An Extension of Stochastic Integrals in the Plane

Eugene Wong; Moshe Zakai

The Annals of Probability, Vol. 5, No. 5 (Oct., 1977), 770-778.

Stable URL:
http://links.jstor.org/sici?sici=0091-1798%28197710%295%3A5%3C770%3AAEOSII%3E2.0.CO%3B2-5

The Annals of Probability is currently published by Institute of Mathematical Statistics.

The Annals of Probability
1977, Vol. 5, No. 5, 770-778

AN EXTENSION OF STOCHASTIC INTEGRALS IN THE PLANE

BY EUGENE WONG[1] AND MOSHE ZAKAI

University of California, Berkeley and Technion-Israel Institute of Technology

For a Wiener process with a two-dimensional parameter $\{W_z, z \in R_+^2\}$, four types of stochastic integrals: $\int \phi\, dW$, $\int \psi\, dW\, dW$, $\int \psi\, dW\, dz$, $\int \psi\, dz\, dW$, have been defined under the condition

$$E \int \phi^2\, dz < \infty \qquad \text{and} \qquad E \int \psi^2\, dz\, dz' < \infty\,.$$

The main purpose of this note is to extend the definition of these stochastic integrals by replacing $E(\cdot) < \infty$ with $(\cdot) < \infty$ a.s. in these conditions. Our results are in fact even more general, allowing W to be replaced by a strong martingale with appropriate properties.

1. Introduction. For two points $a = (a_1, a_2)$, $b = (b_1, b_2)$ in the positive quadrant of the plane, we write $a \prec b$ if $a_1 \leqq b_1$ and $a_2 \leqq b_2$, and we write $a \curlywedge b$ if $a_1 \leqq b_1$ and $a_2 \geqq b_2$. R_b will denote the rectangle $\{a : a < b\}$. A family of σ-fields $(\mathscr{F}_z, z \in R_{z_0})$ is said to be increasing if $a \prec b \Rightarrow \mathscr{F}_a \subset \mathscr{F}_b$. A two-parameter stochastic process $(X_z, \mathscr{F}_z, z \in R_{z_0})$ is said to be a martingale if $E(X_b \mid \mathscr{F}_a) = X_a$, a.s., whenever $b \succ a$. One of the simplest examples of two-parameter martingales is the Wiener process ($\{W_z, z \in R_{z_0}\}$ is a Wiener process if it is Gaussian with zero mean, $E(W_a W_b) = \min(a_1, b_1) \cdot \min(a_2, b_2)$ and almost all its sample functions are continuous).

The following types of stochastic integral have been introduced recently:

$$\int_{R_z} \phi_\zeta\, dM_\zeta \tag{1}$$

$$\int\int_{R_z \times R_z} \psi_{\zeta,\zeta'}\, dM_\zeta\, dM_{\zeta'} \tag{2}$$

$$\int\int_{R_z \times R_z} \psi_{\zeta,\zeta'}\, d\mu_\zeta\, dM_{\zeta'} \left(\int\int \psi_{\zeta,\zeta'}\, dM_\zeta\, d\mu_{\zeta'}\right). \tag{3}$$

The first integral, (1), was introduced in [5], [1] and [12] for the case where M_z is a Wiener process and extended to general martingales in [2]. The second integral, (2) was introduced in [6] and extended to certain general martingales in [4]. An extension of (1) and (2) for N parameter Wiener or Poisson processes appears in [10]. The third integral, (3), was introduced in [7]. Applications of these integrals appear in the references cited and also in [8] and [9]. The stochastic integrals (1), (2), (3), were defined under conditions which (when M_z is

Received December 30, 1975; revised December 17, 1976.

[1] Research supported by the U.S. Army Research Office, Durham Grant DAHCO4-75-G-0189.

AMS 1970 *subject classifications*. Primary 60H05; Secondary 60G45.

Key words and phrases. Stochastic integral, random field, Wiener process, multiparameter process.

a Wiener process) become

$$E\int \phi^2\, d\zeta < \infty\,, \qquad E\int\int \psi^2_{\zeta,\zeta'}\, d\zeta\, d\zeta' < \infty\,, \qquad E\int\int \psi^2_{\zeta,\zeta'}\, d|\mu|_\zeta\, d\zeta' < \infty\,.$$

The purpose of this note is to extend the definition by replacing the requirement $E(\cdot) < \infty$ with $(\cdot) < \infty$ a.s. In the one-parameter case such an extension follows easily by a standard stopping argument; such an argument can still be used for (1), but in general it can not be used in the two (or multi) parameter case for the following reason: Let $T_z = 1$ if $\int_{R_z} \phi_\zeta{}^2\, d\zeta \leqq K$ and zero otherwise, then this does not imply that $\int_{R_{z_0}} T_\zeta \phi_\zeta{}^2\, d\zeta < \infty$ for $z \prec z_0$ (we used this wrong argument in an early draft of [7] and wish to thank R. Cairoli and J. B. Walsh for pointing our mistake out to us). The method of proof given here follows that of [4] (cf. also [3]). The results establish the existence of stochastic integrals and sample continuity along nondecreasing paths, but not sample continuity in the plane.

