Explicit Solutions to a Class of Nonlinear Filtering Problems

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In this paper we obtain the solution of a class of nonlinear filtering problems in the form of a series expansion in terms of multiple Wiener integrals. The solution is explicit in the sense that the kernels of the integrals in the expansion are explicitly determined.

KEY WORDS: Nonlinear filtering, multiple Wiener integrals, orthogonal polynomials

1. INTRODUCTION

Let $Z_t$ be a stochastic process and let $X_t$ be a process of the form

$$X_t = \int_0^t Z_s \, ds + W_t, \quad t \geq 0$$

where $W_t$ is a standard Wiener process independent of $Z_t$. The general filtering problem is to find effective ways of computing the conditional expectation

$$E[f(Z_t) | X_s, 0 \leq s \leq t]$$

for some function $f$.

Except when $Z$ is of finite state, the Gaussian case and some recently discovered examples [4] comprise the entire collection of cases where solutions, in some explicitly computable form, to the nonlinear filtering problem are known. The object of this paper is to add a small but possibly useful class of examples to this collection.

Since Kalman's solution to the linear filtering problem became the
dominant one, there has been a tendency to view filters only in the
differential equation form. An alternative and much older interpretation of
"filters" is that of the representation of the estimator as a functional of the
observed process, e.g., as a convolution. It is in the latter sense that results
of this paper are to be interpreted. While the representation that we shall
derive is not readily implemented as a differential equation, its form is
such that the filter can be implemented, at least in principle, by a lattice of
linear filters and multipliers. Whether such an arrangement can be
reduced to something practical remains to be determined.

2. A WIENER SERIES REPRESENTATION

Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space. Let \(\{Z_t, W_t, 0 \leq t \leq T\}\) be a pair of
independent processes defined on \((\Omega, \mathcal{F}, \mathcal{P})\) such that \(W\) is a standard
Wiener process, and \(Z\) is a strong Markov process that is almost surely
sample square-integrable. Consider an observation process

\[
X_t = \int_0^t Z_s \, ds + W_t, \quad 0 \leq t \leq T, \tag{2.1}
\]

and denote \(\mathcal{F}_{xt} = \sigma(X_s, s \leq t)\). It is well known (see e.g. [6]) that if we
define a probability measure \(\mathcal{P}_0\) by

\[
\frac{d\mathcal{P}_0}{d\mathcal{P}} = \exp \left\{ - \int_0^T Z_s \, dW_s - \frac{1}{2} \int_0^T Z_s^2 \, ds \right\} \tag{2.2}
\]

then \((Z, X)\) has the same distribution under \(\mathcal{P}_0\) as \((Z, W)\) under \(\mathcal{P}\).

For a bounded \(f\) define the unnormalized estimator

\[
\pi_t f = E_0 \left\{ f(Z_t) \frac{d\mathcal{P}}{d\mathcal{P}_0} \bigg| \mathcal{F}_{xt} \right\}. \tag{2.3}
\]

To normalize, one would only need to write

\[
E[f(Z_t) \big| \mathcal{F}_{xt}] = \pi_t f \frac{\pi_t 1}{\pi_t 1} \tag{2.4}
\]

where

\[
\pi_t 1 = L_t = E_0 \left\{ \frac{d\mathcal{P}}{d\mathcal{P}_0} \bigg| \mathcal{F}_{xt} \right\}, \tag{2.5}
\]

is simply the likelihood ratio.
Now, from (2.2) we have

$$\frac{d\mathcal{P}}{d\mathcal{P}_0} = \exp \left\{ \int_0^T Z_s dX_s - \frac{1}{2} \int_0^T Z_s^2 ds \right\}$$  

(2.6)

and the exponential formula for multiple Wiener integrals yields [3]

$$\frac{d\mathcal{P}}{d\mathcal{P}_0} = \sum_{n=0}^{\infty} Z_n \circ X^n$$  

(2.7)

where $Z_0 \circ X^0 \equiv 1$ and for $n > 1$

$$Z_n \circ X^n = \int_{0 < t_1 < \ldots < t_n < T} Z_{t_1}Z_{t_2} \ldots Z_{t_n}X(dt_1) \ldots X(dt_n)$$  

