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ON THE MULTIDIMENSIONAL PREDICTION AND FILTERING PROBLEM AND THE FACTORIZATION OF SPECTRAL MATRICES

BY

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ABSTRACT

Since the linear prediction and filtering of multiple correlated stationary random signals involves the solution of a matrix integral equation, several methods of solution of this equation have been developed. In this paper the direct solution by complex variable techniques is extended from the one-dimensional case. The problem reduces to the factorization of a spectral density matrix and this factorization is the principal topic of the paper.

Under very general conditions, a factorization procedure has been developed by Wiener and Masani (**1**)[3] for discrete time series. Their technique is applied here with some modifications to the factorization of the spectral density matrices of continuous processes. A simplified procedure is developed also for the case where the elements of the spectral density matrix are rational functions of frequency. Examples illustrate the general technique and the simplified procedure.

INTRODUCTION

In the theory of the linear prediction and filtering of correlated stationary random signals, the form of the desired multidimensional network is specified by the solution of a matrix integral equation (**2**)

$$\int_0^\infty \mathbf{R}(\tau - \sigma)\tilde{\mathbf{h}}(\sigma)d\sigma = \mathbf{R}_m(\tau), \qquad 0 \leq \tau \leq \infty, \tag{1}$$

where $\mathbf{R}(\tau - \sigma)$ is a matrix containing as elements the covariance functions of the received waveforms $e_j(t)$, that is,

$$R_{jk}(\tau) = E\{e_j(t)e_k(t+\tau)\}, \tag{2}$$

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Further, if $m_j(t)$ are the desired outputs, then the outputs of the optimum network are given by

$$m_j^*(t) = \sum_k \int_0^\infty h_{jk}(\tau)e_k(t-\tau)d\tau, \tag{3}$$

and the elements of $\mathbf{R}_m(\tau)$ by

$$R_{mjk}(\tau) = E\{e_j(t)m_k(t+\tau)\}. \tag{4}$$

Thus the matrices occurring in Eq. 1 are defined by Eqs. 2, 3, and 4, with the notation $\sim$ denoting transpose in Eq. 1.

The method of undetermined coefficients was proposed originally by Wiener (2) for the solution of Eq. 1. An alternate method of solution (3) is to reduce Eq. 1 to a system of simultaneous differential equations with constant coefficients, and represents a generalization of the work of Zadeh and Ragazzini (4). It is the purpose of this paper to call attention to a third technique which may not be widely known.

A MATRIX WIENER-HOPF APPROACH

In a manner analogous to the one-dimensional case (2, 5) Eq. 1 can be rewritten as

$$\int_0^\infty \mathbf{R}(\tau-\sigma)\tilde{\mathbf{h}}(\sigma)d\sigma - \mathbf{R}_m(\tau) = \mathbf{f}(\tau), \qquad -\infty \le \tau \le \infty, \tag{5}$$

with

$$\mathbf{f}(\tau) = 0, \qquad \tau > 0. \tag{6}$$

The Fourier transforms of $\mathbf{R}(\tau)$, $\mathbf{h}(\tau)$, $\mathbf{R}_m(\tau)$ and $\mathbf{f}(\tau)$ may be denoted by $\mathbf{\Phi}(i\omega)$, $\mathbf{H}(i\omega)$, $\mathbf{\Phi}_m(i\omega)$ and $\mathbf{F}(i\omega)$, respectively, for example,

$$\mathbf{\Phi}(i\omega) \triangleq \int_{-\infty}^{\infty} e^{-i\omega\tau}\mathbf{R}(\tau)d\tau.$$

Taking transforms of both sides of Eq. 5 results in

$$\mathbf{\Phi}(i\omega)\tilde{\mathbf{H}}(i\omega) - \mathbf{\Phi}_m(i\omega) = \mathbf{F}(i\omega). \tag{7}$$

It follows from Eq. 6 that $\mathbf{F}(i\omega)$ is analytic and bounded in the upper half ω-plane (6).

