MARTINGALES ON JUMP PROCESSES.
I: REPRESENTATION RESULTS*

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Abstract. The paper is a contribution to the theory of martingales of processes whose sample
paths are piecewise constant and have finitely many discontinuities in a finite time interval. The
assumption is made that the jump times of the underlying process are totally inaccessible and neces-
sary and sufficient conditions are given for this to be true. It turns out that all martingales are then
discontinuous, and can be represented as stochastic integrals of certain basic martingales. This
representation theorem is used in a companion paper to study various practical problems in com-
monication and control. The results in the two papers constitute a sweeping generalization of recent
work on Poisson processes.

1. Introduction and summary. The theory of martingales has proved to be
successful as a framework for formulating and analyzing many issues in stochastic
control, and in detection and filtering problems [2], [4], [5], [10], [11], [12],
[19], [32], [33], [34]. Three sets of results in the abstract or general theory of
martingales seem to be the most useful ones in these applications. The first set
consists of the optional sampling theorem and the classical martingale inequalities
[17]. The second set consists of the locus of results culminating in the decom-
position theorem for supermartingales [24]. The third set includes the calculus
of stochastic integrals [16], [22], the differentiation formula and its application
to the so-called “exponentiation formula” [15].

In applications one is concerned with martingales which are functionals of
a basic underlying process such as a Wiener or Poisson process, and in order to
use the abstract theory one needs to know how to represent these martingales
usefully and explicitly in terms of the underlying process. Thus the “martingale
representation theorems” serve as a bridge linking the abstract theory and the
concrete applications. Their role is quite analogous to that of matrix representa-
tions of linear operators which serve as the instrument with which one can
apply the abstract theory of linear algebra.

The most familiar of all the basic processes which can arise in practice is the
Wiener process. It is known that every martingale of a Wiener process can be
represented as a stochastic integral of the Wiener process [6], [22]. This funda-
mental representation theorem, together with the exponentiation formula, has
been used to derive solutions of stochastic differential equations [2], [19], [20],
to obtain recursive equations for filters [5], [21], [30], [31] and the likelihood
ratios for some detection problems [10], [18], to mention just a few applications.
These very results combined with the decomposition theorem for supermartingales
form the foundation of an approach to one family of stochastic optimal control

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problems [12]. It turns out that every martingale of a Wiener process has continuous sample paths. This is fortunate because it implies that the martingale is locally square integrable, and hence most of the questions about martingales can be posed within the Hilbert space structure of the space of square integrable random variables.

However, for many processes, e.g., Poisson process, one can have martingales which are not locally square integrable. As Meyer and his co-workers have pointed out [16], [26] the $L^2$ structure is no longer appropriate and one needs to be more careful in defining stochastic integrals and in obtaining the differentiation formula. Indeed the current theory of stochastic integration with respect to such martingales is still not completely satisfactory.

This paper is a contribution to the abstract theory and to its applications for the relatively simple case where the sample functions of the underlying process are step functions which have only a finite number of jumps in every finite time interval. In a sense this is the polar opposite of the Wiener process case since all the martingales are discontinuous, that is, all the continuous martingales have constant sample paths. The most important special cases covered by this paper include the Poisson process, Markov chains and extensions of these, such as processes arising in queueing theory. To some extent the results for some of these special cases are also covered in [4], [5], [10], [11], [29], [30], [31].

The next section gives a precise definition of the underlying process and exhibits some of the important properties of the generated $\sigma$-fields. Conditions are derived which guarantee that the jump times of the process are totally inaccessible stopping times. These preliminary results are used in §3 to show first that there are no nonconstant continuous martingales and then to obtain an integral representation of all martingales. A particular example, which includes most of the special cases mentioned above, is presented in §4. Applications of the results are given in the companion paper [3].