The extension of (1) will be based on Theorems 2.2 and 2.3 of [2], the extension of (2) will be based on Theorems 2.5 and 2.6 of [2], the extension of (3) will be based on Theorem 3.1 of [6]; it will be assumed that the reader is familiar with those theorems and the related definitions and results. The notation of [2] will be followed, and in addition we use:

(a) If $z_1 = (s_1, t_1)$ and $z_2 = (s_2, t_2)$ then $z_1 \otimes z_2$ will denote (s_1, t_2).

(b) If X is a real random variable then $N(X)$ will denote

$$N(X) = E\left[\frac{|X|}{1 + |X|}\right]. \tag{4}$$

Recall that ([2]): (a) $\mathscr{F}^1_{st} \triangleq \mathscr{F}_{s\infty}$; (b) for a right continuous square integrable martingale M_z; $[M]^1_{st}$ denotes the unique process satisfying the following conditions: for each fixed $t \leqq t_0$, $\{[M]^1_{st}, s \leqq s_0\}$ is a one-parameter increasing process which is predictable relative to the family $\{\mathscr{F}_{st}, s \leqq s_0\}$ and such that $\{M^2_{st} - [M]^1_{st}, s \leqq s_0\}$ is a one-parameter martingale.

2. Preliminaries.

LEMMA 1. *Let M_z, $z \prec z_0$, be a right continuous square integrable strong martingale, and let $\{\phi_z, z \prec z_0\}$, be $\mathscr{F}_z{}^1$ predictable and either simple or such that $E[\int_{R_{z_0}} \phi_z{}^2\, d[M]_z{}^1] < \infty$. Then if $(s, t) = z \prec z_0 = (s_0, t_0)$,*

$$N[\sup\nolimits_{0 \leqq s \leqq s_0} |\textstyle\int_{R_z} \phi_\zeta\, dM_\zeta|] \leqq 4N^{\frac{1}{3}}[[\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1]^{\frac{1}{2}}]\,. \tag{5}$$

PROOF. Let I_z, $T_z{}^\varepsilon$, $I_z{}^\varepsilon$ be as follows;

$$I_z = \int_{R_z} \phi_\zeta\, dM_\zeta$$
$$T_z{}^\varepsilon = 1\,, \qquad \text{if} \quad \int_{R_z} \phi_\zeta{}^2\, d[M]_\zeta{}^1 < \varepsilon^2$$
$$= 0\,, \qquad \text{otherwise.}$$

Note that for $z \prec z_0$, $T^\varepsilon_{\zeta\otimes z}$ is $\mathscr{F}^1$ predictable as a function of ζ; this allows us to define:

$$I_z{}^\varepsilon = \int_{R_z} \phi_\zeta T^\varepsilon_{\zeta\otimes z}\, dM_\zeta\,.$$

Then (Theorem 2.3 of [2]),

$$E\{(I_z)^2\} = E\{\textstyle\int_{R_z} \phi_\zeta{}^2 T^\varepsilon_{\zeta\otimes z}\, d[M]_\zeta{}^1 \leqq \varepsilon^2 \tag{6}$$

and $I^\varepsilon_{s,t}$ is, for a fixed t, a one-parameter right continuous martingale in s.

Therefore, for fixed t

$$\begin{aligned} P\{\sup_{0\leqq s\leqq s_0} |I_z| > \theta\} &\leqq P\{(\sup_{0\leqq s\leqq s_0} |I_z{}^\varepsilon| > \theta) \cap (T^\varepsilon_{s_0,t} = 1)\} \\ &\quad + P\{(\sup_{0\leqq s\leqq s_0} |I_z| > \theta) \cap (T^\varepsilon_{s_0,t} < 1)\} \\ &\leqq P\{\sup_{0\leqq s\leqq s_0} |I_z{}^\varepsilon| > \theta\} + P\{T^\varepsilon_{s_0,t} < 1\} \\ &\leqq \frac{\varepsilon^2}{\theta^2} + P\{\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M_\zeta{}^1 > \varepsilon^2\} \\ &\leqq \frac{\varepsilon^2}{\theta^2} + P\left\{\frac{[\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1]^{\frac12}}{1 + [\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1]^{\frac12}} > \frac{\varepsilon}{1+\varepsilon}\right\} \\ &\leqq \frac{\varepsilon^2}{\theta^2} \leqq \frac{1+\varepsilon}{\varepsilon} N[[\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1]^{\frac12}]\,. \end{aligned}$$

Now, for any $\theta \geqq 0$ and any rv X

$$N(X) \leqq \frac{\theta}{1+\theta} P(|X| \leqq \theta) + 1 \cdot P\{|X| > \theta\}\,,$$

hence

$$N(\sup_{0\leqq s\leqq s_0} |I_z|) \leqq \theta + \frac{\varepsilon^2}{\theta^2} + \frac{1+\varepsilon}{\varepsilon} N[[\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1]^{\frac12}]\,.$$

Setting $\theta = \varepsilon^{\frac12}$ and $\varepsilon = N^{\frac23}[[\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1]^{\frac12}]$, and since $N(X) \leqq 1$, we get (5).

