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Explicit Solutions to a Class of Nonlinear Filtering Problems

EUGENE WONG

Department of Electrical Engineering and Computer Sciences, and the Electronics Research Laboratory, University of California, Berkeley, California 94720, U.S.A.

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In this paper we obtain the solution of a class of nonlinear filtering problems in the form of a series expansion in terms of multiple Wiener integrals. The solution is explicit in the sense that the kernels of the integrals in the expansion are explicitly determined.

KEY WORDS: Nonlinear filtering, multiple Wiener integrals, orthogonal polynomials

1. INTRODUCTION

Let Z_t be a stochastic process and let X_t be a process of the form

$$X_t = \int_0^t Z_s \, ds + W_t, \qquad t \geqq 0$$

where W_t is a standard Wiener process independent of Z_t. The general filtering problem is to find effective ways of computing the conditional expectation

$$E[f(Z_t) \mid X_s, 0 \leqq s \leqq t]$$

for some function f.

Except when Z is of finite state, the Gaussian case and some recently discovered examples [4] comprise the entire collection of cases where solutions, in some explicitly computable form, to the nonlinear filtering problem are known. The object of this paper is to add a small but possibly useful class of examples to this collection.

Since Kalman's solution to the linear filtering problem became the dominant one, there has been a tendency to view filters only in the differential equation form. An alternative and much older interpretation of "filters" is that of the representation of the estimator as a functional of the observed process, e.g., as a convolution. It is in the latter sense that results of this paper are to be interpreted. While the representation that we shall derive is not readily implemented as a differential equation, its form is such that the filter can be implemented, at least in principle, by a lattice of linear filters and multipliers. Whether such an arrangement can be reduced to something practical remains to be determined.

2. A WIENER SERIES REPRESENTATION

Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space. Let $\{Z_t, W_t, 0 \leqq t \leqq T\}$ be a pair of independent processes defined on $(\Omega, \mathscr{F}, \mathscr{P})$ such that W is a standard Wiener process, and Z is a strong Markov process that is almost surely sample square-integrable. Consider an observation process

$$X_t = \int_0^t Z_s\, ds + W_t, \qquad 0 \leqq t \leqq T, \tag{2.1}$$

and denote $\mathscr{F}_{xt} = \sigma(X_s, s \leqq t)$. It is well known (see e.g. [6]) that if we define a probability measure $\mathscr{P}_0$ by

$$\frac{d\mathscr{P}_0}{d\mathscr{P}} = \exp\left\{ -\int_0^T Z_s\, dW_s - \tfrac{1}{2}\int_0^T Z_s^2\, ds \right\} \tag{2.2}$$

then (Z, X) has the same distribution under $\mathscr{P}_0$ as (Z, W) under $\mathscr{P}$.

For a bounded f define the unnormalized estimator

$$\pi_t f = E_0\left\{ f(Z_t) \frac{d\mathscr{P}}{d\mathscr{P}_0} \middle| \mathscr{F}_{xt} \right\}. \tag{2.3}$$

To normalize, one would only need to write

$$E[f(Z_t) | \mathscr{F}_{xt}] = \frac{\pi_t f}{\pi_t 1} \tag{2.4}$$

where

$$\pi_t 1 = L_t = E_0\left\{ \frac{d\mathscr{P}}{d\mathscr{P}_0} \middle| \mathscr{F}_{xt} \right. \tag{2.5}$$

is simply the likelihood ratio.

Now, from (2.2) we have

$$\frac{d\mathscr{P}}{d\mathscr{P}_0}=\exp\left\{\int_T Z_s\,dX_s-\tfrac{1}{2}\int_T Z_s^2\,ds\right\} \tag{2.6}$$

and the exponential formula for multiple Wiener integrals yields [3]

$$\frac{d\mathscr{P}}{d\mathscr{P}_0}=\sum_{n=0}^{\infty} Z_n\circ X^n \tag{2.7}$$

where $Z_0\circ X^0\equiv 1$ and for $n>1$

$$Z_n\circ X^n=\int_{0<t_1<\ldots<t_n<T} Z_{t_1}Z_{t_2}\ldots Z_{t_n}X(dt_1)\ldots X(dt_n) \tag{2.8}$$

are desymmeterized multiple Wiener integrals. It now follows that

$$\pi_t f=\sum_{n=0}^{\infty}\int_{0<t_1<\ldots<t_n<t} E_0(Z_{t_1}Z_{t_2}\ldots Z_{t_n}f(Z_t))X(dt_1)\ldots X(dt_n) \tag{2.9}$$

The process Z being identically distributed under either measures, E_0 in (2.9) can also be replaced by E.