2. The basic process and its stopping times. Let $(Z, \mathscr{F})$ be a Blackwell space, that is a measurable space such that $\mathscr{F}$ is a separable $\sigma$-field and every measurable function $f: Z \to R$ maps $Z$ onto an analytic subset of $R$ (see [24, p. 61]). Let $\Omega$ be a family of functions on $R^+ = [0, \infty)$ with values in $Z$, such that each $\omega \in \Omega$ is a step function with only a finite number of jumps in every finite interval, and such that for all $\omega \in \Omega$, $t \in R^+$, $\omega(t) = \omega(t + \varepsilon)$ for all $\varepsilon$ less than some sufficiently small $\varepsilon_0 > 0$. If $Z$ is also a topological space, then each function $\omega$ is right-continuous and has left-hand limits. Let $x_\cdot$ be the evaluation process on $\Omega$, i.e., $x_t(\omega) = \omega(t)$, $t \in R^+$. Let $\mathcal{F}_s$ be the $\sigma$-field on $\Omega$ generated by sets of the form $\{x_s \in B\}$, $B \in \mathscr{F}$, $s \leq t$. Let $\mathcal{F} = \bigvee_{t \in R^+} \mathcal{F}_t$.\footnote{If $A_\cdot$ is a family of subsets then $V_{\cdot}A_\cdot$ denotes the smallest $\sigma$-field containing all the $A_\cdot$.}

Because the positive rationals are dense in $R^+$, it is clear that $\mathcal{F}$ can also be written as $V_n \sigma(x_{r_n})$, where $\sigma(x_{r_n})$ is the $\sigma$-field generated by the function $x_{r_n}$ and $r_n$ is rational. Hence the separability of $\mathscr{F}$ implies the separability of $\mathcal{F}$. Moreover, as will be shown, every real-valued $\mathcal{F}$-measurable function on $\Omega$ will map $\Omega$ onto an analytic subset, hence $(\Omega, \mathcal{F})$ is a Blackwell space. The assertion follows from considering approximations for any measurable $f: \Omega \to R$ of the form
$f^n = g^n \cdot h^n \cdot i$, where $i: (\Omega, \mathcal{F}) \to (Z^N, \mathcal{Z}^N)$ is the natural isomorphism ($\mathbb{N}$ is the set of natural numbers), and $h^n: (Z^N, \mathcal{Z}^N) \to (R^N, \mathcal{B}^N)$ ($\mathcal{B}$ is the Borel field on $R$) consists of measurable components $h^n_1, h^n_2, \ldots$ with $h^n(z_1, z_2, \ldots) = (h^n_1(z_1), h^n_2(z_2), \ldots)$, and finally $g^n$ is a measurable mapping from $(R^N, \mathcal{B}^N)$ into $(R, \mathcal{B})$.

Since the Cartesian product of analytic sets is analytic (see [1]), the image of $Z^N$ in $R^N$ under $h^n$ is an analytic set which is in turn mapped into an analytic subset of $R$ by $g^n$. Since analytic sets form a class closed under countable unions and intersections, this limiting procedure shows that every measurable function $f: \Omega \to R$ maps $\Omega$ onto an analytic set. Since $(\Omega, \mathcal{F})$ is a Blackwell space it follows from [24, §II, Thm. 16] that $(\Omega, \mathcal{F})$ is isomorphic to $(A, \mathcal{B}(A))$ where $A$ is an analytic subset of $R$. Hence the results of [28] can be applied without assuming a topological structure on $Z$ itself.

A $Z$-valued or $R \cup \{\infty\}$-valued function $f$ on $\Omega$ is a random variable (r.v.) if $f^{-1}(B) \in \mathcal{F}$ whenever $B \in \mathcal{Z}$ or whenever $B$ is a Borel subset of $R \cup \{\infty\}$. Unless otherwise stated a r.v. is $R \cup \{\infty\}$-valued. A nonnegative r.v. $T$ is said to be a stopping time (s.t.) if for every $t \in R^+$, $\{T \leq t\} \in \mathcal{F}$. If $T$ is a s.t., then $\mathcal{F}_t$ consists of those sets $A \in \mathcal{F}$ for which $A \cap \{T \leq t\} \in \mathcal{F}$ for each $t \in R_+$, whereas $\mathcal{F}_{t-}$ is the $\sigma$-field generated by $\mathcal{F}$ and sets of the form $A \cap \{t < T\}$, where $A \in \mathcal{F}$, and finally $\mathcal{F}_{T+} = \bigcap_{n>0} \mathcal{F}_{T+1/n}$.

