the discrete Fourier series of the N numbers $x[n], x[n-1], \cdots, x[n-N+1]$, it follows from (23) that

$$s[n] = \frac{1}{N} X_0[n]. \tag{25}$$

It is easy to see that the sequence $X_m[n]$ satisfies the recursion

$$X_m[n] - w^m X_m[n-1] = x[n] - x[n-N]. \tag{26}$$

Hence, this sequence can be computed recursively with the digital filter

$$H_m(z) = \frac{1 - z^{-N}}{1 - w^m z^{-1}} \tag{27}$$

shown in Fig. 2. This filter consists of one delay element, one multiplier, and one shift-register with output $x[n-N]$. The shift-register can be omitted if we have direct access, not only to $x[n]$, but also to $x[n-N]$.

We have introduced $H_m(z)$ as a discrete-Fourier-series analyzer of a running segment of $x[n]$. It can, however, be interpreted also as a frequency domain interpolator [2].

We mention without elaboration that the simplicity of realizing $H_m(z)$ suggests various extensions of the smoothing technique that use, not only the average $X_0[n]$, but also other frequency components of the running segment of the data.

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One-Sided Recursive Filters for Two-Dimensional Random Fields

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***Abstract*—The one-sided (or line-by-line) recursive filtering problem for a two-parameter Gaussian random signal in additive white Gaussian noise is considered. For a reasonably large class of models for the signal dynamics, both the filtering equation and the generalized Riccati equation are explicitly obtained. As an example, the Riccati equation is solved to give the filter gain in a time-invariant case and is compared with the infinite-time limiting solution to the Wiener filter solution obtained by spectral factorization techniques.**

I. INTRODUCTION

Consider an observation equation of the form

$$\zeta(t_1,t_2) = x(t_1,t_2) + \eta(t_1,t_2), \qquad a_1 < t_1 < b_1,\ a_2 < t_2 < b_2, \tag{1.1}$$

where ζ is the observed process, x is a Gaussian two-parameter random field representing the state to be estimated, and $\eta(t_1,t_2)$ is a two-parameter white Gaussian noise with

$$E[\eta(t_1,t_2)\eta(t_1',t_2')] = N_0\delta(t_1 - t_1')\delta(t_2 - t_2'). \tag{1.2}$$

In this paper, we consider the recursive filtering problem associated with the estimator

$$\hat{x}(t_1,t_2) = E[x(t_1,t_2) \mid \zeta(s_1,s_2), a_1 < s_1 < t_1,\ a_2 < s_2 < b_2]. \tag{1.3}$$

We observe that in (1.3), the observation changes in only one direction. Therefore, it is possible to consider this problem to be one involving only a one-dimensional parameter (t_1) but with an infinite dimensional state. A similar approach, but one involving a finite dimensional state, has been applied to image processing in [1], [2] with useful results. However, this previous approach fails to take full advantage of the fact that η is white in both directions, and also obscures any dynamical structure which may exist in the t_2 direction for x. It is also possible to view the problem as one of one-dimensional-time infinite-dimensional-state filtering and to make use of the existing theory in that regard [3], but to do so would again fail to deal in a natural way with the dynamics in the t_2 direction.

In this paper, we shall consider a class of state models for which a recursive equation exists for $x(t_1,t_2)$, and derive this equation together with its associated generalized Riccati equation, while retaining the two-dimensional nature of the parameter space. The simplest example of the state model corresponds to the so-called separable covariance model [1], [2], [4] which has often been used.

When certain time-invariance conditions are satisfied in the t_2 direction, the generalized Riccati equation can be solved analytically. A simple special case corresponds to the half-plane-causal Wiener filter for a signal with spectral density

$$S_x(\nu) = \frac{A}{(1 + \alpha_{11}\nu_1^2 + \alpha_{22}\nu_2^2 + 2\alpha_{12}\nu_1\nu_2)}.$$

The recursive filtering formula derived here will be shown to be equivalent, under suitable stationarity assumptions, to the frequency domain solution obtained by spectral factorization [5].