Now, let Z be a diffusion process, with the density of Z_t being $\mathscr{P}(z,t)$. Introduce an unnormalized conditional density $V(z,t)$ of Z_t given the observation by the relationship [6]

$$\pi_t f=\int_{-\infty}^{\infty} V(z,t)f(z)\,dz \tag{2.10}$$

Then (2.9) reduces to [c.f. 5]

$$V(z,t)=p(z,t)\sum_{n=0}^{\infty} m_n(z,\cdot,t)\circ X^n \tag{2.11}$$

with

$$m_n(z,t_1,t_2,\ldots,t_n,t)=E(Z_{t_1}Z_{t_2}\ldots Z_{t_n}|Z_t=z) \tag{2.12}$$

and

$$m_n(z,\cdot,t)\circ X^n=\int_{0<t_1<\ldots<t_n<t} m_n(z,t_1,\ldots,t_n,t)X(dt_1)\ldots X(dt_n) \tag{2.13}$$

From the Markov property of Z, the functions m_n satisfy the recurrence relationships

$$m_n(z, t_k, t_n, t) = E[Z_{t_n} m_{n-1}(Z_{t_n}, t_1, \ldots, t_n) | Z_t = z]. \tag{2.14}$$

The main result of this paper is an explicit evaluation of these functions for a class of stationary Z.

3. PROCESSES OF THE PEARSON CLASS

We shall restrict our attention to a class of stationary diffusion processes Z_t that have a transition density of the forms

$$p(z, t | z_0, t_0) = p(z) \sum_{k=0}^{\infty} e^{-\lambda_k(t-t_0)} \phi_k(z) \phi_k(z_0) \tag{3.1}$$

where $p(z)$ is the stationary density and ϕ_k are orthonormal polynomials of degree k. Densities of the form (3.1) were introduced by Barrett and Lampard [1]. In [7] diffusion processes with such transition densities were exhaustively studied subject to the additional condition that $p(z)$ is of the Pearson type [2]. It was found that such processes fall into three categories, corresponding to the classical Hermite, Laguerre and Jacobi polynomials respectively. In terms of the Fokker Planck equation for the transition density p

$$\frac{1}{2} \frac{\partial^2}{\partial z^2} [\sigma^2(z) p] - \frac{\partial}{\partial z} [m(z) p] = \frac{\partial}{\partial t} p \tag{3.2}$$

these cases can be summarized as follows:

$$\sigma^2(z) = 2, \ m(z) = -z \tag{3.3a}$$

$\phi_k(z)$ are Hermite polynomials

$$z > 0, \ \sigma^2(z) = 2z, \ m(z) = (\alpha + 1) - z, \ z \geqq 0 \tag{3.3b}$$

$\phi_k(z)$ are Leguerre polynomials

$$|z| < 1, \ \sigma^2(z) = 2(1 - z^2), \ m(z) = (\alpha - \beta) - (\alpha + \beta + 2) z \qquad \alpha, \beta > -1 \tag{3.3c}$$

$\phi_k(z)$ are Jacobi polynomials.

Observe that $z\phi_k(z)$ is a polynomial of degree $k+1$. Furthermore, for any $j \leqq k-2$ $z\phi_j(z)$ is a polynomial of degrees $k-1$ or less and hence is orthonormal to ϕ_k, i.e.,

$$\int p(z) z\phi_k(z)\phi_j(z)\,dz = 0 \qquad j \leqq k-2.$$

It follows that $z\phi_k(z)$ is at most a linear combination of ϕ_k and $\phi_{k\pm 1}$. We shall write

$$z\phi_k(z) = a_{k+1}\phi_{k+1}(z) + b_k\phi_k(z) + c_{k-1}\phi_{k-1}(z) \tag{3.4}$$

for the general 3-term recurrence relationship, and use this to evaluate the conditional moments $m_n(z,\cdot)$ explicitly.