LEMMA 2. *Under the assumptions of Lemma* 1, *and if* $[M]^1$ *as a Borel measure is dominated by* A, *where* A_ζ *is a deterministic increasing right continuous function, then*

$$P[\textstyle\int_{R_{z_0}} I_z{}^2\, d[M]_z{}^1 > \theta] \leqq \frac{A_{z_0}}{\theta^{\frac12}} + P\{\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1 \geqq \theta^{\frac12}\} \tag{7}$$

where $I_z = \int_{R_z} \phi_\zeta\, dM_\zeta$.

PROOF. As in the proof of Lemma 1:

$$\begin{aligned} P\{\textstyle\int_{R_{z_0}} I_\zeta{}^2\, d[M]_\zeta{}^1 > \alpha\} &\leqq P\{\textstyle\int (I^\varepsilon)^2\, d[M]_\zeta{}^1 > \alpha\} + P\{\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1 > \varepsilon^2\} \\ &\leqq \frac{A_{z_0}}{\alpha} + P\{\textstyle\int_{R_{z_0}} \phi_\zeta{}^2\, d[M]_\zeta{}^1 > \varepsilon^2\}\,. \end{aligned}$$

Setting $\alpha = \theta$, $\varepsilon^2 = \theta^{\frac12}$, we get (7).

LEMMA 3. *Let* M_z, $z \prec z_0$, *be a right continuous strong martingale for which* $EM^4_{z_0} < \infty$, *and such that for almost all* ω, $[M]^1$ *and* $[M^2]$ *as Borel measures are dominated by* A, *where* A_ζ *is deterministic increasing right continuous function. Let* $\psi_{\zeta,\zeta'}$ *be predictable and either simple or such that* $E \iint_{R_{z_0}\times R_{z_0}} \psi^2_{\zeta,\zeta'}\, dA_\zeta\, dA_{\zeta'} < \infty$, *and such that* $\psi_{\zeta,\zeta'} = 0$ *if* $\zeta \curlywedge \zeta'$ *is not satisfied. Then*

$$N[[\textstyle\int_{R_{z_0}} [\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, dM_{\zeta'})^2\, dA_\zeta]^{\frac12} \leqq 4N^{\frac13}[[\iint_{R_{z_0}\times R_{z_0}} \psi^2_{\zeta,\zeta'}\, dA_\zeta\, dA_{\zeta'}]^{\frac12}]\,. \tag{8}$$

PROOF. Let b, T_z^ε, b^ε be as follows:

$$b = \int_{R_{z_0}} [\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]^2\, dA_\xi$$
$$T_z^\varepsilon = 1\,, \qquad \text{if} \quad \iint_{R_z \times R_z} \psi^2_{\zeta,\zeta'}\, dA_\zeta\, dA_{\zeta'} < \varepsilon^2$$
$$= 0\,, \qquad \text{otherwise.}$$

Note that if $\zeta' \prec z_0$, $T^\varepsilon_{\zeta \otimes z}$ and $\psi_{\zeta,\zeta'}$, are, as functions of ζ', $\mathscr{F}^1$-predictable. This allows us to define

$$b^\varepsilon = \int_{R_{z_0}} [\int_{R_{z_0}} T^\varepsilon_{(\zeta' \otimes z_0)} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]^2\, dA_\zeta\,.$$

Since A is deterministic and $d[M]^1_{\zeta'} \leqq dA_{\zeta'}$

$$E(b^\varepsilon) = \int_{R_{z_0}} E\{\int_{R_{z_0}} T_{\zeta' \otimes z_0} \psi^2_{\zeta,\zeta'}\, d[M]^1_{\zeta'}\}\, dA_\xi$$
$$\leqq E\{\iint_{R_{z_0} \times R_{z_0}} T\psi^2\, dA\, dA\}\,.$$

Therefore $Eb^\varepsilon \leqq \varepsilon^2$ and, as in Lemma 1,

$$P(b^{\frac12} > \theta) \leqq P[(b^\varepsilon)^{\frac12} > \theta] + P(T^\varepsilon_{z_0} < 1) \leqq \frac{\varepsilon^2}{\theta^2} + P[\iint_{R_{z_0} \times R_{z_0}} \psi^2_{\zeta,\zeta'}\, dA_\zeta\, dA_{\zeta'} \geqq \varepsilon^2]$$

and the rest is the same as in Lemma 1.