Now, let $\mathbf{\Phi}(i\omega)$ be factored into the form

$$\mathbf{\Phi}(i\omega) = \mathbf{\Psi}^\dagger(i\omega)\mathbf{\Psi}(i\omega), \tag{8}$$

where

$$\mathbf{\Psi}^\dagger(i\omega) \triangleq \tilde{\mathbf{\Psi}}(-i\omega). \tag{9}$$

The factorization given by Eq. 8 is to be such that the matrix $\mathbf{\Psi}(i\omega)$ is element-by-element analytic and bounded in the lower half ω-plane, and its determinant Det $\mathbf{\Psi}(i\omega)$ is free of zeros in the lower half ω-plane. These two properties of $\mathbf{\Psi}(i\omega)$ imply that $\mathbf{\Psi}(i\omega)$ and its inverse $\mathbf{\Psi}^{-1}(i\omega)$ both represent transfer functions of physically realizable multidimensional networks.[4] The problem of realizing the factorization of Eq. 8 will be deferred until the next section.

Multiplying Eq. 7 from the left by $[\mathbf{\Psi}^{\dagger}(i\omega)]^{-1}$ yields

$$\mathbf{\Psi}(i\omega)\tilde{\mathbf{H}}(i\omega) - [\mathbf{\Psi}^{\dagger}(i\omega)]^{-1}\mathbf{\Phi}_m(i\omega) = [\mathbf{\Psi}^{\dagger}(i\omega)]^{-1}\mathbf{F}(i\omega). \tag{10}$$

Now, since both $\mathbf{\Psi}(i\omega)$ and $\tilde{\mathbf{H}}(i\omega)$ are analytic and bounded in the lower half ω-plane, the time response of $\mathbf{\Psi}(i\omega)\tilde{\mathbf{H}}(i\omega)$ must vanish for negative time, that is,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathbf{\Psi}(i\omega)\tilde{\mathbf{H}}(i\omega)]e^{i\omega\tau}d\omega = 0, \qquad \tau < 0. \tag{11}$$

Similarly, it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{[\mathbf{\Psi}^{\dagger}(i\omega)]^{-1}\,\mathbf{F}(i\omega)\}e^{i\omega\tau}d\omega = 0, \qquad \tau > 0. \tag{12}$$

Therefore, the time response of $\mathbf{\Psi}(i\omega)\tilde{\mathbf{H}}(i\omega)$ must be equal to the non-negative-time portion of the time response of $[\mathbf{\Psi}^{\dagger}(i\omega)]^{-1}\mathbf{\Phi}_m(i\omega)$. This fact can be expressed by the relationship

$$\mathbf{\Psi}(i\omega)\tilde{\mathbf{H}}(i\omega) = \frac{1}{2\pi}\int_0^{\infty}e^{-i\omega\tau}\left\{\int_{-\infty}^{\infty}e^{iu\tau}[\mathbf{\Psi}^{\dagger}(iu)]^{-1}\mathbf{\Phi}_m(iu)du\right\}d\tau. \tag{13}$$

From Eq. 13, the transposed transfer matrix $\tilde{\mathbf{H}}(i\omega)$ is given by

$$\tilde{\mathbf{H}}(i\omega) = \mathbf{\Psi}^{-1}(i\omega)\int_0^{\infty}e^{-i\omega\tau}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iu\tau}[\mathbf{\Psi}^{\dagger}(iu)]^{-1}\mathbf{\Phi}_m(iu)du\right\}d\tau, \tag{14}$$

which represents a generalization of the one-dimensional solution (5).

THE FACTORIZATION OF THE SPECTRAL DENSITY MATRIX

It can be seen that an explicit solution of Eq. 1 can be found provided that the factorization (Eq. 8) of the spectral density matrix $\mathbf{\Phi}(i\omega)$ can be effected. One factorization technique has been developed recently by Youla (7). Youla's technique is applicable whenever $\mathbf{\Phi}(i\omega)$ is element-by-element a rational function of ω. An earlier factorization procedure was developed by Wiener and Masani (1) for discrete time

[4] Thus $\mathbf{\Psi}(i\omega)$ may be said to be the transfer function of a minimum-phase multi-dimensional network.

series. Their technique is applied here with some modifications to the factorization of the spectral density matrices of continuous processes. The procedure will now be explained and illustrated.

The matrix $\mathbf{\Phi}(i\omega)$ to be factored is assumed to satisfy the following conditions:

(1) $\mathbf{\Phi}(i\omega)$ is Hermitian on the real ω-axis, that is,

$$\mathbf{\Phi}(i\omega) = \mathbf{\Phi}^{\dagger}(i\omega) \tag{15}$$

for ω real.