Define inductively the functions $T_n$:

$$T_0 \equiv 0, \quad T_{n+1}(\omega) = \inf \{t \mid t \geq T_n(\omega) \text{ and } x_t(\omega) \neq x_{T_n(\omega)}(\omega)\},$$

where the infimum over an empty set is taken to be $+\infty$. The next few results characterize the $\sigma$-field $\mathcal{F}$ and demonstrate that the $T_n$ are indeed s.t.s. The key results, Corollary 2.2 and Proposition 2.3, which are the only ones used subsequently, can in fact be proved from first principles assuming only the separability of $\mathcal{Z}$, but it is much more intuitive and easier to rely on the results of [7] and [28].

Let $H: \Omega \to [0, \infty]$ be any function. Then $H$ defines three equivalence relations on $\Omega$ as follows:

$$\omega H \omega' \iff H(\omega) = H(\omega') \text{ and } x_t(\omega) = x_t(\omega') \text{ for } t \leq H(\omega),$$

$$\omega \overset{}{H^-} \omega' \iff H(\omega) = H(\omega') \text{ and there is } \varepsilon > 0 \text{ such that } x_t(\omega) = x_t(\omega') \text{ for } t \leq H(\omega) + \varepsilon,$$

$$\omega \overset{}{H^+} \omega' \iff H(\omega) = H(\omega') \text{ and } x_t(\omega) = x_t(\omega') \text{ for } t < H(\omega).$$

A set $A \subseteq \Omega$ is said to be saturated for $H$, respectively $H^+, H^-$, if $\omega \in A$, and $\omega \overset{}{H_-} \omega'$, respectively $\omega \overset{}{H^+} \omega'$, $\omega \overset{}{H^-} \omega'$, implies $\omega' \in A$. Let $\mathcal{H}_H, \mathcal{H}_H^+, \mathcal{H}_H^-$ denote the family of subsets of $\Omega$ which are saturated for $H, H^+, H^-$ respectively.

**Proposition 2.1.** $\mathcal{F}_t = \mathcal{S}_t \cap \mathcal{F}$, where $\mathcal{S}_t = \mathcal{H}_H$ for $H \equiv t$.

**Proof.** The proof follows from [28, Prop. 1].

**Corollary 2.1.** A nonnegative r.v. $T$ is a s.t. if and only if $\{T \leq t\} \in \mathcal{S}_t$ for all $t \in R_+$.

**Corollary 2.2.** $T_n$ is a s.t. for all $n$.

**Proof.** $T_n$ is obviously a nonnegative r.v. and $\{T_n \leq t\} \in \mathcal{S}_t$ by definition.
PROPOSITION 2.2. Let $T$ be a s.t. Then
\[ \mathcal{F}_T = \mathcal{F}_T \cap \mathcal{F}, \quad \mathcal{F}_{T^+} = \mathcal{F}_T \cap \mathcal{F}, \quad \mathcal{F}_{T^-} = \mathcal{F}_T \cap \mathcal{F}. \]

Proof. This follows from [28, Props. 1, 2].

For a s.t. $T$, $\mathcal{F}_\omega(x_{t \wedge T})$ denotes the $\sigma$-field generated by the $Z$-valued r.v.s $X_{t \wedge T}, t \in R_+$. (If $S, T$ are r.v.s, then $S \wedge T = \{\min S, T\}$.)

PROPOSITION 2.3. Let $T$ be a s.t. Then $\mathcal{F}_T = \mathcal{F}_\omega(x_{t \wedge T})$.

Proof. First of all since every measurable set generated by $x_{t \wedge T}$ clearly belongs to $\mathcal{F}_T$, it follows that $\mathcal{F}_\omega(x_{t \wedge T}) \subset \mathcal{F}_T \cap \mathcal{F} = \mathcal{F}_T$. To prove the reverse inclusion, we begin by noting that $\mathcal{F}_\omega(x_{t \wedge T})$ is separable by the same argument which was used to show that $\mathcal{F} = \mathcal{F}_\omega(x_\omega)$ is separable. By [24, § III, Thm. 17] it follows that $\mathcal{F}_T \subset \mathcal{F}_\omega(x_{t \wedge T})$ since the two families have the same atoms, namely, $\bigcap_n \{x_n \wedge T \in B_n\}$, where $r_n$ is rational and $B_n$ is an atom of $\mathcal{F}$.