II. THE FILTERING EQUATION

As in the one-dimensional case, we can avoid the pathologies of white noise by dealing with its integral. Let η be the Gaussian white noise considered in Section I. It is convenient to define

$$W(A) = \int_A \eta(t_1,t_2)\, dt_1 dt_2 \tag{2.1}$$

for any Borel set A of the plane R^2. The set-parameterized random function W will be called a Wiener process. It is Gaussian with zero mean and covariance

$$E[W(A)W(B)] = N_0 \text{ area } (A \cap B). \tag{2.2}$$

Equation (2.2), together with the Gaussian property, implies that values of W for disjoint areas are independent, and this is a more precise expression of the white noise property. The observation equation can now be rewritten as

$$Z(t_1,t_2) = \int_{a_1}^{t_1} \int_{a_2}^{t_2} [x(s_1,s_2)\, ds_1 ds_2 + W(ds_1 ds_2)], \tag{2.3}$$

where we shall assume that W and x are independent processes.

Now, assume that the state x satisfies an equation of the form

$$d_{t_1}x(t_1,t_2) = \alpha(t_1,t_2)x(t_1,t_2)\, dt_1 + \int_{a_2}^{b_2} \beta(t_1,t_2,s_2)x(t_1,s_2)\, dt_1 ds_2 + \int_{a_2}^{b_2} \gamma(t_1,t_2,s_2)V(dt_1 ds_2), \tag{2.4}$$

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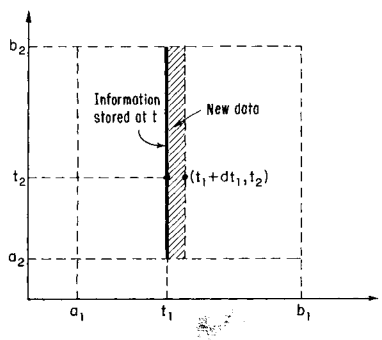

Fig. 1. Nature of recursion for state and estimator.

where V is again a Wiener process with

$$E[V(A)V(B)] = \text{area } (A \cap B), \tag{2.5}$$

and where α, β, γ are deterministic functions. Denote by $\mathcal{F}^z_{t_1}$ the σ-field generated by $\{Z(s_1,s_2): a_1 \le s_1 \le t_1,\ a_2 \le s_2 \le b_2\}$, and define

$$\hat{x}(t_1,t_2) = E[x(t_1,t_2)|\mathcal{F}^z_{t_1}]. \tag{2.6}$$

Then, the basic relationship (see [6, eq. (3.4) and sequel])

$$E[d_{t_1}\hat{x}(t_1,t_2)|\mathcal{F}^z_{t_1}] = E[d_{t_1}x(t_1,t_2)|\mathcal{F}^z_{t_1}] \tag{2.7}$$

yields

$$d_{t_1}\hat{x}(t_1,t_2) = \alpha(t_1,t_2)\hat{x}(t_1,t_2)\,dt_1 + \int_{a_1}^{b_1} \beta(t_1,t_2,s_2)\hat{x}(t_1,s_2)\,ds_2dt_1 + M(dt_1,t_2), \tag{2.8}$$

where $M(t_1,t_2)$ is an $\mathcal{F}^z_{t_1}$-martingale for each t_2.

From the completeness theorem of Wong and Zakai [7] and their more recent result on the transformation of probability measures associated with a two-dimensional Wiener process [8], it is not difficult to show that every $\mathcal{F}^z_{t_1}$-martingale must be of the form

$$M(t_1,t_2) = M(a_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{b_2} k(s_1,t_2,s_2) \cdot [Z(ds_1ds_2) - \hat{x}(s_1,s_2)\,ds_1ds_2], \tag{2.9}$$

so that $\hat{x}$ satisfies the equation

$$d_{t_1}\hat{x}(t_1,t_2) = \alpha(t_1,t_2)\hat{x}(t_1,t_2)\,dt_1 + dt_1\int_{a_2}^{b_2} [\beta(t_1,t_2,s_2) - k(t_1,t_2,s_2)]\hat{x}(t_1,s_2)\,ds_2 + \int_{a_2}^{b_2} k(t_1,t_2,s_2)Z(dt_1ds_2). \tag{2.10}$$

Equation (2.10) is the recursive equation for the estimator $\hat{x}$. The nature of the recursion for both the state equation and the filtering equation is illustrated in Fig. 1.

