A characterization of the kernels associated with the multiple integral representation of some functionals of the Wiener process

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In this paper we present a characterization of those Wiener functionals that are the likelihood ratio for a 'signal plus independent noise' model. The characterization is expressed in terms of the representation of such functionals in a series of multiple Wiener integrals.

Keywords: Wiener functionals, Multiple Wiener integrals, Likelihood ratio, Additive noise model, Radon–Nikodym derivative.

Let \( \{y_s, 0 \leq s < T\} \) be a random process with measurable sample functions satisfying \( |y_s| < K \) a.s. Let \( \{W_s, 0 \leq s < T\} \) be a Wiener process which is independent of the \( \{y_s, 0 \leq s < T\} \) process. Let

\[
A_n = \exp \left\{ \int_0^t y_s \, dW_s - \frac{1}{2} \int_0^t y_s^2 \, ds \right\}.
\]

Then \( A_n \) is the unique solution to the integral equation

\[
A_n = 1 + \int_0^t A_n y_s \, dW_s
\]

and admits the series representation

\[
A_n = \lim_{N \to \infty} \sum_{n=0}^N u_n(t)
\]

where \( u_0(t) = 1 \) and \( u_n(t) = \int_0^t u_{n-1}(s) y_s \, dW_s \).

Let \( \mathcal{F}_t^W \) denote the \( \sigma \)-field generated by \( \{W_s, 0 \leq \theta < t\} \). Since convergence in quadratic mean commutes with conditional expectation, we have (as was observed in [1] and [3])

\[
E(A_n | \mathcal{F}_t^W) = \sum_{n=0}^\infty E(u_n(t) | \mathcal{F}_t^W)
\]

\[
= \sum_{n=0}^\infty \int_0^t \left( \int_0^t \cdots \int_0^t v_n(t_1, \ldots, t_n) \, dW_{t_1} \cdots dW_{t_{n-1}} \right) dW_t
\]

(1)

where

\[
v_n(t_1, \ldots, t_n) = E(y(t_1) y(t_2) \cdots y(t_n))
\]

(2)

and the integrals are iterated Ito integrals. The functional \( g(W) = E(A_n | \mathcal{F}_t^W) \) is a nonnegative functional of the Brownian motion, its Wiener–Ito representation is given by (1) with the kernels \( v_n(\cdots) \) satisfying (2): that is, the \( n \)-th order kernel \( v_n(t_1, \ldots, t_n) \) is the \( n \)-th order moment of a process which is independent of
In this note we consider the converse problem: Let \( g(W) \) be a square integrable functional of the Wiener process \( W \) with the Wiener–Itô representation

\[
g(W) = C + \sum_{n} \int_0^T \left( \int_0^{t_n} \cdots \int_0^{t_2} h_n(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n} \right) dW_{t_n}
\]

where the integrals are iterated Itô integrals. This representation will be abbreviated by

\[
g(W) = C + \sum_{n} h_n \circ W^n \tag{3}'
\]

with \( h_n \circ W^n \) denoting the \( n \)-th order iterated stochastic integral. The problem that we consider is the following: given a square integrable functional of the Brownian motion with representation (3) or (3)', what conditions would ensure the existence of a process \( \{ y(t), 0 \leq t \leq T \} \), independent of \( W \), such that

\[
\Delta h_n(t_1, \ldots, t_n) = E(y(t_1)y(t_2) \cdots y(t_n)) \tag{4}
\]

for all \( n \) and all \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T \)? Another way to state the problem is the following: the functional (1) is the likelihood ratio of a 'signal plus independent noise' with respect to the 'noise only' hypothesis and the problem is to characterize the nonnegative functionals of the Brownian motion that represent the likelihood ratio of a 'signal plus independent noise' with respect to the 'noise only' hypothesis.

Let \( g(\lambda, W) \) denote

\[
g(\lambda, W) = C + \sum_{n} \lambda^n h_n \circ W^n. \tag{5}
\]

It will be shown that \( g(W) \) has the 'signal plus independent noise' representation, i.e. \( h_n \) satisfy (4) if and only if \( g(\lambda, W) \) as defined by (5) is a nonnegative square integrable random variable for every nonnegative \( \lambda \).

**Notation.** For the representation of \( g(W) \), \( h_n(t_1, \ldots, t_n) \) has to be defined for ordered \( n \)-tuples only (cf. (3)), namely for \( t_1, \ldots, t_n \) satisfying \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T \). Define \( h_n(t_1, \ldots, t_n) \) for unordered \( n \)-tuples of distinct times by

\[
h_n(t_1, \ldots, t_n) = h_n(t_{\pi(1)}, \ldots, t_{\pi(n)}) \tag{6}
\]

where \( (t_{\pi(1)}, \ldots, t_{\pi(n)}) \) is the rearrangement of \( (t_1, \ldots, t_n) \) which yields an increasing sequence. We will not distinguish between two kernels \( h_n(t_1, \ldots, t_n) \) and \( h_n'(t_1, \ldots, t_n) \) which are equal almost everywhere (Lebesgue) on \([0, T]^n\). Note that (6) leaves \( h_n(t_1, \ldots, t_n) \) undefined on a Lebesgue set of measure zero in \([0, T]^n\).

