{k+1}(k - \tfrac{1}{2}) + (2m + 1)k + 3(m + 1), \quad m = 0, 1, \cdots, 2^k - 1. \end{aligned} \tag{7}$$

[1] Note that this last condition implies that if there is only one cell no comparison is necessary. This is a consequence of (1).

There is not a unique policy $n^*(N)$ which yields the minimum. In fact, the multiplicity of solutions can be quite large. The complete set of solutions is

$$\begin{aligned} n^*(2^{k+1} + 2m) &= 2^k + j, && j = 0, 1, \cdots, 2m + 1, && m < 2^{k-1} \\ & && j = 2m - 2^k + 1, \cdots, 2^k, && m \geqq 1^{k-1} \\ n^*(2^{k+1} + 2m - 1) &= 2^k + 2j, && j = 0, 1, \cdots, m, && m \leqq 2^{k-1} \\ & && j = m - 2^{k-1}, \cdots, 2^{k-1}, && m > 2^{k-1} \end{aligned} \tag{8}$$

For example, consider $N = 2^4 + 9 = 25$

$$n^*(N) = 10, 12, 14, 16.$$

The policy solution is interesting and somewhat surprising. Intuitively, one would expect that the optimal solution $n^*(N)$ should be such as to divide the remaining $N - 1$ cells into nearly equal subsets, i.e., $N - n^* \cong n^* - 1$. Thus, the large multiplicity of solution is not expected. Furthermore, in some cases the midpoint is in fact not a solution. For example, for $N = 25$, the point $n = 13$ divides the remaining 24 cells equally, but is not among the solutions.

IV. PROOF OF OPTIMALITY

In this section we shall prove that the solutions of (6) are indeed given by (7) and (8). The proof proceeds in three stages. First, it is shown that the right-hand side of (6) is minimized by a specific choice of policy $n^*(N)$. Next, $T(N)$ will be derived. Finally, the multiplicity of the policy solution is found.

A. If we let $f(N) = NT(N)$, (6) is simplified and can be rewritten as

$$f(N) = N + \min_{1 \leqq n \leqq N} \{f(n - 1) + f(N - n)\}. \tag{9}$$

We begin by proving the following theorem:

THEOREM 1: *Under the conditions* $f(0) = f(1) = 0$, *the minimization in* (9) *is achieved with* $n = n^*(N)$, *where for all positive integers* m,

$$n^*(4m - 2) = n^*(4m - 1) = n^*(4m) = n^*(4m + 1) = 2m. \tag{10}$$

Proof: It seen that Theorem 1 is equivalent to the following set of equations with m ranging over all positive integers:

$$f(4m - 2) = 4m - 2 + f(2m - 2) + f(2m - 1) \tag{11a}$$

$$f(4m - 1) = 4m - 1 + f(2m - 1) + f(2m - 1) \tag{11b}$$

$$f(4m) = 4m + f(2m - 1) + f(2m) \tag{11c}$$

$$f(4m + 1) = 4m + 1 + f(2m - 1) + f(2m + 1). \tag{11d}$$

We proceed by induction. First, by enumerating all possibilities, we find that (11) is true for $m = 1$. Next, we assume (11) to be true for $m = 1, \cdots, K$ and prove the following lemma:

LEMMA: *Equation* (11) *being valid for* $m = 1, 2, \cdots, K$, *implies*

(12) $f(n+1) - f(n) \geqq f(n-1) - f(n-2), \qquad n = 2, \cdots, 4K$

(13) $f(n+1) > f(n), \qquad n = 1, 2, \cdots, 4K$

(14) $f(2n) - f(2n-1) > f(2n+1) - f(2n), \qquad n = 1, 2, \cdots, 2K.$

Proof: If (11) is true for $m = 1, \cdots, K$, then

(15)
$$\begin{aligned}&[f(4m+1) - f(4m)] - [f(4m-1) - f(4m-2)]\\ &\quad = [f(2m+1) + f(2m-2)] - [f(2m) + f(2m-1)], \quad m = 1, 2, \cdots, K.\end{aligned}$$