To find the "gain" function $k(t_1,t_2,s_2)$, we note that the increasing property of $\mathcal{F}^z_{t_1}$ implies that (see [6])

$$\begin{aligned} &E\{d_{t_1}[\hat{x}(t_1,t_2)Z(t_1,t_2')]|\mathcal{F}^z_{t_1}\} \\ &= E\{d_{t_1}(E[x(t_1,t_2)Z(t_1,t_2')|\mathcal{F}^z_{t_1}])\mathcal{F}^z_{t_1}\} \\ &= E[d_{t_1}x(t_1,t_2)Z(t_1,t_2')|\mathcal{F}^z_{t_1}]. \end{aligned}$$

Therefore, if we use the differential rule associated with a continuous semi-martingale [8], we get

$$E[\hat{x}(dt_1,t_2)Z(t_1,t_2') + Z(dt_1,t_2')\hat{x}(t_1,t_2)|\mathcal{F}^z_{t_1}] + N_0\int_{a_2}^{t_2'} k(t_1,t_2,s_2)\,ds_2dt_1 = E[x(dt_1,t_2)Z(t_1,t_2') + Z(dt_1,t_2')x(t_1,t_2)|\mathcal{F}^z_{t_1}].$$

Using (2.7), we get

$$dt_1N_0\int_{a_2}^{t_2'} k(t_1,t_2,s_2)\,ds_2 = E[Z(dt_1,t_2')(x(t_1,t_2) - \hat{x}(t_1,t_2))|\mathcal{F}^z_{t_1}] = \int_{a_2}^{t_2'} \rho(t_1,s_2;t_1,t_2)\,ds_2dt_1,$$

where ρ is the covariance

$$\rho(t_1,s_2;t_1,t_2) = E[\epsilon(t_1,s_2)\epsilon(t_1,t_2)|\mathcal{F}^z_{t_1}], \tag{2.11}$$

and where ϵ is the error

$$\epsilon(t_1,t_2) = x(t_1,t_2) - \hat{x}(t_1,t_2). \tag{2.12}$$

It follows that $k(t_1,t_2,s_2) = (1/N_0)\rho(t_1,s_2;t_1,t_2)$ and that (2.10) can be rewritten as

$$d_{t_1}\hat{x}(t_1,t_2) = \alpha(t_1,t_2)\hat{x}(t_1,t_2)\,dt_1 + dt_1\int_{a_2}^{b_2}\left[\beta(t_1,t_2,s_2) - \frac{1}{N_0}\rho(t_1,s_2;t_1,t_2)\right]\hat{x}(t_1,s_2)\,ds_2 + \int_{a_2}^{b_2}\frac{1}{N_0}\rho(t_1,s_1;t_1,t_2)Z(dt_1ds_2). \tag{2.13}$$

To get a generalized Riccati equation for ρ, we first combine (2.13) with (2.4) to get the following equation for ϵ

$$d_{t_1}\epsilon(t_1,t_2) = \alpha(t_1,t_2)\epsilon(t_1,t_2)\,dt_1 + dt_1\int_{a_2}^{b_2}\left[\beta(t_1,t_2,s_2) - \frac{1}{N_0}\rho(t_1,s_2;t_1,t_2)\right]\epsilon(t_1,s_2)\,ds_2 + \int_{a_2}^{b_2}\left[\gamma(t_1,t_2,s_2)V(dt_1ds_2) - \frac{1}{N_0}\rho(t_1,s_2;t_1,t_2)W(dt_1ds_2)\right]. \tag{2.14}$$

It follows, from the differentiation rule for one-dimensional semi-martingales and from the fact that x and Z are jointly Gaussian, that ρ must be deterministic and obey the following equation

$$\begin{aligned} d_{t_1}\rho(t_1,t_2;t_1,\tau_2) &= E[d_{t_1}[\epsilon(t_1,t_2)\epsilon(t_1,\tau_2)]|\mathcal{F}^z_{t_1}] \\ &= E\{\epsilon(t_1,t_2)d_{t_1}\epsilon(t_1,\tau_2) + \epsilon(t_1,\tau_2)d_{t_1}\epsilon(t_1,t_2)|\mathcal{F}^z_{t_1}\} \\ &\quad + \int_{a_2}^{b_2}\gamma(t_1,t_2,s_2)\gamma(t_1,\tau_2,s_2)\,ds_2dt_1 \\ &\quad + \frac{1}{N_0}\int_{a_2}^{b_2}\rho(t_1,s_2;t_1,t_2)\rho(t_1,s_2;t_1,\tau_2)\,ds_2dt_1 \\ &= [\alpha(t_1,\tau_2) + \alpha(t_1,t_2)]\rho(t_1,t_2;t_1,\tau_2)\,dt_1 \\ &\quad + dt_1\int_{a_2}^{b_2}[\beta(t_1,t_2,s_2)\rho(t_1,\tau_2;t_1,s_2) \\ &\quad + \beta(t_1,\tau_2,s_2)\rho(t_1,t_2;t_1,s_2)]\,ds_2 \\ &\quad - dt_1\frac{1}{N_0}\int_{a_2}^{b_2}\rho(t_1,s_2;t_1,t_2)\rho(t_1,s_2;t_1,\tau_2)\,ds_2 \\ &\quad + dt_1\int_{a_2}^{b_2}\gamma(t_1,t_2,s_2)\gamma(t_1,\tau_2,s_2)\,ds_2. \end{aligned} \tag{2.15}$$

Equation (2.15) will be referred to as the generalized Riccati equation for the covariance function ρ.