(2) The determinant Det $\mathbf{\Phi}(i\omega)$ satisfies the inequality

$$\int_{-\infty}^{\infty} \frac{|\ln|\mathrm{Det}\,\mathbf{\Phi}(i\omega)||}{k^2 + \omega^2}\, d\omega < \infty, \tag{16}$$

where $|\ |$ denotes the absolute value, and k is any real finite positive constant (see Appendix).

(3) Let the smallest and largest eigenvalues of $\mathbf{\Phi}(i\omega)$ on the real ω-axis be λ_1 and λ_2, respectively, then

$$0 < \lambda_1 < \lambda_2 < \infty. \tag{17}$$

Conditions (1) and (2) are satisfied for all physical processes. Condition (3) is satisfied if the spectral density matrix $\mathbf{\Phi}(i\omega)$ is determined experimentally, as shown by Wiener and Masani (1), but need not be satisfied by theoretically postulated spectral density matrices.

We note that the diagonal terms $\Phi_{jj}(i\omega)$ of the matrix $\mathbf{\Phi}(i\omega)$ represent auto-spectral densities, and can be factored easily into the form

$$\Phi_{jj}(\omega) = \Psi_{jj}(-i\omega)\Psi_{jj}(i\omega), \tag{18}$$

where the $\Psi_{jj}(i\omega)$ are analytic, bounded and free of zeros in the lower half ω-plane. Now let $\mathbf{\Gamma}(i\omega)$ be a diagonal matrix with elements

$$\Gamma_{jk}(i\omega) \triangleq \frac{1}{\Psi_{jj}(i\omega)}\,\delta_{jk}. \tag{19}$$

It is apparent that the matrix $\mathbf{\Phi}'(i\omega)$, defined by

$$\mathbf{\Phi}'(i\omega) = \mathbf{\Gamma}^{\dagger}(i\omega)\mathbf{\Phi}(i\omega)\mathbf{\Gamma}(i\omega), \tag{20}$$

has diagonal terms which are each equal to unity. Let the matrix $\mathbf{M}(i\omega)$ be obtained by subtracting a unit matrix $\mathbf{1}$ from $\mathbf{\Phi}'(i\omega)$; that is,

$$\mathbf{M}(i\omega) = \mathbf{\Phi}'(i\omega) - \mathbf{1}. \tag{21}$$

The matrix $\mathbf{M}(i\omega)$ can be written in the form

$$\mathbf{M}(i\omega) = \mathbf{M}_0 + \mathbf{M}_+(i\omega) + \mathbf{M}_-(i\omega), \tag{22}$$

where $\mathbf{M}_0$ is a constant matrix, $\mathbf{M}_+(i\omega)$ is element-by-element analytic and bounded in the lower half ω-plane and $\mathbf{M}_-(i\omega)$ is element-by-element analytic and bounded in the upper half ω-plane. It is recognized that these requirements do not uniquely determine the decomposition (18). However, it is shown in the Appendix that different decompositions satisfying the analytic requirements yield equivalent results. In practice it is frequently convenient to add the requirement that

$$\mathbf{M}_+(\infty) = \mathbf{M}_-(\infty) = 0, \tag{23}$$

which makes the decomposition (18) unique. Now let the matrix $\mathbf{N}(i\omega)$ be defined by the series

$$\mathbf{N}(i\omega) \triangleq \mathbf{1} - \mathbf{M}_+ + (\mathbf{M}\mathbf{M}_+)_+ - [\mathbf{M}(\mathbf{M}\mathbf{M}_+)_+]_+ + \cdots, \tag{24}$$

where the subscript $+$ has the same meaning as in Eq. 22. It follows immediately from the work of Wiener and Masani (1) that

(1) The matrices $\mathbf{N}(i\omega)$ and $\mathbf{N}^{-1}(i\omega)$ are element-by-element analytic and bounded in the lower half ω-plane.

(2) The matrix $\mathbf{G}$, defined by

$$\mathbf{G} \triangleq \mathbf{N}^\dagger(i\omega)\mathbf{\Phi}'(i\omega)\mathbf{N}(i\omega), \tag{25}$$

is a constant symmetric matrix.

Furthermore, it is seen from Eq. 24 that

$$\mathbf{N}^\dagger(\infty) = \mathbf{N}(\infty) = \mathbf{1}, \tag{26}$$

if Eq. 23 applies. Therefore,

$$\mathbf{G} = \mathbf{\Phi}'(\infty) \tag{27}$$

since $\mathbf{G}$ is a constant matrix. Thus $\mathbf{G}$ can be found without performing the matrix multiplications implied by Eq. 25.