Let \( \Pi^{(n,m)} \) denote a multinomial of degree \( m \) in \( n \) variables \( x_1, \ldots, x_n \),

\[
\Pi^{(n,m)} = \sum_{|p| \leq m} C^{p_1 \cdots p_n} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}
\]

where \( p_i, i = 1, \ldots, n \), are integers and \( |p| = \sum_i p_i \). A multinomial \( \Pi^{(n,m)} \) is said to be nonnegative if it is nonnegative for all values of its arguments \( x_1, \ldots, x_n \). Let \( W \) be a Wiener process and \( \tau^n \) an \( n \)-tuple of real numbers satisfying \( 0 \leq t_i \leq T, i = 1, \ldots, n \). \( \Pi^{(n,m)}(W, \tau^n) \) will denote the multinomial \( \Pi^{(n,m)} \) evaluated at \( x_i = W(t_i), i = 1, \ldots, n \),

\[
\Pi^{(n,m)}(W, \tau^n) = \sum_{|p| \leq m} C^{p_1 \cdots p_n} (W(t_1))^{p_1} \cdots (W(t_n))^{p_n}
\]

where \( p_i, i = 1, \ldots, n \), are integers and \( |p| = \sum_i p_i \).

\( H_n(\tau^n) \) is defined as

\[
H_n(\tau^n) = H_n(t_1, \ldots, t_n) = \int_0^{t_1} \cdots \int_0^{t_n} h_n(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n. \tag{7}
\]

Finally, \( \hat{\Pi}^{(n,m)}(H, \tau^n) \) is defined as

\[
\hat{\Pi}^{(n,m)}(H, \tau^n) = \sum_{|p| \leq m} C^{p_1 \cdots p_n} H_{p_1}(t_1, t_1, \ldots, t_1, t_2, \ldots, t_n) \tag{8}
\]

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Theorem. Let \( g(W) = C + \sum h_n \Box W^n \) be the Wiener–Itô representation of a square integrable functional of the Wiener process over \([0, T]\). Assume that \( |h_n(t^n)| \leq K \) for all \( n \) and all \( n \)-tuples \( t^n \). Then the following are equivalent:

(a) \( g(\lambda, W) \) is a nonnegative square integrable random variable for every positive real \( \lambda \).

(b) There exists a sequence of positive real numbers \( \lambda_n \), such that \( \lambda_n \rightarrow \infty \) as \( r \rightarrow \infty \) and \( g(W) \) and \( g(\lambda_n, W) \) are nonnegative random variables.

(c) For every nonnegative multinomial \( \Pi^{(n,m)} \) it holds that \( \Pi^{(n,m)}(H, t^n) \) is nonnegative.

(d) There exists a random process \( \{y_s, 0 \leq s \leq T\} \) such that \( |y_s| \leq K \) and

\[
E\left( y(t_1) y(t_2) \cdots y(t_n) \right) = h_n(t_1, \ldots, t_n)
\]

for almost all (Lebesgue) points \( (t_1, \ldots, t_n) \) in \([0, T]^n\) (the probability space on which the \( y \) process is defined is unrelated to the probability space on which the Wiener process \( W \) is defined).

(e) Let \( P_W^X \) denote the probability measure on the space of continuous functions induced by \( W \). Let \( \{z_s, 0 \leq s \leq T\} \) be a random process with measurable sample paths on \([0, T]\). Let \( P_X^{Z+W} \) denote the probability measure induced by \( X_s = \int_0^s z_r \, ds + W_r \) on the space of continuous functions. There exists a process \( \{z_s, 0 \leq s \leq T\} \) such that \( |z_s| \leq K \) a.s. and \( \{z_s, 0 \leq s \leq T\} \) is independent of the Wiener process \( W \) such that \( P_X^{Z+W} \) is equivalent to \( P_W^X \) and the Radon–Nikodym derivative of \( P_X^{Z+W} \) with respect to \( P_W^X \) satisfies

\[
\frac{dP_X^{Z+W}}{dP_W^X}(W) = \frac{1}{C} g(W).
\]