Formally at least, the filtering equation (2.13) and the generalized Riccati equation complete the solution of the filtering problem. However, several questions remain to be addressed. First, (2.4) as a modeling equation for the state appears ad hoc, and is not likely to arise naturally. We need to show that reasonable models for the state dynamics would lead to (2.4). Second, the generalized Riccati equation (2.15) needs to be investigated to discover cases for which analytical solutions may be possible.

III. State Models

Two classes of state models lead to (2.4). First, suppose that the state $x(t_1,t_2)$ satisfies a symmetric partial differential equation

$$\frac{\partial^2 x(t_1,t_2)}{\partial t_1 \partial t_2} = C_2(t_1,t_2)\frac{\partial x(t_1,t_2)}{\partial t_1} + C_1(t_1,t_2)\frac{\partial x(t_1,t_2)}{\partial t_2} + C_0(t_1,t_2)x(t_1,t_2) + D(t_1,t_2)\xi(t_1,t_2), \quad (3.1)$$

where ξ is a two-parameter white Gaussian noise. We can rewrite (3.1) as

$$\frac{\partial}{\partial t_2}\left[\frac{\partial}{\partial t_1}x(t_1,t_2) - C_1(t_1,t_2)x(t_1,t_2)\right] = C_2(t_1,t_2)\left[\frac{\partial}{\partial t_1}x(t_1,t_2) - C_1(t_1,t_2)x(t_1,t_2)\right] + \left[C_0(t_1,t_2) - \frac{\partial}{\partial t_2}C_1(t_1,t_2) + C_1(t_1,t_2)C_2(t_1,t_2)\right]x(t_1,t_2) + D(t_1,t_2)\xi(t_1,t_2). \quad (3.2)$$

A little rearrangement yields

$$\frac{\partial}{\partial t_2}\left\{\exp\left(-\int_{a_2}^{t_2} C_2(t_2,s_2)\,ds_2\right) \cdot \left[\frac{\partial}{\partial t_1}x(t_1,t_2) - C_1(t_1,t_2)x(t_1,t_2)\right]\right\} = \exp\left(-\int_{a_2}^{t_2} C_2(t_1,s_2)\,ds_2\right) \cdot \left\{\left[C_0(t_1,t_2) + C_1(t_1,t_2)C_2(t_1,t_2) - \frac{\partial}{\partial t_2}C_1(t_1,t_2)\right]x(t_1,t_2) + D(t_1,t_2)\xi(t_1,t_2)\right\},$$

which can be integrated to give

$$\left[\frac{\partial}{\partial t_1}x(t_1,t_2) - C_1(t_1,t_2)x(t_1,t_2)\right] - \left[\frac{\partial}{\partial t_1}x(t_1,a_2) - C_1(t_1,a_2)x(t_1,a_2)\right] = \int_{a_2}^{t_2}\exp\left(\int_{\tau_2}^{t_2} C_2(t_1,s_2)\,ds_2\right) \cdot \left\{\left[C_0(t_1,\tau_2) + C_1(t_1,\tau_2)C_2(t_1,\tau_2) - \frac{\partial}{\partial \tau_2}C_1(t_1,\tau_2)\right]x(t_1,\tau_2) + D(t_1,\tau_2)\xi(t_1,\tau_2)\right\}d\tau_2. \quad (3.3)$$

If

$$\frac{\partial}{\partial t_1}x(t_1,a_2) - C_1(t_1,a_2)x(t_1,a_2) \equiv 0,$$

then (3.3) is already in the form of (2.4). If not, we can redefine the state as

$$y(t_1,t_2) = x(t_1,t_2) - x(t_1,a_2),$$

then (2.4) and the resulting filtering equation will be modified only in minor ways. Equation (3.1) is a more natural model than (2.4), and includes as a special case the so-called separable covariance model in which

$$\frac{\partial^2 x(t_1,t_2)}{\partial t_1 \partial t_2} = C_2\frac{\partial x(t_1,t_2)}{\partial t_1} + C_1\frac{\partial x(t_1,t_2)}{\partial t_2} - C_1C_2x(t_1,t_2) + D\xi(t_1,t_2), \quad (3.4)$$

where C_1, C_2, and D are constants.