(2.8)

are desymmetrized multiple Wiener integrals. It now follows that

$$\pi_t f = \sum_{n=0}^{\infty} \int_{0 < t_1 < \ldots < t_n < t} E_0(Z_{t_1}Z_{t_2} \ldots Z_{t_n}f(Z_t))X(dt_1) \ldots X(dt_n)$$  

(2.9)

The process $Z$ being identically distributed under either measures, $E_0$ in (2.9) can also be replaced by $E$.

Now, let $Z$ be a diffusion process, with the density of $Z_t$ being $\mathcal{P}(z, t)$. Introduce an unnormalized conditional density $V(z, t)$ of $Z_t$ given the observation by the relationship [6]

$$\pi_t f = \int_{-\infty}^{\infty} V(z, t)f(z)dz$$  

(2.10)

Then (2.9) reduces to [c.f. 5]

$$V(z, t) = p(z, t) \sum_{n=0}^{\infty} m_n(z, \cdot, t) \circ X^n$$  

(2.11)

with

$$m_n(z, t_1, t_2, \ldots, t_n, t) = E(Z_{t_1}Z_{t_2} \ldots Z_{t_n} | Z_t = z)$$  

(2.12)

and

$$m_n(z, \cdot, t) \circ X^n = \int_{0 < t_1 < \ldots < t_n < t} m_n(z, t_1, \ldots, t_n, t)X(dt_1) \ldots X(dt_n)$$  

(2.13)
From the Markov property of $Z$, the functions $m_n$ satisfy the recurrence relationships

$$m_n(z, t_k, t_n, t) = E[Z_{t_n}m_{n-1}(Z_{t_k}, t_1, \ldots, t_n)|Z_t = z]. \quad (2.14)$$

The main result of this paper is an explicit evaluation of these functions for a class of stationary $Z$.

3. PROCESSES OF THE PEARSON CLASS

We shall restrict our attention to a class of stationary diffusion processes $Z_t$ that have a transition density of the forms

$$p(z, t|z_0, t_0) = p(z) \sum_{k=0}^{\infty} e^{-\lambda_k(t-t_0)} \phi_k(z) \phi_k(z_0) \quad (3.1)$$

where $p(z)$ is the stationary density and $\phi_k$ are orthonormal polynomials of degree $k$. Densities of the form (3.1) were introduced by Barrett and Lampard [1]. In [7] diffusion processes with such transition densities were exhaustively studied subject to the additional condition that $p(z)$ is of the Pearson type [2]. It was found that such processes fall into three categories, corresponding to the classical Hermite, Laguerre and Jacobi polynomials respectively. In terms of the Fokker Planck equation for the transition density $p$

$$\frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z)p] - \frac{\partial}{\partial z} [m(z)p] = \frac{\partial}{\partial t} p \quad (3.2)$$

these cases can be summarized as follows:

$$\sigma^2(z) = 2, \ m(z) = -z \quad (3.3a)$$

$\phi_k(z)$ are Hermite polynomials

$$z > 0, \ \sigma^2(z) = 2z, \ m(z) = (\alpha + 1) - z, \ z \geq 0 \quad (3.3b)$$

$\phi_k(z)$ are Laguerre polynomials

$$|z| < 1, \ \sigma^2(z) = 2(1 - z^2), \ m(z) = (\alpha - \beta) - (\alpha + \beta + 2)z \quad \alpha, \beta > -1 \quad (3.3c)$$

$\phi_k(z)$ are Jacobi polynomials.
Observe that $z\phi_k(z)$ is a polynomial of degree $k+1$. Furthermore, for any $j \leq k-2$ $z\phi_j(z)$ is a polynomial of degrees $k-1$ or less and hence is orthonormal to $\phi_k$, i.e.,

$$\int p(z)z\phi_k(z)\phi_j(z)\,dz = 0 \quad j \leq k-2.$$ 

It follows that $z\phi_k(z)$ is at most a linear combination of $\phi_k$ and $\phi_{k \pm 1}$. We shall write

$$z\phi_k(z) = a_{k+1}\phi_{k+1}(z) + b_k\phi_k(z) + c_{k-1}\phi_{k-1}(z) \quad (3.4)$$

for the general 3-term recurrence relationship, and use this to evaluate the conditional moments $m_n(z, \cdot)$ explicitly.

