A characterization of the kernels associated with the multiple integral representation of some functionals of the Wiener process

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In this paper we present a characterization of those Wiener functionals that are the likelihood ratio for a 'signal plus independent noise' model. The characterization is expressed in terms of the representation of such functionals in a series of multiple Wiener integrals.

Keywords: Wiener functionals, Multiple Wiener integrals, Likelihood ratio, Additive noise model, Radon–Nikodym derivative.

Let $\{y_s,\ 0 \leqslant s \leqslant T\}$ be a random process with measurable sample functions satisfying $|y_s| \leqslant K$ a.s. Let $\{W_s,\ 0 \leqslant s \leqslant T\}$ be a Wiener process which is independent of the $\{y_s,\ 0 \leqslant s \leqslant T\}$ process. Let

$$\Lambda_t = \exp\left\{\int_0^t y_s \,\mathrm{d}W_s - \tfrac{1}{2}\int_0^t y_s^2 \,\mathrm{d}s\right\}.$$

Then Λ_t is the unique solution to the integral equation

$$\Lambda_t = 1 + \int_0^t \Lambda_s y_s \,\mathrm{d}W_s$$

and admits the series representation

$$\Lambda_t = \lim_{N\to\infty} \text{ in q.m. } \sum_{n=0}^{N} u_n(t)$$

where $u_0(t) = 1$ and $u_n(t) = \int_0^t u_{n-1}(s) y_s \,\mathrm{d}W_s$.

Let $\mathcal{F}_t^W$ denote the σ-field generated by $\{W_\theta,\ 0 \leqslant \theta < t\}$. Since convergence in quadratic mean commutes with conditional expectation, we have (as was observed in [1] and [3])

$$\begin{aligned} E(\Lambda_t | \mathcal{F}_t^W) &= \sum_{n=0}^{\infty} E(u_n(t) | \mathcal{F}_t^W) \\ &= \sum_{n=0}^{\infty} \int_0^t \left(\int_0^{t_n} \left(\cdots \int_0^{t_2} v_n(t_1,\ldots,t_n) \,\mathrm{d}W_{t_1} \cdots \right) \mathrm{d}W_{t_{n-1}} \right) \mathrm{d}W_{t_n} \end{aligned} \tag{1}$$

where

$$v_n(t_1,\ldots,t_n) = E(y(t_1) y(t_2) \cdots y(t_n)) \tag{2}$$

and the integrals are iterated Ito integrals. The functional $g(W) = E(\Lambda_T | \mathcal{F}_T^W)$ is a nonnegative functional of the Brownian motion, its Wiener–Ito representation is given by (1) with the kernels $v_n(\cdots)$ satisfying (2); that is, the n-th order kernel $v_n(t_1,\ldots,t_n)$ is the n-th order moment of a process which is independent of

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W. In this note we consider the converse problem: Let $g(W)$ be a square integrable functional of the Wiener process W with the Wiener–Ito representation

$$g(W)=C+\sum_n\int_0^T\left(\int_0^{t_n}\left(\cdots\int_0^{t_2}h_n(t_1,\ldots,t_n)\mathrm{d}W_{t_1}\cdots\right)\mathrm{d}W_{t_{n-1}}\right)\mathrm{d}W_{t_n}$$

where the integrals are iterated Ito integrals. This representation will be abbreviated by

$$g(W)=C+\sum_n h_n\square W^n \tag{3$'$}$$

with $h_n\square W^n$ denoting the n-th order iterated stochastic integral. The problem that we consider is the following: given a square integrable functional of the Brownian motion with representation (3) or (3)$'$, what conditions would ensure the existence of a process $\{y_t,\ 0\leqslant t\leqslant T\}$, independent of W, such that

$$\square h_n(t_1,\ldots,t_n)=E(y(t_1)y(t_2)\cdots y(t_n)) \tag{4}$$

for all n and all $0\leqslant t_1<t_2\cdots<t_n\leqslant T$? Another way to state the problem is the following: the functional (1) is the likelihood ratio of a 'signal plus independent noise' with respect to the 'noise only' hypothesis and the problem is to characterize the nonnegative functionals of the Brownian motion that represent the likelihood ratio of a 'signal plus independent noise' with respect to the 'noise only' hypothesis.