A second class of state models which lead to (2.4) consists of states which are stationary Gaussian random fields with spectral densities of the form

$$S_x(\nu_1,\nu_2) = \frac{|F(\nu_2)|^2}{|i\nu_1 + C_1 + G(\nu_2)|^2}, \quad (3.5)$$

where $F(\nu_2)$ and $G(\nu_2)$ are the Fourier transforms of functions $f(t_2)$ and $g(t_2)$, respectively. If the state x has a spectral density given by (3.5), then the standard argument for one-parameter processes yields the model

$$\frac{\partial x(t_1,t_2)}{\partial t_1} + C_1x(t_1,t_2) + \int_{-\infty}^{\infty} g(t_2 - s_2)x(t_1,s_2)\,ds_2 = \int_{-\infty}^{\infty} f(t_2 - s_2)\xi(t_1,s_2)\,ds_2. \quad (3.6)$$

Identifying ξ as a two-parameter Gaussian white noise, or as the formal derivative of the Wiener process V, we get

$$d_{t_1}x(t_1,t_2) = -C_1x(t_1,t_2)dt_1 + dt_1\int_{-\infty}^{\infty} g(t_2 - s_2)x(t_1,s_2)\,ds_1 + \int_{-\infty}^{\infty} f(t_2 - s_2)V(dt_1ds_2), \quad (3.7)$$

which is clearly a special case of (2.4)

The "separable" model for the stationary case corresponds to a spectral density given by

$$S_x(\nu_1,\nu_2) = \frac{1}{|(i\nu_1 + C_1)(i\nu_2 + C_2)|^2}, \quad (3.8)$$

for which $G(\nu_2) \equiv 0$ and $F(\nu_2) = 1/(i\nu_2 + C_2)$. Equation (3.7) then becomes

$$d_{t_1}x(t_1,t_2) = -C_1x(t_1,t_2)\,dt_1 + \int_{-\infty}^{t_2}\exp\left(-C_2(t_2 - s_2)V(dt_1ds_2)\right). \quad (3.9)$$

IV. Solutions of the Generalized Riccati Equation

If the state model (2.4) is time-invariant in the t_2 direction, i.e., if

$$a_2 = -\infty \qquad b_2 = \infty$$
$$\alpha(t_1,t_2) = \alpha(t_1)$$
$$\beta(t_1,t_2,s_2) = \beta(t_1,t_2 - s_2)$$
$$\gamma(t_1,t_2,s_2) = \gamma(t_1,t_2 - s_2), \quad (4.1)$$

then it is clear that the covariance function $\rho(t_1,t_2;t_1,\tau_2)$ can depend only on t_1 and $t_2 - \tau_2$. Denote this covariance function by $\rho(t_1,t_2 - \tau_2)$. Then the generalized Riccati equation (2.15) becomes

$$\frac{\partial}{\partial t_1}\rho(t_1,t_2 - \tau_2) = 2\alpha(t_1)\rho(t_1,t_2 - \tau_2) + \int_{-\infty}^{\infty}\gamma(t_1,t_2 - s_2)\gamma(t_1,\tau_2 - s_2)\,ds_2 + \int_{-\infty}^{\infty}[\beta(t_1,t_2 - s_2)\rho(t_1,\tau_2 - s_2) + \beta(t_1,\tau_2 - s_2) \cdot \rho(t_1,t_2 - s_2)]\,ds_2 - \frac{1}{N_0}\int_{-\infty}^{\infty}\rho(t_1,t_2 - s_2)\rho(t_1,\tau_2 - s_2)\,ds_2. \quad (4.2)$$

Taking the Fourier transform in the t_2 direction and using $\tilde{f}$ to denote Fourier transform of f, we obtain

$$\frac{\partial}{\partial t_1}\tilde{\rho}(t_1,\nu_2) = 2\alpha(t_1)\tilde{\rho}(t_1,\nu_2) + |\tilde{\gamma}(t_1,\nu_2)|^2 + 2\,\mathrm{Re}\,[\tilde{\beta}(t_1,\nu_2)]\tilde{\rho}(t_1,\nu_2) - \frac{1}{N_0}\tilde{\rho}^2(t_1,\nu_2). \quad (4.3)$$