We note that for any of these cases we have

$$\lambda_0 = 0 \quad \text{and} \quad \phi_0(z) = 1.$$

4. AN EXPLICIT SOLUTION

We begin with the following observation:

THEOREM 4.1 If $Z$ is a stationary Markov process with a transition function of the form (3.1). Then, $m_n(z, \cdot)$ are of the form

$$m_n(z, t_1, \ldots, t_n, t) = \sum_{p=0}^{n} \alpha_{np}(t_2 - t_1, t_3 - t_2, \ldots, t - t_n)\phi_p(z) \quad (4.1)$$

where $\alpha_{np}$ satisfy the recurrence relationship

$$\alpha_{np}(t_2 - t_1, \ldots, t - t_n) = e^{-\lambda_p(t-t_n)}a_{np}\alpha_{n-1, p-1}(t_2 - t_1, \ldots, t_n - t_{n-1})$$

$$+ b_p\alpha_{n-1, p+1}(t_2 - t_1, \ldots, t_n - t_{n-1})$$

$$+ c_p\alpha_{n-1, p+1}(t_2 - t_1, \ldots, t_n - t_{n-1}) n \geq p \geq 0 \quad (4.2)$$

Proof. We note from (3.1) that

$$E[\phi_k(Z_s)|Z_t = z] = e^{-\lambda_k(t-s)}\phi_k(z), \quad t \geq s. \quad (4.3)$$
Hence, from (3.4) we have
\[ m_1(z, t_1, t) = E[Z_{t_1}|Z_t = z] = E[a_1 \phi_1(Z_{t_1}) + b_0 \phi_0(Z_{t_1})|Z_t = z] = a_1 e^{-\lambda_1(t-t_1)} \phi_1(z) + b_0 e^{-\lambda_0(t-t_1)} \phi_0(z) \]
so that (4.1) holds for \( n = 1 \), and we have \( \alpha_{10} = b_0 e^{-\lambda_0(t-t_1)} = b_0 \), \( \alpha_{11} = a_1 e^{-\lambda_1(t-t_1)} \).

Suppose that (4.1) holds for \( k \leq n - 1 \). Then, from (2.14) we have
\[
m_n(z, t_1, \ldots, t_n, t) = \sum_{p=0}^{n-1} \alpha_{n-1, p}(t_2-t_1, \ldots, t_n-t_{n-1}) E[Z_{t_n} \phi_p(Z_{t_n})|Z_t = z] = \sum_{p=0}^{n-1} \alpha_{n-1, p}(t_2-t_1, \ldots, t_n-t_{n-1}) (a_{p+1} e^{-\lambda_{p+1}(t-t_0)} + b_p \phi_p(z) e^{-\lambda_p(t-t_0)} + c_{p-1} \phi_{p-1}(z) e^{-\lambda_{p-1}(t-t_0)})
\]
which is again of the form (4.1).

If we rearrange terms in (4.3), we get (4.2). \( \square \)

In (4.2) let's adopt the convention that \( \alpha_{np} = 0 \) whenever \( p > n \) or \( n < 0 \). Then the equation holds for any \( n \) and \( p \). Observe that when \( n = p \), we have
\[ \alpha_{nn} = e^{-\lambda_n(t-t_n)} a_n \alpha_{n-1, n-1} \]
which can be solved immediately to yield
\[ \alpha_{nn}(\tau_1, \tau_2, \ldots, \tau_n) = \prod_{k=1}^{n} a_k e^{-\lambda_k \tau_k} \]
and that in turn can be used to solve for \( \alpha_{n-1, n-1} \), etc. It is convenient to work with Laplace transforms and make a change in notation as follows:
\[
\hat{\alpha}_p^{(v)}(s_1, s_2, \ldots, s_{p+v}) = \int_0^\infty \cdots \int_0^\infty e^{-(s_1 \tau_1 + \cdots + s_p \tau_p + v)} \times \alpha_{p+v, p}(\tau_1, \tau_2, \ldots, \tau_{p+v}) d\tau_1 \cdots d\tau_{p+v}.
\]
(4.5)
Then, (4.2) becomes