We note that for any of these cases we have

$$\lambda_0 = 0 \quad \text{and} \quad \phi_0(z) = 1.$$

4. AN EXPLICIT SOLUTION

We begin with the following observation:

THEOREM 4.1 *If Z is a stationary Markov process with a transition function of the form* (3.1). *Then, $m_n(z,\cdot)$ are of the form*

$$m_n(z, t_1, \ldots, t_n, t) = \sum_{p=0}^{n} \alpha_{np}(t_2 - t_1, t_3 - t_2, \ldots, t - t_n)\phi_p(z) \tag{4.1}$$

where α_{np} satisfy the recurrence relationship

$$\begin{aligned}\alpha_{np}(t_2 - t_1, \ldots, t - t_n) = e^{-\lambda_p(t-t_n)}\, & a_p\alpha_{n-1,p-1}(t_2 - t_1, \ldots, t_n - t_{n-1}) \\ & + b_p\alpha_{n-1,p}(t_2 - t_1, \ldots, t_n - t_{n-1}) \\ & + c_p\alpha_{n-1,p+1}(t_2 - t_1, \ldots, t_n - t_{n-1})\; n \geqq p \geqq 0 \end{aligned} \tag{4.2}$$

Proof We note from (3.1) that

$$E[\phi_k(Z_s) \,|\, Z_t = z] = e^{-\lambda_k(t-s)}\phi_k(z), \qquad t \geqq s. \tag{4.3}$$

Hence, from (3.4) we have

$$m_1(z, t_1, t) = E[Z_{t_1} | Z_t = z] = E[a_1\phi_1(Z_{t_1}) + b_0\phi_0(Z_{t_1}) | Z_t = z]$$

$$= a_1 e^{-\lambda_1(t-t_1)}\phi_1(z) + b_0 e^{-\lambda_0(t-t_1)}\phi_0(z)$$

so that (4.1) holds for $n=1$, and we have $\alpha_{10} = b_0 e^{-\lambda_0(t-t_1)} = b_0$, $\alpha_{11} = a_1 e^{-\lambda_1(t-t_1)}$.

Suppose that (4.1) holds for $k \leq n-1$. Then, from (2.14) we have

$$m_n(z, t_1, \ldots, t_n, t) = \sum_{p=0}^{n-1} \alpha_{n-1,p}(t_2 - t_1, \ldots, t_n - t_{n-1}) E[Z_{t_n}\phi_p(Z_{t_n}) | Z_t = z]$$

$$= \sum_{p=0}^{n-1} \alpha_{n-1,p}(t_2 - t_1, \ldots, t_n - t_{n-1}) \{a_{p+1}\phi_{p+1}(z) e^{-\lambda_{p+1}(t-t_n)}$$

$$+ b_p\phi_p(z) e^{-\lambda_p(t-t_n)} + c_{p-1}\phi_{p-1}(z) e^{-\lambda_{p-1}(t-t_n)}\} \tag{4.4}$$

which is again of the form (4.1).

If we rearrange terms in (4.3), we get (4.2). □

In (4.2) let's adopt the convention that $\alpha_{np} = 0$ whenever $p > n$ or $n < 0$. Then the equation holds for any n and p. Observe that when $n = p$, we have

$$\alpha_{nn} = e^{-\lambda_n(t-t_n)} a_n \alpha_{n-1,n-1}$$

which can be solved immediately to yield

$$\alpha_{nn}(\tau_1, \tau_2, \ldots, \tau_n) = \prod_{k=1}^{n} a_k e^{-\lambda_k \tau_k}$$

and that in turn can be used to solve for α_{nn-1}, etc. It is convenient to work with Laplace transforms and make a change in notation as follows:

$$\hat{\alpha}_p^{(v)}(s_1, s_2, \ldots, s_{p+v}) = \int_0^\infty \cdots \int_0^\infty e^{-(s_1\tau_1 + \ldots + s_{p+v}\tau_{p+v})}$$

$$\times \alpha_{p+v,p}(\tau_1, \tau_2, \ldots, \tau_{p+v})\, d\tau_1 \ldots d\tau_{p+v}. \tag{4.5}$$