From Eqs. 20 and 25, $\mathbf{\Phi}(i\omega)$ can be written as

$$\mathbf{\Phi}(i\omega) = [\mathbf{\Gamma}^\dagger(i\omega)]^{-1}[\mathbf{N}^\dagger(i\omega)]^{-1}\mathbf{G}\mathbf{N}^{-1}(i\omega)\mathbf{\Gamma}^{-1}(i\omega). \tag{28}$$

All that remains to complete the factorization is to factor $\mathbf{G}$. This factorization is easily accomplished since $\mathbf{G}$ is a constant matrix. For example, $\mathbf{G}$ may be diagonalized by unitary operations and the square root taken of the resultant diagonal matrix. However, there is no need to proceed further than Eq. 28. Using arguments similar to those leading to Eq. 14, it is easy to show that, instead of Eq. 14, the solution can be expressed as

$$\tilde{\mathbf{H}}(i\omega) = \mathbf{\Gamma}(i\omega)\mathbf{N}(i\omega)\mathbf{G}^{-1}\int_0^\infty d\tau e^{-i\omega\tau} \times \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu\tau}\mathbf{N}^\dagger(iu)\mathbf{\Gamma}^\dagger(iu)\mathbf{\Phi}_m(iu)du\right\}. \quad (29)$$

A SIMPLIFIED PROCEDURE FOR RATIONAL SPECTRA

Since Eq. 24 is an infinite series, $\mathbf{N}(i\omega)$ cannot always be obtained in closed form. If $\mathbf{\Phi}'(i\omega)$ is element-by-element a rational function of $i\omega$, an alternate procedure yields $\mathbf{N}(i\omega)$ in a finite number of steps.

To derive this procedure, $\mathbf{\Phi}'(i\omega)$ is written as

$$\mathbf{\Phi}'(i\omega) = \mathbf{1} + \mathbf{M}_0 + \mathbf{M}_+(i\omega) + \mathbf{M}_-(i\omega). \quad (30)$$

It should be noted that the successive terms in the series (24) for $\mathbf{N}(i\omega)$ can be written more explicitly; for example,

$$(\mathbf{M}\mathbf{M}_+)_+ = \mathbf{M}_0\mathbf{M}_+ + \mathbf{M}_+\mathbf{M}_+ + (\mathbf{M}_-\mathbf{M}_+)_+. \quad (31)$$

If the product $\mathbf{\Phi}'(i\omega)\mathbf{N}(i\omega)$ is formed and use is made of Eqs. 30 and 31, it is found that most terms cancel and the product becomes

$$\mathbf{\Phi}'(i\omega)\mathbf{N}(i\omega) = \mathbf{1} + \mathbf{M}_0 + [\mathbf{M}_-(i\omega)\mathbf{N}(i\omega)]_-. \quad (32)$$

From Eq. 32, $\mathbf{N}(i\omega)$ can be written formally as

$$\mathbf{N}(i\omega) = [\mathbf{\Phi}'(i\omega)]^{-1}\{\mathbf{1} + \mathbf{M}_0 + [\mathbf{M}_-(i\omega)\mathbf{N}(i\omega)]_-\}. \quad (33)$$

Since $\mathbf{N}(i\omega)$ is analytic and bounded in the lower half ω-plane and $[\mathbf{M}_-(i\omega)\mathbf{N}(i\omega)]_-$ is analytic and bounded in the upper half ω-plane, the poles of $\mathbf{N}(i\omega)$ must result from the upper half ω-plane poles of $[\mathbf{\Phi}'(i\omega)]^{-1}$.