\[
\hat{\alpha}_p^{(v)}(s_1, s_2, \ldots, s_{p+v}) = \frac{1}{(s_{p+v} + \lambda_p)} \{ a_p \hat{\alpha}_{p-1}^{(v)}(s_1, \ldots, s_{p+v-1}) \\
+ b_p \hat{\alpha}_p^{(v-1)}(s_1, s_2, \ldots, s_{p+v-1}) \\
+ c_p \hat{\alpha}_{p+1}^{(v-2)}(s_1, s_2, \ldots, s_{p+v-1}) \}
\]

(4.6)

which can be solved immediately to yield

\[
\hat{\alpha}_p^{(0)} = \prod_{k=1}^{p} \frac{a_k}{(s_k + \lambda_k)} , \quad \hat{\alpha}_0^{(0)} = 1
\]

(4.7)

verifying the result that we obtained earlier for \( \alpha_{nn} \).

The general solution for \( \hat{\alpha}_p^{(v)} \) is given as follows.

**THEOREM 4.2** Let \( u_k, b_k^{(v)} \) and \( c_k^{(v)} \) be defined as follows:

\[
(k \geq 1, \ v \geq 1)
\]

\[
u_k = \prod_{j=1}^{k} \left( \frac{b_j}{s_j} \right)
\]

(4.8)

\[
b_k^{(v)} = \left( \frac{b_k}{s_k + v + \lambda_k} \right) \prod_{j=1}^{k} \left( \frac{s_j + v + \lambda_j}{s_j + v - 1 + \lambda_j} \right)
\]

(4.9)

\[
c_k^{(v)} = 0 \quad v = 1
\]

(4.10)

\[
\frac{c_{k-1} a_k}{(s_k + v - 1 + \lambda_k - 1)(s_k + v + \lambda_k)} \prod_{j=1}^{k} \left( \frac{s_j + v + \lambda_j}{s_j + v - 2 + \lambda_j} \right) \quad v \geq 2.
\]

For \( v \geq 1, \ p \geq 0 \) and \( 1 \leq k \leq p + 1 \), define a \( v \)-dimensional row vector \( a_{pk}^{(v)} \) as follows:

\[
a_{p1}^{(v)} = \left( b_1^{(v)}, u_v \left( \frac{s_v c_1^{(v)}}{u_{v-1}} \right), u_v \left( \frac{s_{v-1} c_1^{(v-1)}}{u_{v-2}} \right), \ldots, u_v \left( \frac{s_2 c_1^{(v)}}{u_1} \right) \right)
\]

\[
a_{pk}^{(v)} = (b_k^{(v)}, c_k^{(v)}, 0 \ldots 0), \quad 2 \leq k \leq p
\]

\[
a_{pp+1}^{(v)} = (0, c_{p+1}^{(v)}, 0 \ldots 0).
\]
Finally, define $v+1$ by $v$ matrices

$$A^{(v)}_{pk} = \begin{bmatrix} a^{(v)}_{pk} \\ \delta_{pk} I_v \end{bmatrix}$$  \hspace{1cm} (4.12)$$

where $I_v$ is the $v \times v$ identity matrix.