For a fixed ν_2, (4.3) is an ordinary Riccati equation and can be linearized by the transformation

$$\tilde{p}(t_1,\nu_2) = N_0 \frac{1}{u(t_1,\nu_2)} \frac{\partial}{\partial t_1} u(t_1,\nu_2). \tag{4.4}$$

Substitution of (4.4) into (4.3) results in the equation

$$\frac{\partial^2}{\partial t_1^2} u(t_1,\nu_2) = 2[\alpha(t_1) + \mathrm{Re}\,(\tilde{\beta}(t_1,\nu_2))] \frac{\partial}{\partial t_1} u(t_1,\nu_2) + \frac{1}{N_0} |\tilde{\gamma}(t_1,\nu_2)|^2 u(t_1,\nu_2). \tag{4.5}$$

Specializing still further, we consider the case where x is also t_1-invariant. Then α, $\tilde{\beta}$, and $\tilde{\gamma}$ are all independent of t_1. Indeed, if we assume the spectral density function to have the form given by (3.5), then (4.5) becomes

$$\frac{\partial^2}{\partial t_1^2} u(t_1,\nu_2) + 2[C_1 + \mathrm{Re}\, G(\nu_2)] \frac{\partial u(t_1,\nu_2)}{\partial t_1} - \frac{1}{N_0} |F(\nu_2)|^2 u(t_1,\nu_2) = 0. \tag{4.6}$$

For this case, we have

$$\tilde{p}(t_1,\nu_2) = \frac{\rho(a_1,\nu_2) + \left[\frac{|F(\nu_2)|^2}{m(\nu_2)} - \frac{\rho(a_1,\nu_2)}{m(\nu_2)}(C_1 + \mathrm{Re}\, G(\nu_2))\right] \tanh\,(t_1 - a_1)}{1 + \frac{1}{m(\nu_2)}\left[C_1 + \mathrm{Re}\, G(\nu_2) + \frac{1}{N_0}\rho(a_1,\nu_2)\right] \tanh\,(t_1 - a_1)}, \tag{4.7}$$

where

$$m(\nu_2) = \sqrt{[C_1 + \mathrm{Re}\, G(\nu_2)]^2 + \frac{1}{N_0}|F(\nu_2)|^2}. \tag{4.8}$$

If $a_1 = -\infty$, then $\tilde{p}(t_1,\nu_2)$ is stationary and independent of t_1, and (4.7) reduces to

$$\tilde{p}(\nu_2) = N_0[m(\nu) - C_1 - \mathrm{Re}\, G(\nu_2)]. \tag{4.9}$$

As a simple example, let $N_0 = 1$ and

$$S_x(\nu) = \frac{4C_1C_2}{|(i\nu_1 + C_1)(i\nu_2 + C_2) - 1|^2}.$$

Then

$$|F(\nu_2)|^2 = \frac{4C_1C_2}{\nu_2^2 + C_2^2}$$

$$G(\nu_2) = -\frac{1}{i\nu_2 + C_2}$$

$$\mathrm{Re}\, G(\nu_2) = -\left(\frac{C_2}{C_2^2 + \nu_2^2}\right)$$

$$m(\nu_2) = \sqrt{\left(C_1 - \frac{C_2}{C_2^2 + \nu_2^2}\right)^2 + \frac{4C_1C_2}{C_2^2 + \nu_2^2}} = C_1 + \frac{C_2}{C_2^2 + \nu_2^2},$$

and from (4.9)

$$\tilde{p}(\nu_2) = \frac{2C_2}{C_2^2 + \nu_2^2}$$

or

$$\rho(t_2) = e^{-C_2|t_2|}.$$

We observe that if $x(t_1,t_2)$ has a spectral density given by (3.5), then the half-plane Wiener filter can be obtained by spectral factorization [5], or, alternatively, by using (4.9) and (3.6) in the filtering equation (2.13). Now if we identify the observed process as $\zeta(t_1,t_2)$ of (1.1) and use (3.6) in (2.13), we get

$$\frac{\partial}{\partial t_1}\hat{x}(t_1,t_2) + C_1\,\hat{x}(t_1,t_2) + \int_{-\infty}^{\infty}\left[g(t_2 - s_2) + \frac{1}{N_0}\rho(t_2 - s_2)\right]\hat{x}(t_1,s_2)\,ds_2 = \frac{1}{N_0}\int_{-\infty}^{\infty}\rho(t_2 - s_2)\zeta(t_1,s_2)\,ds_2.$$