Let $g(\lambda,W)$ denote

$$g(\lambda,W)=C+\sum_n\lambda^n h_n\square W^n. \tag{5}$$

It will be shown that $g(W)$ has the 'signal plus independent noise' representation, i.e. h_n satisfy (4) if and only if $g(\lambda,W)$ as defined by (5) is a nonnegative square integrable random variable for every nonnegative λ.

Notation. For the representation of $g(W)$, $h_n(t_1,\ldots,t_n)$ has to be defined for ordered n-tuples only (cf. (3)), namely for $t_1,\ldots,t_n$ satisfying $0\leqslant t_1<t_2<\cdots<t_n\leqslant T$. Define $h_n(t_1,\ldots,t_n)$ for unordered n-tuples of *distinct* times by

$$h_n(t_1,\ldots,t_n)=h_n(t_{i_1},\ldots,t_{i_n}) \tag{6}$$

where $(t_{i_1},\ldots,t_{i_n})$ is the rearrangement of $(t_1,\ldots,t_n)$ which yields an increasing sequence. We will not distinguish between two kernels $h_n(t_1,\ldots,t_n)$ and $h'_n(t_1,\ldots,t_n)$ which are equal almost everywhere (Lebesgue) on $[0,T]^n$. Note that (6) leaves $h_n(t_1,\ldots,t_n)$ undefined on a Lebesgue set of measure zero in $[0,T]^n$.

Let $\Pi^{(n,m)}$ denote a multinomial of degree m in n variables $x_1,\ldots,x_n$,

$$\Pi^{(n,m)}=\sum_{|p|\leqslant m}C^{p_1,p_2,\ldots,p_n}x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}$$

where p_i, $i=1,\ldots,n$, are integers and $|p|=\Sigma_i p_i$. A multinomial $\Pi^{(n,m)}$ is said to be nonnegative if it is nonnegative for all values of its arguments $x_1,\ldots,x_n$. Let W be a Wiener process and t^n an n-tuple of real numbers satisfying $0\leqslant t_i\leqslant T$, $i=1,\ldots,n$. $\Pi^{(n,m)}(W,t^n)$ will denote the multinomial $\Pi^{(n,m)}$ evaluated at $x_i=W(t_i)$, $i=1,\ldots,n$,

$$\Pi^{(n,m)}(W,t^n)=\sum_{|p|\leqslant m}C^{p_1,\ldots,p_n}(W(t_1))^{p_1}\cdots(W(t_n))^{p_n}$$

where p_i, $i=1,\ldots,n$, are integers and $|p|=\Sigma_i p_i$.

$H_n(t^n)$ is defined as

$$H_n(t^n)=H_n(t_1,\ldots,t_n)=\int_0^{t_1}\cdots\int_0^{t_n}h_n(\theta_1,\ldots,\theta_n)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n. \tag{7}$$

Finally, $\tilde{\Pi}^{(n,m)}(H,t^n)$ is defined as

$$\tilde{\Pi}^{(n,m)}(H,t^n)=\sum_{|p|\leqslant m}C^{p_1,\ldots,p_n}H_{|p|}(\underbrace{t_1,t_1,\ldots,t_1}_{p_1},t_2,t_2,\ldots,\underbrace{t_n,\ldots,t_n}_{p_n})$$

Theorem. *Let $g(W) = C + \Sigma h_n \square W^n$ be the Wiener–Ito representation of a square integrable functional of the Wiener process over* $[0, T]$. *Assume that $|h_n(t^n)| \leq K$ for all n and all n-tuples t^n. Then the following are equivalent*:

(a) *$g(\lambda, W)$ is a nonnegative square integrable random variable for every positive real λ.*

(b) *There exists a sequence of positive real numbers λ_r such that $\lambda_r \to \infty$ as $r \to \infty$ and $g(W)$ and $g(\lambda_r, W)$ are nonnegative random variables.*

(c) *For every nonnegative multinomial $\Pi^{(n,m)}$ it holds that $\Pi^{(n,m)}(H, t^n)$ is nonnegative.*

(d) *There exists a random process $\{y_s, 0 \leq s \leq T\}$ such that $|y_s| \leq K$ and*

$$E(y(t_1)y(t_2)\cdots y(t_n)) = h_n(t_1,\ldots,t_n)$$

for almost all (Lebesgue) points $(t_1,\ldots,t_n)$ in $[0, T]^n$ (the probability space on which the y process is defined is unrelated to the probability space on which the Wiener process W is defined).

(e) *Let P_W^X denote the probability measure on the space of continuous functions induced by W. Let $\{z_t, 0 \leq t \leq T\}$ be a random process with measurable sample paths on $[0, T]$. Let P_{Z+W}^X denote the probability measure induced by $X_t = \int_0^t z_s\,\mathrm{d}s + W_t$ on the space of continuous functions. There exists a process $\{z_s, 0 \leq s \leq T\}$ such that $|z_s| \leq K$ a.s. and $\{z_s, 0 \leq s \leq T\}$ is independent of the Wiener process W such that P_{Z+W}^X is equivalent to P_W^X and the Radon–Nikodym derivative of P_{Z+W}^X with respect to P_W^X satisfies*

$$\frac{\mathrm{d}P_{Z+W}^X}{\mathrm{d}P_W^X}(W) = \frac{1}{C}g(W).$$

Proof. Obviously (a) implies (b). We turn now to the proof that (b) implies (c). Recall first that for a deterministic square integrable function $\phi(\cdot)$ we have (cf. equation (3.4) of [2]):

$$\left(\int_0^T \phi(\theta)\,\mathrm{d}W_\theta\right)\left(\int_0^T\cdots\int_0^T h_n(\theta_1,\ldots,\theta_n)\,\mathrm{d}W_{\theta_1}\cdots\mathrm{d}W_{\theta_n}\right)$$
$$= \int_0^T\cdots\int_0^T \phi(\theta_1)h_n(\theta_2,\ldots,\theta_{n+1})\,\mathrm{d}W_{\theta_1}\cdots\mathrm{d}W_{\theta_{n+1}} + \sum_{q=0}^{n-1}\psi_q \square W^q. \tag{8}$$

The exact form of ψ will not interest us. It follows by repeated applications of this result that

$$W(t_1)W(t_2)\cdots W(t_n) = \int_0^T\cdots\int_0^T \chi_{t_1}(\theta_1)\chi_{t_2}(\theta_2)\cdots\chi_{t_n}(\theta_n)\,\mathrm{d}W_{\theta_1}\cdots\mathrm{d}W_{\theta_n} + \sum_{q=0}^{n-2}\psi_q \square W^q$$

for some deterministic ψ_q, $0 \leq q \leq n-2$ where $\chi_t(\theta)$ denotes the characteristic function ($\chi_t(\theta) = 1$ for $\theta \leq t$ and zero otherwise). Note that in equation (8) and in the last equation the integrals over $[0, T]^n$ are multiple Wiener–Ito integrals. Rewriting $h_n \square W^n$ as a multiple Wiener–Ito integral we have

$$h_n \square W^n = \frac{1}{n!}\int_0^T\cdots\int_0^T h_n(\theta_1,\ldots,\theta_2)\,\mathrm{d}W_{\theta_1}\cdots\mathrm{d}W_{\theta_n}$$

with h_n extended to $[0, T]^n$ by (6). Consequently, by the orthogonality properties of the Wiener–Ito integrals

$$E\{W(t_1)W(t_2)\cdots W(t_n)(h_n \square W^n)\} = \int_0^T\cdots\int_0^T \chi_{t_1}(\theta_1)\cdots\chi_{t_n}(\theta_n)h_n(\theta_1,\ldots,\theta_n)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n$$
$$= H_n(t_1,\ldots,t_n). \tag{9}$$