Then, $\hat{\alpha}^{(v)}_p$ are given as follows:

$$\begin{bmatrix} \hat{\alpha}^{(v)}_p \\ \hat{\alpha}^{(0)}_p \end{bmatrix} = \prod_{j=1}^{p} \frac{a_j}{(s_{j+v} + \lambda_j)}$$

$$ \times \left[ \sum_{k=0}^{v} \sum_{m_v = 1}^{p+1} \sum_{m_{v-1} = 1}^{m_v + 1} \cdots \sum_{m_{k+1} = 1}^{m_{k+2} + 1} A^{(v)}_{pm_v} A^{(v-1)}_{m_v, m_{v-1}} \cdots A^{(k+1)}_{m_{k+2}, m_{k+1}} 1_{k+1} \right]$$  \hspace{1cm} (4.13)$$

when $1_k$ is the $k$-dimensional unit column vector.

**Proof** We begin by iterating (4.6) in $p$ and get

$$\hat{\alpha}^{(v)}_p = \prod_{j=1}^{p} \frac{a_j}{(s_{j+v} + \lambda_j)} \hat{\alpha}^{(v)}_0 + \sum_{m=1}^{p} \frac{1}{\prod_{j=1}^{m} \frac{a_j}{(s_{j+v} + \lambda_j)}}$$

$$ \times \left( \frac{b_m}{s_{m+v} + \eta_m} \right) \hat{\alpha}^{(v-1)}_m + \left( \frac{c_m}{s_{m+v} + \lambda_m} \right) \hat{\alpha}^{(v-2)}_{m+1}$$  \hspace{1cm} (4.14)$$

for $p \geq 1$ and

$$\hat{\alpha}^{(v)}_0 = \frac{b_0}{(s_v + \lambda_0)} \hat{\alpha}^{(v-1)}_0 + \frac{c_0}{(s_v + \lambda_0)} \hat{\alpha}^{(v-a)}_0$$  \hspace{1cm} (4.15)$$

Now, denote for $p \geq 1$

$$\hat{\alpha}^{(v)}_p = \left[ \prod_{j=1}^{p} \frac{a_j}{(s_{j+v} + \lambda_j)} \right]_{l_p}^{(v)}$$  \hspace{1cm} (4.15)$$
Then, we have

\[ \gamma_p^{(v)} = \delta_0^{(v)} + \sum_{m=1}^{p} \frac{1}{\prod_{j=1}^{m} (s_{j+v} + \lambda_j)} \left\{ \left( \frac{b_m}{s_{m+v} + \lambda_m} \right) \prod_{j=1}^{m} \frac{a_j}{(s_{j+v-1} + \lambda_j)} \gamma_m^{(v-1)} \right\} \]

\[ + \left( \frac{c_m}{s_{m+v} + \lambda_m} \right) \prod_{j=1}^{m+1} \frac{a_j}{(s_{j+v-2} + \lambda_j)} \gamma_m^{(v-2)} \]  

(4.16)

which simplifies to yield

\[ \gamma_p^{(v)} = \delta_0^{(v)} + \sum_{m=1}^{p} b_m^{(v)} \gamma_{m-1}^{(v)} + \sum_{m=1}^{p+1} c_m^{(v)} \gamma_{m+1}^{(v-2)} \]  

(4.17)

where \( b_m^{(v)} \) and \( c_m^{(v)} \) are as defined in (4.9) and (4.10).

Equation (4.15) can be iterated to yield

\[ \delta_0^{(v)} = \frac{b_0}{\prod_{j=1}^{v} (s_j + \lambda_0)} + \sum_{k=0}^{v-2} \frac{c_0 a_1 b_0^{-k-2}}{\prod_{j=k+1}^{v} (s_j + \lambda_0)} \left( \frac{s_{k+1} + \lambda_0}{s_{k+1} + \lambda_1} \right) \gamma_1^{(k)} \]  

(4.18)

which is of the form

\[ \delta_0^{(v)} = u_v + \sum_{k=0}^{v-2} s_{k+2} c_1^{(k+2)} \left( \frac{u_v}{u_{k+1}} \right) \gamma_1^{(k)} \]  

(4.19)