COROLLARY 2.3. $\mathcal{F}_T = \sigma(x_T, T_i; 0 \leq i \leq n)$.

Proof. The proof follows from Proposition 2.3 since
\[ \mathcal{F}_\omega(x_{t \wedge T_n}) = \sigma(x_{T_i \wedge T_n}, T_i \wedge T_n; 0 \leq i \leq n) \]
\[ = \sigma(x_{T_i}, T_i; 0 \leq i \leq n). \]

COROLLARY 2.4. $\mathcal{F}_t = \sigma(x_{T_i \wedge T}, T_i \wedge t, 0 \leq i \leq n)$.

COROLLARY 2.5. Let $T$ be a s.t. Then $\mathcal{F}_T = \mathcal{F}_T$. 

Proof. Since the sample functions are piecewise constant and $\omega(t) = \omega(t^+)$, it follows that $\mathcal{F}_T = \mathcal{F}_T$, and then the result follows from Proposition 2.2.

PROPOSITION 2.4. $\mathcal{F}_{T_n} = \sigma(x_{T_i}, T_i, 0 \leq i \leq n - 1)$.

Proof. This proof is similar to the proof of Proposition 2.3, with both $\sigma$-fields having the atoms $\{x_{T_i} \in A_i, T_{i+1} \in B_i; 0 \leq i \leq n - 1\}$, where $A_i$ is an atom of $Z$ and $B_i$ is an atom of $R$.

PROPOSITION 2.5. Let $n \geq 1$, and $\delta > 0$. Let $T = (T_{n-1} + \delta) \wedge T_n$, and let $\mathfrak{A} \in \mathcal{F}_T$. Then there exists $A^0 \in \mathcal{F}_{T_{n-1}}$ such that $A \cap \{T < T_n\} = A^0 \cap \{T < T_n\}$.

Proof. By Proposition 2.3, $\mathcal{F}_T = \mathcal{F}_\omega(x_{t \wedge T})$ and it is easy to see that the latter coincides with the $\sigma$-field generated by the r.v.s $\{x_{t \wedge T}, T_i \wedge T; i = 0, 1, 2, \cdots\}$. Hence there exists a function $g$ measurable in its arguments such that
\[ I_\mathfrak{A}(\omega) = g(x_{T_0 \wedge T}, T_0 \wedge T(\omega), \cdots, x_{T_{n-1} \wedge T}, T_{n-1} \wedge T(\omega), x_{T_n \wedge T}, T_n \wedge T(\omega), \cdots) \]
\[ = g(x_{T_0}, T_0(\omega), \cdots, x_{T_{n-1}}, T_{n-1}(\omega), x_{T_n}, T_n(\omega), T_n \wedge T(\omega), \cdots). \]

Define the measurable function $g^0$ by
\[ g^0(x_0, t_0, \cdots, x_{n-1}, t_{n-1}) = g(x_0, t_0, \cdots, x_{n-1}, t_{n-1}, x_{n-1} + \delta, x_{n-1}, \cdots, t_{n-1} + \delta, \cdots). \]

Now if $T_{n-1} \leq T < T_n$, then $x_{T_{n-1} \wedge T} = x_{T_{n-1}}(\omega)$ and $T_{n+k} \wedge T(\omega) = T_{n-k}(\omega) + \delta$ for all $k \geq 0$. Therefore,
\[ I_\mathfrak{A}(\omega)I_{\{T < T_n\}}(\omega) = g^0(x_{T_0}(\omega), T_0(\omega), \cdots, x_{T_{n-1}}(\omega), T_{n-1}(\omega))(I_{\{T < T_n\}}(\omega)), \]
so that the set $A^0 = \{\omega | g^0(x_{T_0}(\omega), \cdots, T_{n-1}(\omega)) = 1\}$ satisfies the assertion.
**Lemma 2.1.** Let \( n \geq 1 \), and let \( S \) be a s.t. Then there exists a r.v.f, measurable with respect to \( \mathcal{F}_{T_{n-1}} \) such that \( SI_{(S < T_n)} = f I_{(S < T_n)} \).