Let $[\mathbf{\Phi}'(i\omega)]^{-1}$ and $\mathbf{M}_-(i\omega)$ be written as

$$[\mathbf{\Phi}'(i\omega)]^{-1} = [\mathbf{\Phi}'(\infty)]^{-1} + \sum_{m=1}^{K}\mathbf{A}_m(\sigma_m + i\omega)^{-1} + \sum_{m=1}^{K}\tilde{\mathbf{A}}_m(\sigma_m - i\omega)^{-1}, \quad (34)$$

and

$$\mathbf{M}_-(i\omega) = \sum_{n=1}^{L}\mathbf{B}_n(\tau_n - i\omega)^{-1}. \quad (35)$$

From Eq. 34 it is apparent that $\mathbf{N}(i\omega)$ can be written as

$$\mathbf{N}(i\omega) = \mathbf{1} + \sum_{m=1}^{K} \mathbf{C}_m(\sigma_m + i\omega)^{-1}. \tag{36}$$

Substituting Eqs. 34 through 36 into Eq. 33 and equating coefficients of the upper half ω-plane poles yields

$$\mathbf{C}_m = \mathbf{A}_m(\mathbf{1} + \mathbf{M}_0) + \mathbf{A}_m \sum_{n=1}^{L} \mathbf{B}_n(\sigma_m + \tau_n)^{-1}$$
$$+ \mathbf{A}_m \sum_{n=1}^{L} \sum_{k=1}^{K} \mathbf{B}_n\mathbf{C}_k(\sigma_k + \tau_n)^{-1}(\sigma_m + \tau_n)^{-1}, \quad m = 1, 2, \cdots, K. \tag{37}$$

Equation 37 represents a set of K matrix equations from which the K unknown constant matrices $\mathbf{C}_m$ can be found.

Example I

A two-dimensional prediction problem is considered in some detail. Let two correlated signals $e_1(\tau)$ and $e_2(\tau)$ be received in the interval $-\infty \leq \tau \leq t$. It is desired to estimate $e_1(t + \alpha)$ and $e_2(t + \alpha)$ for $\alpha > 0$. The spectral density matrix of $e_1(\tau)$ and $e_2(\tau)$ is given as[5]

$$\mathbf{\Phi}(i\omega) = \begin{bmatrix} \dfrac{1}{1 + \omega^2} & \dfrac{\epsilon}{(1 - i\omega)^2} \\ \dfrac{\epsilon}{(1 + i\omega)^2} & \dfrac{1}{1 + \omega^2} \end{bmatrix}, \qquad |\epsilon| < 1. \tag{38}$$

The matrix $\mathbf{\Phi}_m(i\omega)$ for this case is simply

$$\mathbf{\Phi}_m(i\omega) = e^{i\omega\alpha}\mathbf{\Phi}(i\omega). \tag{39}$$

From Eq. 20 it is found that

$$\mathbf{\Phi}'(i\omega) = \begin{bmatrix} 1 & \epsilon\dfrac{1 + i\omega}{1 - i\omega} \\ \epsilon\dfrac{1 - i\omega}{1 + i\omega} & 1 \end{bmatrix}. \tag{40}$$

In this example the power spectral density matrix is a rational function of $i\omega$; hence the simplified procedure previously described can be applied. For comparison the problem is solved using first the general approach and then this simplified procedure.

[5] This example appeared in (2) and was solved by the method of undetermined coefficients.

The unit matrix **1** may be subtracted from Eq. 40 to yield

$$\mathbf{M}(i\omega) = \begin{bmatrix} 0 & \epsilon\dfrac{1+i\omega}{1-i\omega} \\ \epsilon\dfrac{1-i\omega}{1+i\omega} & 0 \end{bmatrix}, \tag{41}$$

and, finally,

$$\mathbf{M}_{+}(i\omega) = \begin{bmatrix} 0 & 0 \\ \dfrac{2\epsilon}{1+i\omega} & 0 \end{bmatrix}. \tag{42}$$

For this simple problem the series (24) for $\mathbf{N}(\omega)$ terminates after two terms, and the result is

$$\mathbf{N}(i\omega) = \begin{bmatrix} 1 & 0 \\ \dfrac{-2\epsilon}{1+i\omega} & 1 \end{bmatrix}. \tag{43}$$

From Eq. 25 the matrix **G** is given by

$$\mathbf{G} = \begin{bmatrix} 1 & \dfrac{-2\epsilon}{1-i\omega} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon\dfrac{1+i\omega}{1-i\omega} \\ \epsilon\dfrac{1-i\omega}{1+i\omega} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{-2\epsilon}{1+i\omega} & 1 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{bmatrix}. \tag{44}$$