With the use of (4.19), we can now rewrite (4.16) in the form of

\[ \begin{bmatrix} \gamma_p^{(v)} \\ \gamma_p^{(v-1)} \\ \vdots \\ \gamma_p^{(0)} \end{bmatrix} = \sum_{m=1}^{p+1} A_{pm}^{(v)} \begin{bmatrix} \gamma_{m}^{(v-1)} \\ \vdots \\ \gamma_{m}^{(0)} \end{bmatrix} + u_v \mathbf{1}_{v+1} \]  

(4.20)
where \( A_{pm}^{(v)} \) are as defined by (4.12) and (4.11). Equation (4.20) can now be iterated in \( v \). With \( \gamma_p^{(0)} = 1 \) we get

\[
\begin{bmatrix}
\gamma_p^{(v)} \\
\vdots \\
\gamma_p^{(0)}
\end{bmatrix} = \sum_{k=0}^{r} u_k \left[ \sum_{m_v=1}^{p+1} \sum_{m_{v-1}=1}^{m_v+1} \cdots \sum_{m_{k+1}=1}^{m_k+2} \right] \\
\times \left\{ A_{pm_v}^{(v)} A_{m_{v-1}}^{(v-1)} \cdots A_{m_{k+2}}^{(k+1)} \right\}
\]

whence the desired result (4.13) follows immediately using (4.15).

\[
\Box
\]

### 5. THE SYMMETRIC CASE

There are some cases for which the polynomials \( \phi_n(z) \) contain only even or odd terms according as \( n \) is even or odd respectively. This is the situation, for example, for Gegenbauer polynomials (which include both Chebyshev and Legendre polynomials), and most importantly for Hermite polynomials which correspond to \( Z_t \) being a Gaussian process. We shall refer to these cases collectively as the symmetric case.

For the symmetric case the coefficient \( b_k \) in the recurrence relationship (3.4) is necessarily zero for every \( k \). It follows from (4.9) that \( b_k^{(v)} \) are identically zero, and the result of Theorem 4.2 simplifies a great deal as is indicated as follows:

**Theorem 5.1** For the symmetric case we have

\[
\alpha_p^{(2v+1)} = 0
\]

\[
\alpha_p^{(2v)} = \prod_{j=1}^{n} \frac{a_j}{(s_{2v} + j + \lambda_j)} \left\{ \sum_{m_v=1}^{p+1} \sum_{m_{v-1}=1}^{m_v+1} \cdots \sum_{m_1=1}^{m_2+1} c_m^{(2v)} c_{m_{v-1}}^{(2v-2)} \cdots c_{m_1}^{(2)} \right\}. \quad (5.1)
\]

**Proof** Since \( b_k^{(v)} \equiv 0 \), (4.17) becomes

\[
\gamma_p^{(v)} = \sum_{m=2}^{p+1} c_m^{(v)} c_{m-2}^{(v-2)} + \alpha_0^{(v)} \quad (5.2)
\]

and (4.18) now takes the form

\[
\alpha_0^{(v)} = \left( \frac{c_0}{s_v} \right) \alpha_1^{(v-2)} \quad (5.3)
\]
With the use of (5.2) for $\hat{a}_0^{(v)}$, (5.2) can be rewritten as

$$\gamma_p^{(v)} = \sum_{m=1}^{p+1} c_m^{(v)} \gamma_m^{(v-2)}$$

where $c_m^{(v)}$ is given by (4.10). Since $\gamma_p^{(0)} = 1$ and $\gamma_p^{(1)} = 0$, we have $\gamma_p^{(v)} = 0$ for all $v$ odd, and

$$\gamma_p^{(2v)} = \sum_{m_v=1}^{p+1} \sum_{m_{v-1}=1}^{m_v+1} \sum_{m_1=1}^{m_{v-1}+1} c_m^{(2v)} c_{m_{v-1}}^{(2v-2)} \cdots c_m^{(2)}$$

whence (5.1) follows. □

It is interesting to note that in the Gaussian case (c.f. 33a) the terms $c_k^{(v)}$ are given by

$$c_k^{(v)} = \frac{k}{(s_v+1)(s_v+2)} \prod_{j=1}^{k-2} \left( \frac{s_{j+v} + j}{s_{j+v} + j + 2} \right).$$

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References