**Proof.** \( SI_{(S < T_n)} = SI_{(S < T_{n-1})} + SI_{(T_{n-1} \leq S < T_n)} \), and \( SI_{(S < T_{n-1})} \) are \( \mathcal{F}_{T_{n-1}} \)-measurable so that by replacing \( S \) by \( S \lor T_{n-1} \) if necessary, one can assume that \( S \geq T_{n-1} \). Let \( \Gamma = \{ S < T_n \} \). Then \( \Gamma = \bigcup_m \Gamma_m \), where

\[
\Gamma_m = \bigcup_k \{ S \leq T_{n-1} + k2^{-m} \} \cap \{ T_{n-1} + k2^{-m} < T_n \}.
\]

Fix \( \delta = 2^{-m} \). By Proposition 2.5 there exist sets \( A_k \in \mathcal{F}_{T_{n-1}} \) such that

\[
\{ S \leq T_{n-1} + k\delta \} \cap \{ T_{n-1} + k\delta < T_n \} = A_k \cap \{ T_{n-1} + k\delta < T_n \}, \quad k \geq 1.
\]

Define sets \( B_k \) by

\[
B_1 = A_1 \quad \text{and} \quad B_k = \{ \omega \in A_k | \omega \notin A_i \text{ for } i < k \} \quad \text{for } k > 1,
\]

and then define the function \( f_m : \Omega \rightarrow [0, \infty] \) by

\[
f_m(\omega) = T_{n-1} + k\delta \quad \text{if } \omega \in B_k \quad \text{and} \quad f_m(\omega) = T_{n-1}(\omega) \quad \text{if } \omega \notin \bigcup_k B_k.
\]

Certainly \( f_m \) is \( \mathcal{F}_{T_{n-1}} \)-measurable. Also

\[
f_m(\omega) - \delta \leq S(\omega) \leq f_m(\omega) < T_n(\omega) \quad \text{for } \omega \in \Gamma_m.
\]

To see this note first that if \( \omega \in A_1 \cup \{ T_{n-1} + \delta < T_n \} \), then clearly \( T_{n-1}(\omega) = f_m(\omega) - \delta \leq S(\omega) < f_m(\omega) < T_n(\omega) \). Next, as an induction hypothesis, suppose that the inequalities in (2.1) hold for

\[
\omega \in \bigcup_{k=1}^N A_k \cap \{ T_{n-1} + k\delta < T_n \},
\]

and let

\[
\omega \in A_{N+1} \cap \{ T_{n-1} + (N + 1)\delta < T_n \}, \quad \omega \notin \bigcup_{k=1}^N A_k \cap \{ T_{n-1} + k\delta < T_n \}.
\]

Let \( k \leq N + 1 \) be the smallest integer such that \( \omega \in B_k \). Suppose \( k \leq N \). Then, since \( B_k \subset A_k \), and since from (2.2), \( T_n > T_{n-1} + k\delta \), it follows that

\[
\omega \in A_k \cap \{ T_{n-1} + k\delta < T_n \}
\]

which contradicts the second condition of (2.2). Hence \( \omega \in B_{N+1} \) and so \( T_{n-1}(\omega) + N\delta \leq S(\omega) \leq T_n(\omega) + (N + 1)\delta = f_m(\omega) < T_n(\omega) \). Therefore (2.1) holds by induction. Finally, define the \( \mathcal{F}_{T_{n-1}} \)-measurable function \( f \) by \( f(\omega) = \lim_m \inf f_m(\omega) \). The obvious inclusion \( \Gamma_m \subset \Gamma_{m+1} \) implies that if \( \omega \in \Gamma_m \), then \( f_m(\omega) - 2^{-(m+k)} \leq S(\omega) \leq f_{m+k}(\omega) \) for all \( k \geq 0 \). Hence \( f(\omega) = S(\omega) \) and the assertion is proved. \( \square \)

To prove further it is convenient to introduce a probability measure on \((\Omega, \mathcal{F})^2\). **Throughout this paper let** \( P \) **denote a fixed probability measure on** \((\Omega, \mathcal{F})\). **Recall the following important classification of stopping times** [25].