Then, (4.2) becomes

$$\hat{\alpha}_p^{(v)}(s_1, s_2, \ldots, s_{p+v}) = \frac{1}{(s_{p+v}+\lambda_p)} \{a_p \hat{\alpha}_{p-1}^{(v)}(s_1, \ldots, s_{p+v-1})$$

$$+ b_p \hat{\alpha}_p^{(v-1)}(s_1, s_2, \ldots, s_{p+v-1})$$

$$+ c_p \hat{\alpha}_{p+1}^{(v-2)}(s_1, s_2, \ldots, s_{p+v-1})\} \tag{4.6}$$

which can be solved immediately to yield

$$\hat{\alpha}_p^{(0)} = \prod_{k=1}^{p} \frac{a_k}{(s_k+\lambda_k)}, \; \hat{\alpha}_0^{(0)} = 1 \tag{4.7}$$

verifying the result that we obtained earlier for α_{nn}.

The general solution for $\hat{\alpha}_p^{(v)}$ is given as follows.

THEOREM 4.2 *Let u_k, $b_k^{(v)}$ and $c_k^{(v)}$ be defined as follows:*

$$(k \geqq 1, \; v \geqq 1)$$

$$u_k = \prod_{j=1}^{k} \left(\frac{b_0}{s_j}\right) \tag{4.8}$$

$$b_k^{(v)} = \left(\frac{b_k}{s_{k+v}+\lambda_k}\right) \prod_{j=1}^{k} \left(\frac{s_{j+v}+\lambda_j}{s_{j+v-1}+\lambda_j}\right) \tag{4.9}$$

$$c_k^{(v)} = 0 \qquad v = 1 \tag{4.10}$$

$$= \frac{c_{k-1} a_k}{(s_{k+v-1}+\lambda_{k-1})(s_{k+v}+\lambda_k)} \prod_{j=1}^{k} \left(\frac{s_{j+v}+\lambda_j}{s_{j+v-2}+\lambda_j}\right) \qquad v \geqq 2.$$

For $v \geqq 1$, $p \geqq 0$ and $1 \leqq k \leqq p+1$, define a v-dimensional row vector $a_{pk}^{(v)}$ as follows:

$$a_{p1}^{(v)} = \left(b_1^{(v)}, u_v\left(\frac{s_v c_1^{(v)}}{u_{v-1}}\right), u_v\left(\frac{s_{v-1} c_1^{(v-1)}}{u_{v-2}}\right), \ldots, u_v\left(\frac{s_2 c_1^{(v)}}{u_1}\right)\right)$$

$$a_{pk}^{(v)} = (b_k^{(v)}, c_k^{(v)}, 0 \ldots 0), \qquad 2 \leqq k \leqq p$$

$$a_{pp+1}^{(v)} = (0, c_{p+1}^{(v)}, 0 \ldots 0).$$

Finally, define $v+1$ by v matrices

$$A_{pk}^{(v)}=\begin{bmatrix} a_{pk}^{(v)} \\ \delta_{pk} I_v \end{bmatrix} \tag{4.12}$$

where I_v is the $v\times v$ identity matrix.

Then, $\hat{\alpha}_p^{(v)}$ are given as follows:

$$\begin{bmatrix} \hat{\alpha}_p^{(v)} \\ \vdots \\ \hat{\alpha}_p^{(0)} \end{bmatrix} = \prod_{j=1}^{p} \frac{a_j}{(s_{j+v}+\lambda_j)}$$

$$\times\left[\sum_{k=0}^{v} u_k \left\{\sum_{m_v=1}^{p+1}\sum_{m_{v-1}=1}^{m_v+1}\cdots\sum_{m_{k+1}=1}^{m_{k+2}+1} A_{pm_v}^{(v)} A_{m_v m_{v-1}}^{(v-1)}\cdots A_{m_{k+2}m_{k+1}}^{(k+1)} 1_{k+1}\right\}\right] \tag{4.13}$$

when 1_k is the k-dimensional unit column vector.