LEMMA 4. *Under the assumptions of Lemma* 3:

(9) $$N[\sup_{0 \leqq t \leqq t_0} |\iint_{R_{(s_0,t)} \times R_{(s_0,t)}} \psi_{\zeta,\zeta'}\, dM_\zeta\, dM_{\zeta'}|]$$
$$\leqq 4N^{\frac13}[[\int_{R_{z_0}} [\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]^2\, dA_\zeta]^{\frac12}]$$
$$\leqq 7N^{\frac13}[[\iint_{R_{z_0} \times R_{z_0}} \psi^2_{\zeta,\zeta'}\, dA_\zeta\, dA_{\zeta'}]^{\frac12}]\,.$$

PROOF. The second line follows from the first line by Lemma 3. To porve the first line, let

$$a_z = \iint_{R_z \times R_z} \psi_{\zeta,\zeta'}\, dM_\zeta\, dM_{\zeta'}$$
$$S_z^\varepsilon = 1\,, \qquad \text{if} \quad \int_{R_z} [\int_{R_z} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]^2\, dA_\zeta \leqq \varepsilon^2$$
$$= 0\,, \qquad \text{otherwise}$$
$$a_z^\varepsilon = \int_{R_z} S_{(z \otimes \zeta)} [\int_{R_z} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]\, dM_\zeta\,.$$

Then,

(10) $$E(a_z^\varepsilon)^2 \leqq \varepsilon^2\,.$$

Therefore,

$$P\{\sup_{0 \leqq t \leqq t_0} |a_{(s_0,t)}| > \theta\} \leqq P\{\sup_{0 \leqq t \leqq t_0} |a^\varepsilon_{(s_0,t)}| > \theta\}$$
$$+ P\{\int_{R_{z_0}} [\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]^2\, dA_\zeta > \varepsilon^2\}\,.$$

By (10) and following the arguments of the proof of Lemma 1 we get (9).

LEMMA 5. *Under the assumptions of Lemma* 3

(11) $$P\{\int_{R_{z_0}} a_z^2\, dA_z > \theta\} \leqq A_{z_0} \theta^{-\frac12} + \theta^{-\frac14} + P\{\iint_{R_{z_0} \times R_{z_0}} \psi^2_{\zeta,\zeta'}\, dA_\zeta\, dA_{\zeta'} > \theta^{\frac14}\}$$

where $a_z = \iint_{R_z \times R_z} \psi_{\zeta,\zeta'}\, dM_\zeta\, dM_{\zeta'}$.

PROOF. Following the notation of Lemma 4,

$$P\{\textstyle\int_{R_{z_0}} a_z^2\, dA_z > \theta\} \leqq P\{\int_{R_{z_0}} (a_z^\varepsilon)^2\, dA_z > \theta\} + P\{S_{z_0}^\varepsilon < 1\}$$
$$\leqq \frac{\varepsilon^2 A_{z_0}}{\theta} + P\{\textstyle\int_{R_{z_0}}[\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, dM_{\zeta'}]^2\, dA_\zeta > \varepsilon^2\}\,.$$

Turning, now, to the notation of Lemma 3

$$P\{\textstyle\int_{R_{z_0}} a_z^2\, dA_z > \theta\} \leqq \frac{\varepsilon^2 A_{z_0}}{\theta} + P\{b > \varepsilon^2\}$$
$$\leqq \frac{\varepsilon^2 A_{z_0}}{\theta} + P\{b^{\varepsilon_1} > \varepsilon^2\} + P\{T_{z_0}^{\varepsilon_1} < 1\}$$
$$\leqq \frac{\varepsilon^2 A_{z_0}}{\theta} + \frac{\varepsilon_1^2}{\varepsilon^2} + P\{\textstyle\int\int_{R_{z_0}\times R_{z_0}} \psi_{\zeta,\zeta'}^2\, dA_\zeta\, dA_{\zeta'} > \varepsilon_1^2\}\,.$$

Setting $\varepsilon^2 = \theta^{\frac{1}{2}}$ and $\varepsilon_1^2 = \theta^{\frac{3}{4}}$ yields (11).

LEMMA 6. *Let M_z, $z \prec z_0$ be a continuous square integrable strong martingale. Let μ_z be a continuous random function of bounded variation adapted to $\mathscr{F}_z$. Let $|\mu|_z$ denote the variation of $\mu_z(|\mu|_z = \int_{R_z} |d\mu_\zeta|)$ and assume that $|\mu|_{z_0} \leqq \mu^0$* a.s., *where $\mu^0 < \infty$ is a nonrandom constant. Let $\psi_{\zeta,\zeta'}$ be predictable and either bounded or such that $E\int\int_{R_{z_0}\times R_{z_0}} \psi_{\zeta,\zeta'}^2\, d|\mu|_\zeta\, d[M]_{\zeta'}^1 < \infty$), and such that $\psi_{\zeta,\zeta'} = 0$ unless $\zeta \curlywedge \zeta'$. Then*

$$N[[\textstyle\int_{R_{z_0}}[\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, d\mu_\zeta]^2\, d[M]_{\zeta'}^1]^{\frac{1}{2}}] \qquad (12)$$
$$\leqq 4(\mu^0)^{\frac{1}{2}} N^{\frac{1}{2}}[[\textstyle\int\int_{R_{z_0}\times R_{z_0}} \psi_{\zeta,\zeta'}^2\, d|\mu|_\zeta\, d[M]_{\zeta'}^1]^{\frac{1}{2}}]\,.$$