Now under the same assumption, [compare (11c) and (9)],

(16) $$f(2m) + f(2m-1) = \min_{1 \leqq n \leqq 4m} \{f(n-1) + f(4m-n)\}, \quad 1 \leqq m \leqq K.$$

Therefore, it follows that

(17) $$f(2m+1) + f(2m-2) \geqq f(2m) + f(2m-1), \qquad 1 \leqq m \leqq K,$$

and

(18) $$f(4m+1) - f(4m) \geqq f(4m-1) - f(4m-2), \qquad 1 \leqq m \leqq K.$$

Similarly, we find that

(19) $$f(4m-1) - f(4m-2) = f(4m-3) - f(4m-4)$$

(20) $$f(4m) - f(4m-1) \geqq f(4m-2) - f(4m-3)$$

(21) $$f(4m-2) - f(4m-3) = f(4m-4) - f(4m-5), \qquad m \leqq K.$$

Relationships (18) through (21) imply (12). Further, since $f(2) - f(1) > 0$ and $f(3) - f(2) > 0$, inequality (12) implies that $f(n+1) - f(n) > 0$, which is inequality (13).

To prove (14), we note that if (11) is valid for $m = 1, \cdots, K$, then

$$f(4m-2) - f(4m-3) = 1 + f(2m-2) - f(2m-3)$$

$$f(4m-1) - f(4m-2) = 1 + f(2m-1) - f(2m-2)$$

$$f(4m) - f(4m-1) = 1 + f(2m) - f(2m-1)$$

$$f(4m+1) - f(4m) = 1 + f(2m+1) - f(2m).$$

It is clear that $f(2m) - f(2m-1) > f(2m+1) - f(2m)$ for $m = 1, 2, \cdots, K$ implies

$$f(4m-2) - f(4m-3) > f(4m-1) - f(4m-2),$$

and

$$f(4m) - f(4m-1) > f(4m+1) - f(4m), \qquad m = 1, 2, \cdots, K.$$

Therefore, inequality (14) follows from $f(2) - f(1) > f(3) - f(2)$. This latter is easily verified.

Proceeding with the main part of the proof for Theorem 1, we write $f(4K+2)$ as

$$f(4K+2) = 4K+2+\min_{2\leqq n\leqq 4K+1}[f(n-1)+f(4K+2-n)]$$

$$(22)\qquad = 2K+2+\min\Big\{\min_{1\leqq n\leqq K}[f(2n-1)+f(4k+2-2n)],$$

$$\min_{1\leqq n\leqq K}[f(2n)+f(4K+1-2n)]\Big\}.$$

Using (12), we find that

$$f(n+1)+f(n-2)\geqq f(n)+f(n-1),\quad n=2,\cdots,4K.$$

Thus,

$$f(2K+2)+f(2K-1)\leqq f(2K+3)+f(2K-2)\leqq\cdots f(4K)+f(1),$$

and

$$f(2K+2)+f(2K-1)=\min_{1\leqq n\leqq K}[f(2n-1)+f(4K+2-2n)].$$

Similarly, we find that

$$f(2K)+f(2K+1)=\min_{1\leqq n\leqq K}[f(2n)+f(4K+1-2n)].$$

Therefore, (22) is reduced to

$$f(4K+2)=4K+2+\min\{[f(2K+2)+f(2K-1)],[f(2K)+f(2K+1)]\}$$
$$=4K+2+f(2K)+f(2K+1),$$

where the last step follows again from (12). We note that we have extended (11a) to $m = K + 1$, and (12) to $n = 4K + 1$.

Using (12) and (13) and arguments similarly to those in reducing (22), we can write $f(4K + 3)$ as

$$(23)\qquad f(4K+3)=4K+3+\min\{2f(2K+1),f(2K)+f(2K+2)\}$$

It follows from (14) that

$$f(2K+2)-f(2K+1)\geqq f(2K+3)-f(2K+2),$$

and it follows from (12) that

$$f(2K+3)-f(2K+2)>f(2K+1)-f(2K).$$

Therefore,

$$f(2K+2)+f(2K)>2f(2K+1),$$

and from (23)

$$(24)\qquad f(4K+3)=4K+3+2f(2K+1).$$

Following a procedure nearly identical to the above, we can show that

$$f(4K + 4) = 4K + 4 + f(2K + 1) + f(2K + 2), \tag{25}$$

and

$$f(4K + 5) = 4K + 5 + f(2K + 1) + f(2K + 3). \tag{26}$$

By induction, Theorem 1 follows.