Proof We begin by iterating (4.6) in p and get

$$\hat{\alpha}_p^{(v)}=\prod_{j=1}^{p}\frac{a_j}{(s_{j+v}+\lambda_j)}\hat{\alpha}_0^{(v)}+\sum_{m=1}^{p}\frac{1}{\prod_{j=1}^{m}\frac{a_j}{(s_{j+v}+\lambda_j)}}$$

$$\times\left(\frac{b_m}{s_{m+v}+\eta_m}\right)\hat{\alpha}_m^{(v-1)}+\left(\frac{c_m}{s_{m+v}+\lambda_m}\right)\hat{\alpha}_{m+1}^{(v-2)} \tag{4.14}$$

for $p\geqq 1$ and

$$\hat{\alpha}_0^{(v)}=\frac{b_0}{(s_v+\lambda_0)}\hat{\alpha}_0^{(v-1)}+\frac{c_0}{(s_v+\lambda_0)}\hat{\alpha}_1^{(v-a)} \tag{4.15}$$

Now, denote for $p\geqq 1$

$$\hat{\alpha}_p^{(v)}=\left[\prod_{j=1}^{p}\frac{a_j}{(s_{j+v}+\lambda_j)}\gamma_p^{(v)}.\right] \tag{4.15}$$

Then, we have

$$\gamma_p^{(\nu)}=\hat{\alpha}_0^{(\nu)}+\sum_{m=1}^{p}\frac{1}{\prod_{j=1}^{m}\frac{a_j}{(s_{j+\nu}+\lambda_j)}}\left\{\left(\frac{b_m}{s_{m+\nu}+\lambda_m}\right)\prod_{j=1}^{m}\frac{a_j}{(s_{j+\nu-1}+\lambda_j)}\gamma_m^{(\nu-1)}\right.$$

$$\left.+\left(\frac{c_m}{s_{m+\nu}+\lambda_m}\right)\prod_{j=1}^{m+1}\frac{a_j}{(s_{j+\nu-2}+\lambda_j)}\gamma_{m+1}^{(\nu-2)}\right\} \tag{4.16}$$

which simplifies to yield

$$\gamma_p^{(\nu)}=\hat{\alpha}_0^{(\nu)}+\sum_{m=1}^{p} b_m^{(\nu)}\gamma_m^{(\nu-1)}+\sum_{m=1}^{p+1} c_m^{(\nu)}\gamma_m^{(\nu-2)} \tag{4.17}$$

where $b_m^{(\nu)}$ and $c_m^{(\nu)}$ are as defined in (4.9) and (4.10).

Equation (4.15) can be iterated to yield

$$\hat{\alpha}_0^{(\nu)}=\frac{b_0}{\prod_{j=1}^{\nu}(s_j+\lambda_0)}+\sum_{k=0}^{\nu-2}\frac{c_0 a_1 b_0^{\nu-k-2}}{\prod_{j=k+1}^{\nu}(s_j+\lambda_0)}\left(\frac{s_{k+1}+\lambda_0}{s_{k+1}+\lambda_1}\right)\gamma_1^{(k)} \tag{4.18}$$

which is of the form

$$\hat{\alpha}_0^{(\nu)}=u_\nu+\sum_{k=0}^{\nu-2} s_{k+2}c_1^{(k+2)}\left(\frac{u_\nu}{u_{k+1}}\right)\gamma_1^{(k)}. \tag{4.19}$$