PROOF. Let

$$b = \textstyle\int_{R_{z_0}}[\int_{R_{z_0}} \psi_{\zeta,\zeta'}\, d\mu_\zeta]^2\, d[M]_{\zeta'}^1\,;$$
$$T_z^\varepsilon = 1\,, \quad \text{if} \quad \textstyle\int\int_{R_z\times R_z} \psi_{\zeta,\zeta'}^2\, d|\mu|_\zeta\, d[M]_{\zeta'}^1 \leqq \frac{\varepsilon^2}{\mu^0}\,;$$
$$= 0\,, \quad \text{otherwise.}$$

Then, as we have seen in the previous lemmas:

$$b^\varepsilon = \textstyle\int_{R_{z_0}}[\int_{R_{z_0}} T_{(\zeta'\otimes z_0)}^\varepsilon \psi_{\zeta,\zeta'}\, d\mu_\zeta]^2\, d[M]_{\zeta'}^1\,,$$

and

$$E(b^\varepsilon) \leqq \varepsilon^2\,.$$

Therefore,

$$P\{b^{\frac{1}{2}} > \theta\} \leqq P\{(b^\varepsilon)^{\frac{1}{2}} > \theta\} + P\{T_{z_0}^\varepsilon < 1\}$$
$$\leqq \frac{\varepsilon^2}{\theta^2} + P\left\{\frac{[\int\int_{R_{z_0}\times R_{z_0}} \psi^2\, d|\mu|\, d[M]^1]^{\frac{1}{2}}}{1 + [\int\int_{R_{z_0}\times R_{z_0}} \psi^2\, d|\mu|\, d[M]^1]^{\frac{1}{2}}} > \frac{\varepsilon/(\mu^0)^{-\frac{1}{2}}}{1 + \varepsilon/(\mu^0)^{-\frac{1}{2}}}\right\}$$
$$\leqq \frac{\varepsilon^2}{\theta^2} + \frac{1 + \varepsilon(\mu^0)^{-\frac{1}{2}}}{\varepsilon(\mu^0)^{-\frac{1}{2}}} N[[\textstyle\int\int_{R_{z_0}\times R_{z_0}} \psi^2\, d|\mu|\, d[M]^1]^{\frac{1}{2}}]\,.$$

Hence,

$$N(b^{\frac{1}{2}}) \leqq \theta + \frac{\varepsilon^2}{\theta^2} + \frac{(\mu^0)^{\frac{1}{2}}}{\varepsilon} N[[\textstyle\int\int_{R_{z_0}\times R_{z_0}} \phi^2 \, d|\mu| \, d[M]^1]^{\frac{1}{2}}]$$

and (12) follows as in Lemma 1.

LEMMA 7. *Under the assumptions of Lemma* 6

$$\begin{aligned} &N[\sup_{0\leqq t\leqq t_0} |\textstyle\int\int_{R_{(s_0,t)}\times R_{(s_0,t)}} \phi_{\zeta,\zeta'} \, d\mu_\zeta \, dM_{\zeta'}|] \\ &\qquad \leqq 4(\mu^0)^{\frac{1}{6}} N^{\frac{1}{3}}[[\textstyle\int_{R_{z_0}} [\int_{R_{z_0}} \phi_{\zeta,\zeta'} \, d\mu_\zeta]^2 \, d[M]^1]^{\frac{1}{2}}] \\ &\qquad \leqq 7(\mu^0)^{\frac{1}{18}} N^{\frac{1}{3}}[[\textstyle\int\int_{R_{z_0}\times R_{z_0}} \phi^2_{\zeta,\zeta'} \, d|\mu|_\zeta \, d[M]_\zeta{}^1]^{\frac{1}{2}}] \,. \end{aligned} \tag{13}$$

The proof follows along the same lines as the proofs of Lemmas 4 and 5 and is therefore omitted.

3. Stochastic integrals. If X is a rv then $N(X)$ is a quasinorm on the space of random variables and $N(X_n - X) \to 0$, $n \to \infty$ is equivalent to $X_n \to X$ in probability (cf. [11]). If $N(X_{n+1} - X_n) \leqq n^{-4}$, and since for any rv Y

$$P(|Y| > \varepsilon) = P\left[\frac{|Y|}{1+|Y|} > \frac{\varepsilon}{1+\varepsilon}\right] \leqq \frac{1+\varepsilon}{\varepsilon} N(Y) \,, \tag{14}$$

it follows that $P(|X_{n+1} - X_n| > n^{-2}) \leqq 2n^{-2}$. Therefore, by the Borel–Cantelli lemma X_n converges a.s. to a unique random variable.