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[2] It may be of interest to note that Lemmas 2.2, 2.3 and 2.4 below can be proved without imposing a probability measure \( P \) by using the algebraic definition of a predictable s.t. of [28]. Then a predictable s.t. in the sense used here is simply a nonnegative r.v. which is a.s. \( P \) equal to a predictable s.t. in the sense of [28].
Let $T$ be a s.t. $T$ is said to be **totally inaccessible** if $T > 0$ a.s. and if for every increasing sequence of s.t.s $S_1 \leq S_2 \leq \cdots$,

$$
P\{S_k(\omega) < T(\omega) \text{ for all } k \text{ and } \lim_{k \to \infty} S_k(\omega) = T(\omega) < \infty\} = 0;
$$

whereas $T$ is said to be **predictable** if there exists an increasing sequence of s.t.s $S_1 \leq S_2 \leq \cdots$ such that

$$
P\{T = 0, \text{ or } S_k < T \text{ for all } k \text{ and } \lim_{k \to \infty} S_k = T\} = 1.
$$

The next three lemmas relate this classification to the properties of the jump times $T_n$ of the process $x$.

**Lemma 2.2.** Let $T$ be a totally inaccessible s.t. Then

$$
TI_{\{T < \infty\}} = \left[ \sum_{n=1}^{\infty} T_n I_{\{T = T_n\}} \right] I_{\{T < \infty\}} \quad \text{a.s.}
$$

**Proof.** The equality above holds if and only if $P\{T_{n-1} < T < T_n\} = 0$ for each $n \geq 1$. Let $n$ be fixed. By Lemma 2.1 there exists a $\mathcal{F}_{T_{n-1}}$-measurable function $f$ such that $f(\omega) = T(\omega)$ for $\omega \in \{T_{n-1} < T < T_n\}$. Let $S_k = T_{n-1} \vee (f - 1/k)$. Then $S_k \geq T_{n-1}$ and $S_k$ is $\mathcal{F}_{T_{n-1}}$-measurable so that it is a s.t. Also $S_k$ is increasing and clearly

$$
\{T_{n-1} < T < T_n\} \subset \left\{ S_k < T \text{ for all } k \text{ and } \lim_{k \to \infty} S_k = T < \infty \right\}.
$$

Since $T$ is totally inaccessible, the set on the right has probability measure zero. The assertion is proved.

**Lemma 2.3.** Let $T$ be a s.t. such that for all $n \geq 1$, $P\{T = T_n < \infty\} = 0$. Then $T$ is predictable.

**Proof.** Let $h$ be a function measurable in its arguments and taking values in the set $\{0,1\}$ such that the process $I_{T \leq t}$ has the representation

$$
I_{T \leq t} = h(t, x_{T_0 \wedge t}, T_0 \wedge t, \ldots, x_{T_n \wedge t}, T_n \wedge t, \ldots).
$$

Since $I_{T \leq t} = \max_{s \leq t} I_{T \leq s}$, by modifying $h$ if necessary it can be assumed that

$$
h(t, \xi) = \max_s h(s, \xi).
$$

The r.v. $(h(t + \epsilon), x_{T_0 \wedge t}, T_0 \wedge t, \ldots)$ is $\mathcal{F}_{t}$-measurable and so the r.v.

$$
T_{\epsilon}(\omega) = \inf\{t| h(t + \epsilon, x_{T_0 \wedge t}, T_0 \wedge t, \ldots) = 1\}
$$

is a s.t., and it is immediate that for $\epsilon > 0$,

$$
T_{\epsilon}(\omega) < T(\omega) \quad \text{for } \omega \in \{0 < T < \infty\}.
$$