It is seen that **G** is indeed a constant matrix. From Eq. 29 and 39 the solution is found to be

$$\tilde{\mathbf{H}}(i\omega) = \mathbf{\Gamma}(i\omega)\mathbf{N}(i\omega)\int_0^\infty d\tau e^{-i\omega\tau}\left[\frac{1}{2\pi}\int_{-\infty}^\infty e^{iu\tau}\mathbf{N}^{-1}(iu)\mathbf{\Gamma}^{-1}(iu)du\right] \tag{45}$$

$$= e^{-\alpha}\begin{bmatrix} 1 & 0 \\ 2\epsilon\alpha & 1 \end{bmatrix}.$$

Following the simplified procedure of the previous section, Eqs. 30 and 40 are used to obtain

$$\mathbf{\Phi}'(\infty) = \mathbf{1} + \mathbf{M}_0 = \begin{bmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{bmatrix}. \tag{46}$$

The matrix given by Eq. 40 may be inverted to yield

$$[\mathbf{\Phi}'(i\omega)]^{-1} = \frac{1}{1-\epsilon^2}\begin{bmatrix} 1 & -\epsilon\dfrac{1+i\omega}{1-i\omega} \\ -\epsilon\dfrac{1-i\omega}{1+i\omega} & 1 \end{bmatrix}. \tag{47}$$

A comparison of Eqs. 34 and 47 shows that $\sigma_1 = 1$ and that

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ \dfrac{-2\epsilon}{1-\epsilon^2} & 0 \end{bmatrix}. \tag{48}$$

From Eq. 41, $\mathbf{M}_-(i\omega)$ is given by

$$\mathbf{M}_-(i\omega) = \begin{bmatrix} 0 & \dfrac{2\epsilon}{1-i\omega} \\ 0 & 0 \end{bmatrix}, \tag{49}$$

and Eq. 35 yields $\tau_1 = 1$ and

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 2\epsilon \\ 0 & 0 \end{bmatrix}. \tag{50}$$

Substitution of Eqs. 46, 48, and 50 into Eq. 37 yields a matrix equation which may be solved for $\mathbf{C}_1$ to give

$$\mathbf{C}_1 = \begin{bmatrix} 0 & 0 \\ -2\epsilon & 0 \end{bmatrix}. \tag{51}$$

From Eq. 36, $\mathbf{N}(i\omega)$ may be written as

$$\mathbf{N}(i\omega) = \mathbf{1} + \begin{bmatrix} 0 & 0 \\ -2\epsilon & 0 \end{bmatrix} \frac{1}{1+i\omega}, \tag{52}$$

which is identical to Eq. 43 obtained by the general procedure.

Example II

In this example a somewhat more complicated spectral density matrix will be factored. Let this matrix $\mathbf{\Phi}(i\omega)$ be

$$\mathbf{\Phi}(i\omega) = \begin{bmatrix} \dfrac{\alpha^2}{\alpha^2 + \omega^2} & \dfrac{\epsilon\alpha\beta}{(\alpha + i\omega)(\beta - i\omega)} \\ \dfrac{\epsilon\alpha\beta}{(\alpha - i\omega)(\beta + i\omega)} & \dfrac{\beta^2}{\beta^2 + \omega^2} \end{bmatrix}, \qquad |\epsilon| < 1. \tag{53}$$

The matrix $\mathbf{\Gamma}(i\omega)$ of Eq. 19 is found, by factoring the diagonal terms, to be

$$\mathbf{\Gamma}(i\omega) = \begin{bmatrix} \dfrac{\alpha + i\omega}{\alpha} & 0 \\ 0 & \dfrac{\beta + i\omega}{\beta} \end{bmatrix}. \tag{54}$$

From Eqs. 20 and 54, $\mathbf{\Phi}'(i\omega)$ is found to be

$$\mathbf{\Phi}'(i\omega) = \begin{bmatrix} 1 & \epsilon\dfrac{\alpha - i\omega}{\alpha + i\omega}\dfrac{\beta + i\omega}{\beta - i\omega} \\ \epsilon\dfrac{\alpha + i\omega}{\alpha - i\omega}\dfrac{\beta - i\omega}{\beta + i\omega} & 1 \end{bmatrix}. \tag{55}$$

Thus $\mathbf{M}(i\omega)$ of Eq. 21 is given by

$$\mathbf{M}(i\omega) = \begin{bmatrix} 0 & \epsilon\dfrac{\alpha - i\omega}{\alpha + i\omega}\dfrac{\beta + i\omega}{\beta - i\omega} \\ \epsilon\dfrac{\alpha + i\omega}{\alpha - i\omega}\dfrac{\beta - i\omega}{\beta + i\omega} & 0 \end{bmatrix}. \tag{56}$$