It follows that $\hat{x}$ can be viewed as the output of a two-dimensional filter with ζ as the input, and with the transfer function of the filter given by

$$H(\nu_1,\nu_2) = \frac{\frac{1}{N_0}\tilde{p}(\nu_2)}{i\nu_1 + C_1 + G(\nu_2) + \frac{1}{N_0}\tilde{p}(\nu_2)} = \frac{m(\nu_2) - |C_2 + \mathrm{Re}\, G(\nu_2)|}{i\nu_1 + i\,\mathrm{Im}\,[G(\nu_2)] + m(\nu_2)}$$

where

$$m(\nu_2) = \sqrt{[C_1 + \mathrm{Re}\, G(\nu_2)]^2 + \frac{1}{N_0}|F(\nu_2)|^2}.$$

To verify this solution by spectral factorization, we write

$$N_0 + S_x(\nu_1,\nu_2) = |K(\nu_1,\nu_2)|^2,$$

where

$$K(\nu_1,\nu_2) = \sqrt{N_0}\left[\frac{i\nu_1 + i\,\mathrm{Im}\,[G(\nu_2)] + m(\nu_2)}{i\nu_1 + i\,\mathrm{Im}\,[G(\nu_2)] + |C_1 + \mathrm{Re}\, G(\nu_2)|}\right].$$

Then the transfer function of the Wiener filter can be written as

$$H(\nu_1,\nu_2) = \frac{1}{K(\nu_1,\nu_2)}\left[\frac{S_x(\nu_1,\nu_2)}{\overline{K}(\nu_1,\nu_2)}\right]_+,$$

where $\overline{K}$ denotes the complex conjugate of K and $[\cdot]_+$ denotes the half-plane causal portion. Since

$$\frac{S_x(\nu_1,\nu_2)}{\overline{K}(\nu_1,\nu_2)} = \frac{1}{\sqrt{N_0}}\frac{|F(\nu_2)|^2}{[-i\nu_1 - i\,\mathrm{Im}\, G(\nu_2) + m(\nu_2)][i\nu_1 + i\,\mathrm{Im}\, G(\nu_2) + |C_1 + \mathrm{Re}\, G(\nu_2)|]},$$

we have

$$\left[\frac{S_x(\nu_1,\nu_2)}{\overline{K}(\nu_1,\nu_2)}\right] = \frac{1}{\sqrt{N_0}}\frac{|F(\nu_2)|^2}{[|C_1 + \mathrm{Re}\ G(\nu_2)| + m(\nu_2)][i\nu_1 + i\,\mathrm{Im}\, G(\nu_2) + |C_1 + \mathrm{Re}\, G(\nu_1)|]}$$

and

$$H(\nu_1,\nu_2) = \frac{1}{N_0} \cdot \frac{|F(\nu_2)|^2}{[|C_1 + \mathrm{Re}\, G(\nu_2)| + m(\nu_2)][i\nu_1 + i\,\mathrm{Im}\,[G(\nu_2)] + m(\nu_2)]} = \frac{m(\nu_2) - |C_2 + \mathrm{Re}\, G(\nu_2)|}{[i\nu_1 + i\,\mathrm{Im}\,[G(\nu_2)] + m(\nu_2)]}.$$

V. Summary of Results

The filtering formula (2.10) and the generalized Riccati equation (2.15) represent the complete solution to the continuous space Gauss–Markovian field filtering problem for first-order dynamics in t_1. Generalization to arbitrary order dynamics in the t_1 direction appears straightforward using one-dimensional vector methods. In the stationary (in t_1,t_2) and first-order separable case, it appears that the "gain" function cannot be obtained analytically from its Fourier transform. However, processing can be done by taking the fast Fourier transform (FFT) of the observations for each new "line" and performing the "gain" weighting in the Fourier domain. Alternatively, one can numerically obtain the "gain" by performing an inverse FFT and appropriately truncating the result to minimize computation. This can lead to good suboptimal estimates in the case where the random field is not highly correlated over the whole observation space. Recursive smoothing formulas are presently being investigated to complete the two-parameter recursive Wiener filtering problem for continuous fields.

The theory presented here differs from previous work [1], [2] in that the fields considered are continuous rather than discrete, but also more fundamentally, in that we obtain scalar filter equations rather than vector equations for the first-order separable covariance model. This work also differs fundamentally from the infinite dimensional filtering theory of [3] in that the observation and state noise models considered here are two-parameter Wiener processes, but are restricted to be only one-parameter Wiener process in the canonical formulation of [3].