PROPOSITION 1. *Let* $\{M_z, z \in R_{z_0}\}$ *be a right continuous* (*continuous*) *square integrable strong martingale. Let* $\mathscr{L}^a$ *be the linear space of all* $\mathscr{F}_z$ *predictable processes* $\{\phi_z, z \in R_{z_0}\}$ *such that*

$$\textstyle\int_{R_{z_0}} \phi^2 \, d[M]_\zeta{}^i < \infty \quad \text{a.s.}\,, \qquad i = 1 \text{ or } 2\,. \tag{15}$$

Then $I_z = \int_{Rz} \phi_\zeta \, dM_\zeta$ *can be extended uniquely to all* $\phi \in \mathscr{L}^a$, *the resulting integral satisfies* (5) *if* $i = 1$ (*or an obvious modification of* (5) *if* $i = 2$). *Furthermore, let* (15) *be satisfied for* $i = 1$ *and* $i = 2$, *and let* Γ *be a nondecreasing path from* (0, 0) *to* z_0. *Then a version of* I_z *has* a.s. *right continuous* (*continuous*) *samples on the path* Γ; $\{I_z, z \in \Gamma\}$ *is a one-parameter locally square integrable martingale on* Γ. *If* $[M]^1$ *satisfies the condition introduced in Lemma* 2, *then* (7) *holds.*

PROOF. If $\phi_\zeta \in \mathscr{L}^a$ then there exists a sequence of simple functions $\phi_\zeta{}^n \in \mathscr{L}^a$ so that

$$\textstyle\int_{R_{z_0}} (\phi_\zeta{}^n - \phi_\zeta)^2 \, d[M]_\zeta{}^i \to 0 \text{ in probability as } n \to \infty \,,$$

hence

$$N\{[\textstyle\int_{R_{z_0}} (\phi_\zeta - \phi_\zeta{}^n)^2 \, d[M]_\zeta{}^i]^{\frac{1}{2}}\} \to 0 \,.$$

By choosing a subsequence, if necessary, we can assume that

$$N\{[\textstyle\int_{R_{z_0}} (\phi_\zeta{}^{n+1} - \phi_\zeta{}^n)^2 \, d[M]_\zeta{}^i]^{\frac{1}{2}}\} \leqq n^{-4} \,.$$

Hence, by Lemma 1 and the Borel–Cantelli lemma $\int_{R_z} \phi_\zeta{}^n \, dM_\zeta$ converges a.s. to an rv I_z which is independent of the particular choice of the approximating sequence ϕ^n. Also by Lemma 1, and since Lemma 1 implies uniform convergence along horizontal lines, there exists a version of I_z which is right continuous (continuous if M is continuous) on a preselected horizontal (if $i = 1$), or vertical (if $i = 2$) path which satisfies (5) for $i = 1$.

Let Γ be a nondecreasing path from (0, 0) to z_0. Let ζ_Γ denote the smallest $z \in \Gamma$, such that $\zeta \prec z$. Let $\zeta = (\sigma, \tau)$, $\zeta_\Gamma = (\sigma_\Gamma, \tau_\Gamma)$; then, if $\zeta \in \Gamma$, $\sigma = \sigma_\Gamma$, $\tau = \tau_\Gamma$, in general either $\sigma = \sigma_\Gamma$ or $\tau = \tau_\Gamma$. Define $A_1 = \{\zeta \in R_{z_0} : \tau < \tau_\Gamma\}$, $A_2 = \{\zeta \in R_{z_0} : \sigma < \sigma_\Gamma\}$. Set

$$\begin{aligned} \phi_\zeta^\alpha &= \phi_\zeta\,, \quad \text{if} \quad \zeta \in A_1\,, & \phi_\zeta^\beta &= \phi_\zeta\,, \quad \text{if} \quad \zeta \in A_2\,, \\ &= 0\,, \quad \text{otherwise}\,; & &= 0\,, \quad \text{otherwise}\,; \end{aligned}$$

then

$$I_z = \textstyle\int_{R_z} \phi_\zeta^\alpha \, dM_\zeta + \int_{R_z} \phi_\zeta^\beta \, dM_\zeta = I_z^\alpha + I_z^\beta\,.$$

$(I_z^\alpha, \mathscr{F}_z, z \in \Gamma)$ and $(I_z^\beta, \mathscr{F}_z, z \in \Gamma)$ are both one-parameter locally square integrable martingales and so is $I_z = I_z^\alpha + I_z^\beta$. Finally, if $[M] \leqq [A]$, then by Lemma 2, $\int \phi^n \, dM$ satisfies (7), and therefore $\int \phi \, dM$ also satisfies (7).