From Eqs. 22, 23, and 56 the matrix $\mathbf{M}_+(i\omega)$ is

$$\mathbf{M}_+(i\omega) = \begin{bmatrix} 0 & \epsilon\dfrac{2\alpha}{\alpha + i\omega}\dfrac{\beta - \alpha}{\beta + \alpha} \\ \epsilon\dfrac{2\beta}{\beta + i\omega}\dfrac{\alpha - \beta}{\alpha + \beta} & 0 \end{bmatrix}. \tag{57}$$

The series (24) for $\mathbf{N}(i\omega)$ in this example is a geometric series, and can be evaluated to yield

$$\mathbf{N}(i\omega) = \begin{bmatrix} 1 + \dfrac{k_1}{\alpha + i\omega} & k_2 \dfrac{\alpha}{\alpha + i\omega} \\ -k_2 \dfrac{\beta}{\beta + i\omega} & 1 - \dfrac{k_1}{\beta + i\omega} \end{bmatrix}, \tag{58}$$

where the constants k_1 and k_2 are given by

$$k_1 = \frac{4\epsilon^2\alpha\beta(\alpha - \beta)}{(\alpha + \beta)^2 - 4\epsilon^2\alpha\beta}, \tag{59}$$

and

$$k_2 = \frac{2\epsilon(\alpha^2 - \beta^2)}{(\alpha + \beta)^2 - 4\epsilon^2\alpha\beta}. \tag{60}$$

Application of Eq. 27 yields immediately

$$\mathbf{G} = \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix}. \tag{61}$$

The matrix **G** could be obtained also from Eq. 25. Since this example has a rational spectral density, it could be solved by the simplified procedure previously discussed.

APPENDIX

In order that continuous processes may be brought within the framework of the work of Wiener and Masani (1), a mapping of the ω-plane into some z-plane must be effected such that the real axis of the ω-plane is transformed into the unit circle in the z-plane, the upper half ω-plane into the region outside the unit circle in the z-plane and the lower half ω-plane into the region inside the unit circle. Such a mapping can be obtained by the transformation

$$z = \frac{ik + \omega}{ik - \omega}, \qquad 0 < k < \infty. \tag{62}$$

Physically, the constant k can be interpreted as a scale factor which translates the dimensionless time scale of the discrete case into the time scale of the continuous case. It will be shown that the transformation (62) also serves to give a precise definition to the decomposition (22).

Let $F(z)$ be a function which can be represented by the expansion

$$F(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{63}$$

Then the functions $F_+(z)$, $F_-(z)$, and F_0 are defined as

$$F_0 \triangleq a_0 \tag{64}$$

$$F_+(z) \triangleq \sum_{n=1}^{\infty} a_n z^n \tag{65}$$

$$F_-(z) \triangleq \sum_{n=1}^{\infty} a_{-n} z^{-n}. \tag{66}$$

It is apparent from Eqs. 65 and 66 that

$$F_+(0) = F_-(\infty) = 0. \tag{67}$$

From Eq. 62 it is seen that the points $z = 0,\ \infty$ correspond to $\omega = -ik,\ ik$, respectively. Therefore, the condition to be imposed on the decomposition (22) is

$$\mathbf{M}_+(-ik) = \mathbf{M}_-(ik) = 0, \qquad 0 < k < \infty. \tag{68}$$

It may appear from Eq. 68 that the condition (23) is inapplicable, since k is restricted to be finite. This, however, is not the case. Consider the alternative decompositions of $\mathbf{M}(i\omega)$

$$\mathbf{M}(i\omega) = \mathbf{M}_+(i\omega) + \mathbf{M}_-(i\omega) + \mathbf{M}_0 = \mathbf{M}^+(i\omega) + \mathbf{M}^-(i\omega) + \mathbf{M}^0, \tag{69}$$

where $\mathbf{M}_+$ and $\mathbf{M}^+$ are analytic in the lower half ω-plane, $\mathbf{M}_-$ and $\mathbf{M}^-$ analytic in the upper half ω-plane and $\mathbf{M}_0$ and $\mathbf{M}^0$ are constants. Furthermore, require that