Proof. Obviously (a) implies (b). We turn now to the proof that (b) implies (c). Recall first that for a deterministic square integrable function \( \phi(\cdot) \) we have (cf. equation (3.4) of [2]):

\[
\left( \int_0^T \phi(\theta) \, dW_\theta \right) \left( \int_0^T \cdots \int_0^T h_n(\theta_1, \ldots, \theta_n) \, dW_{\theta_1} \cdots dW_{\theta_n} \right)
\]

\[
= \int_0^T \cdots \int_0^T \phi(\theta_1) h_n(\theta_2, \ldots, \theta_{n+1}) \, dW_{\theta_1} \cdots dW_{\theta_{n+1}} + \sum_{q=0}^{n-1} \psi_q \Box W^q.
\]

(8)

The exact form of \( \psi \) will not interest us. It follows by repeated applications of this result that

\[
W(t_1) W(t_2) \cdots W(t_n) = \int_0^T \cdots \int_0^T \int_0^T \int_0^T \cdots \int_0^T h_n(\theta_1, \ldots, \theta_n) \, dW_{\theta_1} \cdots dW_{\theta_n} + \sum_{q=0}^{n-2} \psi_q \Box W^q
\]

for some deterministic \( \psi_q, 0 \leq q \leq n-2 \) where \( \chi_q(\theta) \) denotes the characteristic function (\( \chi_q(\theta) = 1 \) for \( \theta \leq t \) and zero otherwise). Note that in equation (8) and in the last equation the integrals over \([0, T]^n\) are multiple Wiener–Itô integrals. Rewriting \( h_n \Box W^n \) as a multiple Wiener–Itô integral we have

\[
h_n \Box W^n = \frac{1}{n!} \int_0^T \cdots \int_0^T h_n(\theta_1, \ldots, \theta_2) \, dW_{\theta_1} \cdots dW_{\theta_n}
\]

with \( h_n \) extended to \([0, T]^n\) by (6). Consequently, by the orthogonality properties of the Wiener–Itô integrals

\[
E\left( W(t_1) W(t_2) \cdots W(t_n) (h_n \Box W^n) \right) = \int_0^T \cdots \int_0^T \chi(t_1) \cdots \chi(t_n) h_n(\theta_1, \ldots, \theta_n) \, d\theta_1 \cdots d\theta_n
\]

\[
= H_n(t_1, \ldots, t_n).
\]

(9)

Now, let \( \Pi^{(n,m)} \) be a nonnegative multinomial, that is,

\[
\sum_{|p| \leq m} C^{p_1, \ldots, p_m} X_1^{p_1} \cdots X_m^{p_m} \geq 0
\]
for all values of \(x_1, \ldots, x_n\), then, replacing \(x_i\) by \(x_i/\lambda\) it follows that

\[
\sum_{|p| \leq m} \lambda^{m-|p|} C^{p_1, \ldots, p_n} x_1^{p_1} \cdots x_n^{p_n} > 0
\]

for all \(\lambda > 0\). Therefore, for all values of \(\lambda\), for which (b) is satisfied we have

\[
E \left( g(\lambda, W) \sum_{|p| \leq m} \lambda^{m-|p|} C^{p_1, \ldots, p_n} (W(t_1))^p_1 \cdots (W(t_n))^p_n \right) > 0.
\]

We can rewrite the last equation as

\[
E \left( \left( 1 + \sum \lambda^{m-|p|} C^{p_1, \ldots, p_n} \int_0^t \cdots \int_0^t \prod_{i \leq |p|} \psi_{i}(\theta_i) dW_{\theta_1} \cdots dW_{\theta_{|p|}} \right) \right) > 0.
\]

The result will be a polynomial of order \(m\) in \(\lambda\). Note that terms of the form

\[
E \left( \sum \lambda^{m-|p|} C^{p_1, \ldots, p_n} \psi_{i}(\theta_i) W^{i-2} \right)
\]

will contribute to the coefficient of \(\lambda^{m-2}\) but not to the coefficient of \(\lambda^m\). The coefficient of \((\lambda^m)^m\) will, therefore, be

\[
\sum_{|p| \leq m} C^{p_1, \ldots, p_n} H_{|p|}(t^{p_1, \ldots, p_n})
\]

where \(t^{p_1, \ldots, p_n}\) denotes the \(|p|\)-tuple

\[
t^{p_1, \ldots, p_n} = (t_{1_1}, t_{1_2}, \ldots, t_{1_n}, \ldots, t_{n_1}, \ldots, t_{n_n})
\]

\[
p_1 \quad p_n
\]