With the use of (4.19), we can now rewrite (4.16) in the form of

$$\begin{bmatrix}\gamma_p^{(\nu)}\\ \gamma_p^{(\nu-1)}\\ \vdots\\ \gamma_p^{(0)}\end{bmatrix}=\sum_{m=1}^{p+1}A_{pm}^{(\nu)}\begin{bmatrix}\gamma_m^{(\nu-1)}\\ \vdots\\ \vdots\\ \gamma_m^{(0)}\end{bmatrix}+u_\nu 1_{\nu+1} \tag{4.20}$$

where $A_{pm}^{(\nu)}$ are as defined by (4.12) and (4.11). Equation (4.20) can now be iterated in ν. With $\gamma_p^{(0)}=1$ we get

$$\begin{bmatrix} \gamma_p^{(\nu)} \\ \vdots \\ \gamma_p^{(0)} \end{bmatrix} = \sum_{k=0}^{\nu} u_k \Bigg[\sum_{m_\nu=1}^{p+1} \sum_{m_{\nu-1}=1}^{m_\nu+1} \cdots \sum_{m_{k+1}=1}^{m_{k+2}+1} \times \left\{ A_{pm_\nu}^{(\nu)} A_{m_\nu m_{\nu-1}}^{(\nu-1)} \cdots A_{m_{k+2}m_{k+1}}^{(k+1)} 1_{k+1} \right\} \Bigg]$$

whence the desired result (4.13) follows immediately using (4.15). □

5. THE SYMMETRIC CASE

There are some cases for which the polynomials $\phi_n(z)$ contain only even or odd terms according as n is even or odd respectively. This is the situation, for example, for Gegenbauer polynomials (which include both Chebyshev and Legendre polynomials), and most importantly for Hermite polynomials which correspond to Z_t being a Gaussian process. We shall refer to these cases collectively as the symmetric case.

For the symmetric case the coefficient b_k in the recurrence relationship (3.4) is necessarily zero for every k. It follows from (4.9) that $b_k^{(\nu)}$ are identically zero, and the result of Theorem 4.2 simplifies a great deal as is indicated as follows:

THEOREM 5.1 *For the symmetric case we have*

$$\alpha_p^{(2\nu+1)}=0$$

$$\alpha_p^{(2\nu)} = \prod_{j=1}^{p} \frac{a_j}{(s_{2\nu+j}+\lambda_j)} \left\{ \sum_{m_\nu=1}^{p+1} \sum_{m_{\nu-1}=1}^{m_\nu+1} \cdots \sum_{m_1=1}^{m_2+1} c_{m_\nu}^{(2\nu)} c_{m_{\nu-1}}^{(2\nu-2)} \cdots c_{m_1}^{(2)} \right\}. \quad (5.1)$$

Proof Since $b_k^{(\nu)} \equiv 0$, (4.17) becomes

$$\gamma_p^{(\nu)} = \sum_{m=2}^{p+1} c_m^{(\nu)} \gamma_m^{(\nu-2)} + \hat{\alpha}_0^{(\nu)} \quad (5.2)$$

and (4.18) now takes the form

$$\hat{\alpha}_0^{(\nu)} = \left(\frac{c_0}{s_\nu} \right) \hat{\alpha}_1^{(\nu-2)} \quad (5.3)$$

With the use of (5.2) for $\hat{\alpha}_0^{(\nu)}$, (5.2) can be rewritten as

$$\gamma_p^{(\nu)} = \sum_{m=1}^{p+1} c_m^{(\nu)} \gamma_m^{(\nu-2)} \tag{5.4}$$

where $c_m^{(\nu)}$ is given by (4.10). Since $\gamma_p^{(0)} = 1$ and $\gamma_p^{(1)} = 0$, we have $\gamma_p^{(\nu)} = 0$ for all ν odd, and

$$\gamma_p^{(2\nu)} = \sum_{m_\nu=1}^{p+1} \sum_{m_{\nu-1}=1}^{m_\nu+1} \sum_{m_1=1}^{m_2+1} c_{m_\nu}^{(2\nu)} c_{m_{\nu-1}}^{(2\nu-2)} \ldots c_{m_1}^{(2)} \tag{5.5}$$

whence (5.1) follows. □

It is interesting to note that in the Gaussian case (c.f. 33a) the terms $c_k^{(\nu)}$ are given by

$$c_k^{(\nu)} = \frac{k}{(s_{\nu-1}+1)(s_\nu+2)} \prod_{j=1}^{k-2} \left(\frac{s_{j+\nu}+j}{s_{j+\nu}+j+2} \right). \tag{5.6}$$

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