$$\mathbf{M}_+(-ik) = \mathbf{M}_-(ik) = 0, \qquad 0 < k < \infty, \tag{70}$$

and

$$\mathbf{M}^+(\infty) = \mathbf{M}^-(\infty) = 0. \tag{71}$$

It is apparent that $\mathbf{M}^+$ and $\mathbf{M}_+$, thus defined, differ only by an additive matrix constant. Now let $\mathbf{N}(i\omega)$ be given by

$$\mathbf{N}(i\omega) = 1 - \mathbf{M}_+ + (\mathbf{M}\mathbf{M}_+)_+ - [\mathbf{M}(\mathbf{M}\mathbf{M}_+)_+]_+ + \cdots, \tag{72}$$

and let $\mathbf{N}'(i\omega)$ be given by

$$\mathbf{N}'(i\omega) = 1 - \mathbf{M}^+ + (\mathbf{M}\mathbf{M}^+)^+ - [\mathbf{M}(\mathbf{M}\mathbf{M}^+)^+]^+ + \cdots. \tag{73}$$

Equation 73 can be rewritten as

$$\begin{aligned}\mathbf{N}'(i\omega) &= 1 - (\mathbf{M}_+ + \mathbf{K}_1) + \{[\mathbf{M}(\mathbf{M}_+ + \mathbf{K}_1)]_+ + \mathbf{K}_2\} \cdots \\ &= \{1 - \mathbf{M}_+ + (\mathbf{M}\mathbf{M}_+)_+ - [\mathbf{M}(\mathbf{M}\mathbf{M}_+)_+]_+ + \cdots\}\{1 - \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3 + \cdots\} \\ &= \mathbf{N}(i\omega)\mathbf{K}.\end{aligned} \tag{74}$$

Thus the matrices $\mathbf{N}'(i\omega)$ and $\mathbf{N}(i\omega)$ differ only by a multiplicative constant matrix. This difference does not affect the final result given by Eq. 29. To see that such is the case, instead of Eq. 29, write

$$\tilde{\mathbf{H}}'(i\omega) = \mathbf{\Gamma}(i\omega)\mathbf{N}'(i\omega)[\mathbf{G}']^{-1}\int_0^\infty d\tau e^{-i\omega\tau}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu\tau}[\mathbf{N}'(u)]^\dagger \mathbf{\Gamma}^\dagger(u)\mathbf{\Phi}_m(u)du\right\}. \tag{75}$$

From (74), it is found that

$$[\mathbf{N}'(i\omega)]^\dagger = \mathbf{K}^\dagger \mathbf{N}^\dagger(i\omega), \tag{76}$$

$$\begin{aligned}\mathbf{G}' &= [\mathbf{N}'(i\omega)]^\dagger \mathbf{\Phi}'(i\omega)\mathbf{N}'(i\omega) \\ &= \mathbf{K}^\dagger \mathbf{N}^\dagger(i\omega)\mathbf{\Phi}'(i\omega)\mathbf{N}(\omega)\mathbf{K} \\ &= \mathbf{K}^\dagger \mathbf{G}\mathbf{K},\end{aligned} \tag{77}$$

and

$$[\mathbf{G}']^{-1} = \mathbf{K}^{-1}\mathbf{G}^{-1}[\mathbf{K}^\dagger]^{-1}. \tag{78}$$

Therefore,

$$\begin{aligned}\tilde{\mathbf{H}}'(i\omega) &= \mathbf{\Gamma}(i\omega)\mathbf{N}(i\omega)\mathbf{K}\mathbf{K}^{-1}\mathbf{G}^{-1}[\mathbf{K}^\dagger]^{-1} \\ &\quad\times \int_0^\infty d\tau e^{-i\omega\tau}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu\tau}\mathbf{K}^\dagger \mathbf{N}^\dagger(iu)\mathbf{\Gamma}^\dagger(iu)\mathbf{\Phi}_m(iu)du\right\} \qquad (79) \\ &= \mathbf{\Gamma}(i\omega)\mathbf{N}(i\omega)\mathbf{G}^{-1}\int_0^\infty d\tau e^{-i\omega\tau}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu\tau}\mathbf{N}^\dagger(iu)\mathbf{\Gamma}^\dagger(iu)\mathbf{\Phi}_m(iu)du\right\} \\ &= \tilde{\mathbf{H}}(i\omega).\end{aligned}$$

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