PROPOSITION 2. *Let M_z, $z \prec z_0$, be a right continuous (continuous) strong martingale for which $EM_z^4 < \infty$ and such that for almost all ω, $[M]_\zeta^1 \leqq A_\zeta$ and $[M]_\zeta^2 \leqq A_\zeta$ for all $\zeta \prec z_0$ where A_ζ is a deterministic right continuous increasing function. Let $\mathscr{L}^b$ be the linear space of all $\mathscr{F}_{z \vee z'}$ predictable processes $\psi_{z,z'}$, $(z, z') \in R_{z_0} \times R_{z_0}$ such that*

$$\iint_{R_{z_0} \times R_{z_0}} \psi_{\zeta,\zeta'}^2 \, dA_\zeta \, dA_{\zeta'} < \infty \quad \text{a.s.}\,, \tag{16}$$

and $\psi_{\zeta,\zeta'} = 0$ if $\zeta \curlywedge \zeta'$ is not satisfied. Then $I_z = \iint_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dM_\zeta \, dM_{\zeta'}$ can be extended uniquely to all $\psi \in \mathscr{L}^b$, the resulting integral satisfies (9) *and* (11). *Furthermore, let Γ be a nondecreasing path from* (0, 0) *to z_0, then a version of I_z has* a.s. *right continuous (continuous) samples on Γ; I_z, $z \in \Gamma$ is a one-parameter locally square integrable martingale on Γ.*

PROOF. The proof follows from Lemmas 4 and 5 along the same lines as the proof of Proposition 1, with ϕ^α and ϕ^β replaced by ψ^α and ψ^β:

$$\begin{aligned} \psi_{\zeta,\zeta'}^\alpha &= \psi_{\zeta,\zeta'}\,, \quad \text{if} \quad \zeta \vee \zeta' \in A^\alpha\,, & \psi_{\zeta,\zeta'}^\beta &= \psi_{\zeta,\zeta'}\,, \quad \text{if} \quad \zeta \vee \zeta' \in A^\beta\,, \\ &= 0\,, \quad \text{otherwise}\,; & &= 0\,, \quad \text{otherwise.} \end{aligned}$$

The details are, therefore, omitted.

The proof of the next proposition follows from Lemma 7 along the same lines as the proof of Proposition 1, and is also omitted.

PROPOSITION 3. *Let M_z and μ_z be as in Lemma 6, let $\mathscr{L}^c$ be the linear space of all $\mathscr{F}_{z \vee z'}$ predictable processes $\psi_{z,z'}$, $(z, z') \in R_{z_0} \times R_{z_0}$, such that*

$$\iint_{R_{z_0} \times R_{z_0}} \psi_{\zeta,\zeta'}^2 \, d|\mu|_\zeta \, d[M]_{\zeta'}^1 < \infty \quad \text{a.s.} \tag{17}$$

and $\psi_{\zeta,\zeta'} = 0$ if $\zeta \curlywedge \zeta'$ is not satisfied. Then $I_z = \iint_{R_{z_0} \times R_{z_0}} \psi_{\zeta,\zeta'} \, d\mu_\zeta \, dM_{\zeta'}$ can be *extended uniquely to all* $\psi \in \mathscr{L}^c$ *and satisfies* (13). *Furthermore, let* H *be a horizontal path in* R_{z_0}. *Then a version of* I_z *has* a.s. *continuous samples on* H *and* I_z, $z \in H$ *is a one-parameter locally square integrable martingale on* H.

PROPOSITION 4 (dominated convergence lemma for stochastic integrals).

(a) *Let* M_z *satisfy the the assumptions of Proposition* 1 *and let* $\phi_\zeta{}^n \in \mathscr{L}^a$, $\phi_\zeta \in \mathscr{L}^a$ *and* $|\phi_\zeta{}^n| \leqq \phi_\zeta$ *for all* $\zeta \in R_{z_0}$. *If* $\phi_\zeta{}^n$ *converges in probability to* ϕ_ζ *for all* $\zeta \in R_{z_0}$, *then*

$$\int \phi_\zeta{}^n \, dM_\zeta \to_P \int \phi_\zeta \, dM_\zeta \, . \tag{18}$$

(a′) *Let* M_z *be as in Proposition* 1, *and in addition* $[M]_\zeta{}^1 \leqq A_\zeta$, *where* A_ζ *is deterministic and finite. Let* $\phi_\zeta{}^n \in \mathscr{L}^a$, $\phi_\zeta \in \mathscr{L}^a$ *and*

$$\int_{R_{z_0}} (\phi_\zeta{}^n)^2 \, dA_\zeta < \infty \, , \qquad \int_{R_{z_0}} (\phi_\zeta)^2 \, dA_\zeta < \infty \quad \text{a.s.}$$

If $|\phi_\zeta{}^n| \leqq \phi_\zeta$ *for almost all* $\zeta \in R_{z_0}$ *where "almost all" refers to the* A_ζ *measure, and if* $\phi_\zeta{}^n$ *converges in probability to* ϕ_ζ *for almost all* (A_ζ *measure*), $\zeta \in R_{z_0}$ *holds.*