B. The functional form of $f(N)$ is given by the following theorem:

THEOREM 2. *Equation* (9) *is satisfied if and only if*

$$f(2^{k+1} + 2m - 1) = 2^{k+1}(k - \tfrac{1}{2}) + 2mk + 3m + 1, \quad k = 0, 1 \cdots, \quad m = 0, 1, \cdots, 2^k, \tag{27a}$$

$$f(2^{k+1} + 2m) = 2^{k+1}(k - \tfrac{1}{2}) + (2m + 1)k + 3m + 3, \quad k = 0, 1, \cdots \quad m = 0, 1, \cdots, 2^{k-1}. \tag{27b}$$

Proof: The "only if" part follows simply from the fact that no two functions can both be the minimum without being equal. To prove (27), we again use induction. That is, we verify (27) for $k = 0$ and assume it to be valid for $k = 0, 1, \cdots, K - 1$. If it follows thereby that (27) is valid for $k = K$, then (27) must be true for all k. The detailed proof involves substitution of (27) in (11) and elementary manipulation, and will be omitted here.

C. Theorem 1 is strengthened by the following result:

$$f[n^*(N) - 1] + f[N - n^*(N)] = \min_{1 \leqq n \leqq n} \{f(n) + f(N - n)\} \tag{28}$$

if and only if

$$\begin{aligned} n^*(2^{k+1} + 2m) &= 2^k + j, \quad j = 0, 1, \cdots, 2m + 1, \quad 0 \leqq m < 2^{k-1} \\ & \qquad j = 2m - 2^k + 1, \cdots, 2^k, \quad 2^{k-1} \leqq m < 2^k - 1, \end{aligned} \tag{29a}$$

$$\begin{aligned} n^*(2^{k+1} + 2\mathrm{m} - 1) &= 2^k + 2j, \quad j = 0, 1, \cdots, m, \quad 0 \leqq m \leqq 2^{k-1} \\ & \qquad j = m - 2^{k-1}, \cdots, 2^{2k-1}, 2^{k-1} \leqq m \leqq 2^k. \end{aligned} \tag{29b}$$

Proof: The "if" part is proved by substituting (29) and (27) in (28) and verifying. In the process it is also shown that for $2^{k+1} \leqq N \leqq 2^{k+2} - 1$, the only solution in the range $2^k \leqq n^* \leqq 2^{k+1}$ are those given by (29). Thus, it remains only to show that no value of n^* greater than 2^{k+1} or less than 2^k is a solution.

Consider $N = 2^{k+1} + 2m$, $0 \leqq m < 2^{k-1}$. Since we know that $n^* = 2^k$ is a solution, we need only to show that (similiar results follow for $n^* > 2^{k+1}$ by symmetry)

$$\begin{aligned} f(2^k - 1) + f(2^k + 2m) &< f(2^k - 2) + f(2^k + 2m + 1) \\ &\leqq f(2^k - 3) + f(2^k + 2m + 2) \leqq \cdots . \end{aligned} \tag{30}$$

The first of the inequalities in (30) is easily verified using (27). The remaining inequalities follow from (12). For $2^{k-1} \leqq m \leqq 2^k$, we use $n^* = 2^{k+1}$, and from

(27) and (12) show that

$$f(2^{k+1} - 1) + f(2m) < f(2^{k+1}) + f(2m - 1) \leqq f(2^{k+1} + 1) + f(2m - 2) \leqq \cdots . \tag{31}$$

Inequalities (30) and (31) suffice to prove (29a).

For $N = 2^{k+1} + 2m - 1$, the proof of (29b) follows nearly identical lines and will not be reproduced here.

REFERENCES

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2. R. BELLMAN, *Dynamic Programming*, Princeton University Press, Princeton, N. J., 1957.