Since this is the coefficient of the highest term of a nonnegative polynomial, it must be nonnegative and this proves (c). The proof that (c) implies (d) is based on an infinite dimensional extension of the fundamental result on the existence of a solution to the moment problem (Theorem 1.1 of [4]). L.A. Shepp, in an unpublished memorandum, extended Theorem 1.1 of [4] and derived conditions for the existence of a probability measure on function space with given moments. His arguments will be repeated here. Let \(\mathcal{K} = \{X(t)\}\) denote the space of real valued functions on \([0, T]\) satisfying \(X(t) = \int_0^t x_s ds\) where \(x_s\) is measurable on \([0, T]\) and ess-sup \(|x_s| \leq K\) (the ess. is with respect to the Lebesgue measure). Let \(\Gamma = \{\gamma\}\) be the collection of bounded continuous functions on \(\mathcal{K}\) with the norm

\[
|\gamma| = \sup_{X \in \mathcal{K}} |\gamma(X)|.
\]

For a multinomial \(\Pi^{(m,n)}\) and \(n\)-tuple \(t^n\) set

\[
\gamma_{t^n}(X) = \sum_{|p| \leq m} C^{p_1, \ldots, p_n} (X(t_1))^{p_1} \cdots (X(t_n))^{p_n}.
\]

Let \(\Gamma_{t^n}\) denote the collection of functions on \(\mathcal{K}\) which are of the form \(\gamma_{t^n}\), and note that this is a linear collection of bounded and continuous functions on \(\mathcal{K}\). To each \(\gamma_{t^n}\) in \(\Gamma_{t^n}\) associate the functional

\[
F(\gamma_{t^n}) = \Pi^{(m,n)}(H, t^n).
\]

This functional is linear and continuous hence bounded. Therefore, by the Hahn–Banach theorem, there exists a bounded linear extension of \(F(\cdot)\) to all functions \(\gamma\) in \(\Gamma\) and this extension is nonnegative since \(\Pi^{(m,n)}(H, t^n)\) was assumed to be nonnegative. By the Riesz representation theorem there exists a nonnegative measure \(\mu\) on \(\mathcal{K}\) such that

\[
F(\gamma) = \int_{\mathcal{K}} \gamma(X) d\mu(X).
\]
Since $\mu(\cdot)$ is nonnegative with $\mu(\mathcal{X}) = 1$, $\mu$ is a probability measure on $\mathcal{X}$. The measure $\mu(\cdot)$ therefore defines a process $\{Y(t), 0 \leq t \leq T\}$ such that

$$E(Y(t_1) \cdots Y(t_n)) = H_n(t_1, \ldots, t_n)$$

and this measure induces a measure on the space of bounded measurable functions $\{y_s, 0 \leq s \leq T\}$, $|y_s| \leq K$ and

$$E(y(t_1) \cdots y(t_n)) = h_n(t_1, \ldots, t_n)$$

which completes the proof of (d). The proof that (d) implies (e) is given in the introduction and (a) follows by replacing the 'signal' $\int_0^t y_s \; ds + W_t$ by $\int_0^t \lambda_s \; ds + W_t$.

Remarks. (a) The question arises whether every 'reasonable' nonnegative functional of the Brownian motion satisfies condition (a) or (b) of the theorem. The answer is negative as the following two simple examples show. The first example, due to L.A. Shepp is as follows: Let

$$g(W) = \frac{1}{T} W^2(T) = 1 + \frac{2}{T} \int_0^T \int_0^T dW_{\eta} dW_\eta.$$  

In this case $h_n = 0$ for $n > 2$ and therefore part (d) of the theorem cannot be true for this $g(W)$. Note that $g(W) = T^{-1} W^2(T)$ is a continuous nonnegative functional on the space of continuous functions and $T^{-1}(\lambda W(T))^2$ is also nonnegative, however, in this case

$$g(\lambda \cdot W) \neq g(\lambda, W)$$

(the representation (3) is not a continuous functional on the space of continuous functions). The second example is due to B. Hajek: Condition (e) of the theorem implies that

$$E_1(W(T))^2 = E_0 \left( \int_0^T y_s \; ds + W_T \right)^2 = E_0 \left( \int_0^T y_s \; ds \right)^2 + E_0 W^2(T) \geq E_0 W^2(T)$$

(10)

where $E_0$ denotes expectation with respect to the $P_W$ measure and $E_1$ denotes expectation with respect to $P_{Y+\mu}$. On the other hand, if $P_1$ is the measure induced by $dX_t = -\alpha X_t \; dt + dW_t$, $\alpha > 0$, $X_0 = 0$; then, by Ito's formula

$$E(\lambda(W(T))^2 = -\alpha^2 E_0 \int_0^T X_t^2 \; ds + T.$$  

Hence $E(\lambda(W(T))^2 < E W^2(T)$ which contradicts (10). Therefore $dP_1/dP_W$ does not satisfy condition (e) of the theorem.

(b) The extension of the results of this note to the case of multiparameter Wiener processes $W(t_1, \ldots, t_n)$ is straightforward and therefore omitted (cf. [1] and [2]).

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