Finally, we observe that everything in this paper can be easily generalized to n-dimensional random fields, provided that the dynamics of the filtering remains one-dimensional. The only increase in complexity would be notational.

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A Generalized Likelihood Ratio Formula: Arbitrary Noise Statistics for Doubly Composite Hypotheses

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***Abstract*—A relationship between the likelihood ratio and a generalized causal conditional mean estimator is presented for the doubly composite hypotheses problem. The observation statistics are arbitrary and need not be Gaussian. The relationship parallels the well-known relationship for the Gaussian noise and composite signal hypothesis case, and the likelihood ratio can still be viewed as an estimator-correlator operation.**

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I. Introduction

Relationships between detection and estimation of signals in additive white Gaussian noise for a composite signal hypothesis problem have been studied extensively by many authors. Esposito [1] showed an explicit relationship between the likelihood ratio and a noncausal conditional mean estimate of the signal in additive white Gaussian noise. Kailath [2] obtained the likelihood ratio for the continuous model in the form of an estimator-correlator with a causal conditional mean estimate of the signal. Hatsell and Nolte [3] interpreted the likelihood ratio and the conditional mean signal estimator geometrically. Jaffer and Gupta [4] obtained a relationship between the likelihood ratio and the conditional mean signal estimator in sequential form for additive independent Gaussian noise. A point of commonality among these works is that they all hinge on the additive Gaussian noise assumption and a single hypothesis problem.

Birdsall and Gobien [5] showed how detection and estimation occur simultaneously in a natural way when one adopts a Bayesian viewpoint for a doubly composite hypotheses problem. However, they gave no explicit relationship between the estimator and the likelihood ratio. Recently, Schwartz [6] showed that the likelihood ratio for a composite signal hypothesis problem is completely determined by the *a posteriori* conditional estimate of the unknown parameter when the conditional probability density function (pdf) of the data, conditional on the unknown parameter, is drawn from the exponential family.

In this paper we shall show that the estimator-correlator operation can be viewed as an inherent feature of the likelihood ratio for doubly composite hypotheses detection problems for arbitrary noise statistics which are not necessarily Gaussian but are defined by an arbitrary probability density function. The estimators are causal and conditional mean, but not necessarily direct signal estimators. For the doubly composite hypotheses problem, the likelihood ratio can be realized as an estimator-correlator in which the difference of two generalized causal conditional mean estimators, one for each hypothesis, is used in the estimator-correlator implementation. For a single composite hypothesis problem and independent Gaussian statistics, the generalized estimator reduces to the familiar conditional mean estimate of the signal.

II. Likelihood Ratio: Gaussian Noise for Composite Signal Hypothesis

It has been shown [3], [4] that the discrete-time conditional mean estimate of the signal can be expressed in terms of the likelihood ratio, for composite signal hypothesis problems in additive white Gaussian noise with unit variance , as

$$\frac{\partial}{\partial \mathbf{Z}_k} \ln \ell(\mathbf{Z}_k) = \hat{\mathbf{S}}(\mathbf{Z}_k), \tag{1}$$

where

$\mathbf{Z}_k$	observation vector, $Z_k^T = [z_1, z_2, \cdots, z_k]$ (T denotes transpose),
$s_i(\boldsymbol{\theta})$	ith signal component,
$\boldsymbol{\theta}$	vector of unknown parameters of the signal,
$\hat{s}(\mathbf{Z}_i)$	causal conditional mean estimate of $s_i(\boldsymbol{\theta})$, expressed as $\hat{s}(\mathbf{Z}_i) = E\{s_i(\boldsymbol{\theta}) \mid \mathbf{Z}_i, H_1\}$,
$\hat{\mathbf{S}}(\mathbf{Z}_k)^T$	$= [\hat{s}(\mathbf{Z}_1), \hat{s}(\mathbf{Z}_2), \cdots, \hat{s}(\mathbf{Z}_k)]$,

and where the likelihood ratio is obtained by integrating $\hat{S}(\mathbf{Z}_k)$ over the data $\mathbf{Z}_k$ as

$$\ln \ell(\mathbf{Z}_k) = \oint_0^{\mathbf{Z}_k} \hat{\mathbf{S}}^T(\mathbf{Z}_k')\, d\mathbf{Z}_k', \tag{2}$$