Now, let $\Pi^{(n,m)}$ be a nonnegative multinomial, that is,

$$\sum_{|p| \leq m} C^{p_1,\ldots,p_n} x_1^{p_1}\cdots x_n^{p_n} \geq 0$$

for all values of $x_1,\ldots,x_n$, then, replacing x_i by x_i/λ it follows that

$$\sum_{|p|\leq m} \lambda^{m-|p|} C^{p_1,\ldots,p_n} x_1^{p_1}\cdots x_n^{p_n} \geq 0$$

for all $\lambda > 0$. Therefore, for all values of λ_r for which (b) is satisfied we have

$$E\left\{ g(\lambda_r, W) \sum_{|p|\leq m} \lambda_r^{m-|p|} C^{p_1,\ldots,p_n} (W(t_1))^{p_1}\cdots (W(t_n))^{p_n} \right\} \geq 0.$$

We can rewrite the last equation as

$$E\left\{ \left(1+\sum_q \lambda_r^q h_q \Box W^q\right)\left(\sum_{|p|\leq m} \lambda_r^{m-|p|} C^{p_1,\ldots,p_n} \int_0^T \cdots \int_0^T \prod_{i\leq|p|} \chi_{t_i}(\theta_i)\,\mathrm{d}W_{\theta_1}\cdots \mathrm{d}W_{\theta_{|p|}} \right.\right.$$
$$\left.\left. + \sum_{i\leq m} \lambda_r^{m-i} \psi_i \Box W^{i-2} \right)\right\} \geq 0.$$

The result will be a polynomial of order m in λ_r. Note that terms of the form

$$E\left\{ (\lambda_r^q h_q \Box W^q)(\lambda_r^{m-i} \psi_i \Box W^{i-2}) \right\}$$

will contribute to the coefficient of λ_r^{m-2} but not to the coefficient of λ_r^m. The coefficient of $(\lambda_r)^m$ will, therefore, be

$$\sum_{|p|\leq m} C^{p_1,\ldots,p_n} H_{|p|}(t^{p_1,\ldots,p_n})$$

where $t^{p_1,p_2,\ldots,p_n}$ denotes the $|p|$-tuple

$$t^{p_1,\ldots,p_n} = (\underbrace{t_1, t_1,\ldots,t_1}_{p_1}, t_2,\ldots, \underbrace{t_n,\ldots,t_n}_{p_n})$$

Since this is the coefficient of the highest term of a nonnegative polynomial, it must be nonnegative and this proves (c). The proof that (c) implies (d) is based on an infinite dimensional extension of the fundamental result on the existence of a solution to the moment problem (Theorem 1.1 of [4]). L.A. Shepp, in an unpublished memorandum, extended Theorem 1.1 of [4] and derived conditions for the existence of a probability measure on function space with given moments. His arguments will be repeated here. Let $\mathcal{X} = \{X(t)\}$ denote the space of real valued functions on $[0, T]$ satisfying $X(t) = \int_0^t x_s\,\mathrm{d}s$ where x_s is measurable on $[0, T]$ and ess-sup $|x_t| \leq K$ (the ess. is with respect to the Lebesgue measure). Let $\Gamma = \{\gamma\}$ be the collection of bounded continuous functions on $\mathcal{X}$ with the norm

$$|\gamma| = \sup_{X\in\mathcal{X}} |\gamma(X)|.$$

For a multinomial $\Pi^{(m,n)}$ and n-tuple t^n set

$$\gamma_\pi(X) = \sum_{|p|\leq m} C^{p_1,\ldots,p_n} (X(t_1))^{p_1}\cdots (X(t_n))^{p_n}.$$