(b) *Let* M_z *be as in Proposition* 2 *and let* $\psi^n_{\zeta,\zeta'}$, $\psi_{\zeta,\zeta'} \in \mathscr{L}^b$ *and* $|\psi^n_{\zeta,\zeta'}| \leqq \psi_{\zeta,\zeta'}$ *for almost all* (*with respect to the* $A_\zeta \times A_\zeta$ *measure*) $(\zeta, \zeta') \in R_{z_0} \times R_{z_0}$. *If* $\psi^n_{\zeta,\zeta'}$ *converges in probability to* $\psi_{\zeta,\zeta'}$ *for almost all* ($A \times A$ *measure*), (ζ, ζ') *in* $R_{z_0} \times R_{z_0}$, *then*

$$\int\int \psi^n_{\zeta,\zeta'} \, dM_\zeta \, dM_{\zeta'} \to_P \int\int \psi_{\zeta,\zeta'} \, dM_\zeta \, dM_{\zeta'} \, .$$

(c) *Let* M_z, μ_z *be as in Proposition* 3. *If* $\psi^n_{\zeta,\zeta'} \in \mathscr{L}^c$, $\psi_{\zeta,\zeta'} \in \mathscr{L}^c$, $|\psi^n_{\zeta,\zeta'}| \leqq \psi_{\zeta,\zeta'}$ *for all* (ζ, ζ') *in* $R_{z_0} \times R_{z_0}$, *and* $\psi^n_{\zeta,\zeta'}$ *converges in probabibity to* $\psi_{\zeta,\zeta}$ *for all* (ζ, ζ') *in* $R_{z_0} \times R_{z_0}$, *then*

$$\int\int \psi^n_{\zeta,\zeta'} \, d\mu_\zeta \, dM_{\zeta'} \to_P \int\int \psi_{\zeta,\zeta'} \, d\mu_\zeta \, dM_{\zeta'} \, .$$

The proof follows directly from the fact that the stochastic integrals satisfy (5) for (a), (a′); (9) for (b); and (13) for (c); we omit the details.

REMARK. If M_z is continuous, then under the assumptions of Proposition 1 (Proposition 2), the existence of a version of $\int \phi \, dM$ ($\int\int \psi \, dM \, dM$) which is continuous on a single nondecreasing path implies the existence of a version which is continuous on countably many paths. The question whether the integrals of Propositions 1, 2, 3 have continuous (or even bounded) versions in R_{z_0} (M_z being continuous) is open.* A simple case where $\int \phi \, dM$ ($\int \psi \, dM \, dM$) has continuous versions in R_{z_0} is the case where almost all samples of ϕ_ζ are bounded on R_{z_0} or essentially bounded with respect to the A_z measure introduced in Proposition 4 ($\psi_{\zeta,\zeta'}$ is essentially bounded $A_z \times A_z$ for almost all ω), a similar remark holds for $\int\int \psi \, d\mu \, DM$. The proof follows easily from an obvious modification of Lemmas 1, 4 and 7, or directly from the results for $E \int \phi^2 \, d[M]^i < \infty$ ($E \int\int \psi^2 \, d[M]^i \, d[M]^i < \infty$, $E \int\int \psi^2 \, d|\mu| \, d[M]^i < \infty$) by truncating the integrands and passing to the limit.

* Added in proof. This question has been resolved in the affirmative in a forthcoming paper to be published in this journal.

REFERENCES

[1] Cairoli, R. (1972). Sur une équation differentielle stochastique. *C.R. Acad. Sci. Paris, Sér. A* **274** 1939–1952.

[2] Cairoli, R. and Walsh, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183.

[3] Courrège, P. (1962–1963). Intégrales stochastiques et martingales de carré intégrable. Seminaire Brelot-Choquet-Deny, No. 7.

[4] Itô, K. (1961). Lectures on stochastic process. (Notes by K. M. Rao.) Tata Institute of Fundamental Research, Bombay, 181.

[5] Wong, E. (1971). The dynamics of random fields. *Proc. U.S.-Japan Joint Seminar on Stochastic Methods in Dynamical Problems.* Kyoto.

[6] Wong, E. and Zakai, M. (1974). Martingales and stochastic integrals for processes with a multidimensional parameter. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **29** 109–122.

[7] Wong, E. and Zakai, M. (1976). Weak martingales and stochastic integrals in the plane. *Ann. Probability* **4** 570–586.

[8] Wong, E. and Zakai, M. (1977). Differentiation formulas for stochastic integrals in the plane. *Stochastic Processes and Their Applications*, in press.

[9] Wong, E. and Zakai, M. (1977). Likelihood ratios and transformations of probability measures associated with two parameter Wiener processes. Unpublished manuscript.

[10] Yor, M. (1975). Representation des martingales de carré integrable relative au processus de Wiener et de Poisson à N parametres. *C.R. Acad. Sci. Paris.*

[11] Yoshida, K. (1965). *Functional Analysis.* Academic Press, New York.

[12] Zimmerman, G. J. (1972). Some sample function properties of the two-parameter Gaussian process. *Ann. Math. Statist.* **43** 1235–1246.

Department of Electrical Engineering
Technion-Israel Institute of Technology
Haifa, Israel

Department of Electrical Engineering and Computer Sciences
University of California
Berkeley, California 94720