Let Γ_π denote the collection of functions on $\mathcal{X}$ which are of the form γ_π, and note that this is a linear collection of bounded and continuous functions on $\mathcal{X}$. To each γ_π in Γ_π associate the functional

$$F(\gamma_\pi) = \tilde{\Pi}^{(m,n)}(H, t^n).$$

This functional is linear and continuous hence bounded. Therefore, by the Hahn–Banach theorem, there exists a bounded linear extension of $F(\cdot)$ to all functions γ in Γ and this extension is nonnegative since $\Pi^{(m,n)}(H, t^n)$ was assumed to be nonnegative. By the Riesz representation theorem there exists a nonnegative measure μ on $\mathcal{X}$ such that

$$F(\gamma) = \int_{\mathcal{X}} \gamma(X)\,\mathrm{d}\mu(X).$$

Since $\mu(\cdot)$ is nonnegative with $\mu(\mathfrak{X})=1$, μ is a probability measure on $\mathfrak{X}$. The measure $\mu(\cdot)$ therefore defines a process $\{Y(t),\ 0\leqslant t\leqslant T\}$ such that

$$E(Y(t_1)\cdots Y(t_n))=H_n(t_1,\ldots,t_n)$$

and this measure induces a measure on the space of bounded measurable functions $\{y_s,\ 0\leqslant s\leqslant T\}$, $|y_s|\leqslant K$ and

$$E(y(t_1)\cdots y(t_n))=h_n(t_1,\ldots,t_n)$$

which completes the proof of (d). The proof that (d) implies (e) is given in the introduction and (a) follows by replacing the 'signal' $\int_0^t y_s\,\mathrm{d}s+W_t$ by $\lambda\int_0^t\lambda_s\,\mathrm{d}s+W_t$.

Remarks. (a) The question arises whether every 'reasonable' nonnegative functional of the Brownian motion satisfies condition (a) or (b) of the theorem. The answer is negative as the following two simple examples show. The first example, due to L.A. Shepp is as follows: Let

$$g(W)=\frac{1}{T}W^2(T)=1+\frac{2}{T}\int_0^T\int_0^\theta \mathrm{d}W_\eta\,\mathrm{d}W_\theta.$$

In this case $h_n=0$ for $n>2$ and therefore part (d) of the theorem cannot be true for this $g(W)$. Note that $g(W)=T^{-1}W^2(T)$ is a continuous nonnegative functional on the space of continuous functions and $T^{-1}(\lambda W(T))^2$ is also nonnegative, however, in this case

$$g(\lambda\cdot W)\neq g(\lambda,W)$$

(the representation (3) is not a continuous functional on the space of continuous functions). The second example is due to B. Hajek: Condition (e) of the theorem implies that

$$E_1(W(T))^2=E_0\Big(\int_0^T y_s\,\mathrm{d}s+W_T\Big)^2=E\Big(\int_0^T y_s\,\mathrm{d}s\Big)^2+E_0W^2(T)\geqslant E_0W^2(T) \tag{10}$$

where E_0 denotes expectation with respect to the P_W measure and E_1 denotes expectation with respect to P_{Y+W}. On the other hand, if P_1 is the measure induced by $\mathrm{d}X_t=-\alpha X_t\,\mathrm{d}t+\mathrm{d}W_t$, $\alpha>0$, $X_0=0$; then, by Ito's formula

$$E(X(T))^2=-\alpha^2E\int_0^T X_s^2\,\mathrm{d}s+T.$$

Hence $E(X(T))^2<EW^2(T)$ which contradicts (10). Therefore $\mathrm{d}P_1/\mathrm{d}P_W$ does not satisfy condition (e) of the theorem.

(b) The extension of the results of this note to the case of multiparameter Wiener processes $W(t_1,\ldots,t_n)$ is straightforward and therefore omitted (cf. [1] and [2]).

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