Furthermore $T_{\epsilon} \leq T_{\epsilon'}$ if $\epsilon' \leq \epsilon$. Define the s.t.s $S_k$ by $S_k = T_{1/k} \wedge k$. It will now be shown that

$$
\lim_{k \to \infty} S_k(\omega) = T(\omega) \quad \text{for } \omega \in \bigcup_{n=1}^{\infty} \{T_{n-1} < T < T_n\}.
$$
Let $\omega \in \{ T_{n-1} < T < T_n \}$. Then

$$h(t, x_{T_0 \wedge t}(\omega), T_0 \wedge t(\omega), \ldots, x_{T_n \wedge t}(\omega), T_n \wedge t(\omega) \ldots)$$

$$= \begin{cases} 0 & \text{for } T_{n-1}(\omega) < t < T(\omega), \\ 1 & \text{for } t \geq T(\omega), \end{cases}$$

so that

$$h \left( t + \frac{1}{k}, x_{T_0 \wedge t}(\omega), T_0 \wedge t(\omega), \ldots \right)$$

$$= \begin{cases} 0 & \text{for } T_{n-1}(\omega) < t + (1/k) < T(\omega) \text{ or } T_{n-1}(\omega) < t < T(\omega), \\ 1 & \text{for } t \geq T(\omega). \end{cases}$$

Hence $T_{1/k}(\omega) = T(\omega) - 1/k$ for $1/k < T(\omega) - T_{n-1}(\omega)$. It follows that $S_k(\omega)$ converges to $T(\omega)$ and the assertion follows.

**Lemma 2.4.** $T_n$ is totally inaccessible if and only if for every $\mathcal{F}_{T_{n-1}}$-measurable function $f$, $P\{ T_n = f < \infty \} = 0$.

**Proof.** Suppose $P\{ T_n = f < \infty \} > 0$. Let $S_k = T_{n-1} \vee (f - 1/k)$. Then $S_k$ is an increasing sequence of s.t.s and

$$\{ T_n = f < \infty \} \subset \left\{ S_k < T_n \text{ for all } k \text{ and } \lim_{k \to \infty} S_k = T_n < \infty \right\}$$

so that $T_n$ cannot be totally inaccessible thereby proving necessity. To prove sufficiency suppose that $T_n$ is not totally inaccessible so that there is an increasing sequence of s.t.s $S_k$ such that

$$P\{ \Gamma \} = P\left\{ S_k < T_n \text{ for all } k \text{ and } \lim_{k \to \infty} S_k = T_n < \infty \right\} > 0.$$

By Lemma 2.1 there exist functions $f_k$, measurable with respect to $\mathcal{F}_{T_{n-1}}$, such that $S_k(\omega) = f_k(\omega)$ for $\omega \in S_k < T_n$. Let $f = \inf f_k$. Then from (2.3) it follows that $f(\omega) = T_n(\omega)$ for $\omega \in \Gamma$ so that $P\{ f = T_n < \infty \} > 0$ and sufficiency is proved.

From the lemma above the following intuitive sufficient condition follows immediately.

**Theorem 2.1.** Let $F(t|x_0, t_0, \ldots, x_{n-1}, t_{n-1})$ be the conditional probability distribution of $T_n$ given $x_{T_0}, T_0, \ldots, x_{T_{n-1}}, T_{n-1}$. Suppose that $F$ is continuous in $t_n$ for all values of $(x_0, t_0, \ldots, x_{n-1}, t_{n-1})$. Then $T_n$ is totally inaccessible.

As an application of Theorem 2.1 note that if $x_n$ is a Poisson process, then $F(t_n|x_0, t_0, \ldots, x_{n-1}, t_{n-1}) = (1 - \exp - (t_n - t_{n-1}))I_{t_n \geq t_{n-1}}$ is continuous. Hence the jump times of a Poisson process are totally inaccessible.

### 3. The martingale representation theorem.**

It will be necessary from now on to complete the $\sigma$-fields $\mathcal{F}$ and $\mathcal{F}$ with respect to the measure $P$. An additional condition is also imposed.

**Assumptions.** (i) The $\sigma$-fields $\mathcal{F}$, $\mathcal{F}$ are augmented so as to be complete with respect to $P$. (ii) The stopping times $T_n$ are totally inaccessible for $n \geq 1$.

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3 If $Z$ is a Borel subset of $\mathcal{B}$ and $\mathcal{F}$ contains all Borel subsets of $Z$, then the conditional probability $P$ exists by [23, p. 361].
Note that after completion of the space \((\Omega, \mathcal{F})\) it ceases to be a Blackwell space. But, of course, the results of §2 continue to hold if the relevant equalities are interpreted as being true almost surely \(P\).

The family \(\mathcal{F}_t\) is said to be free of times of discontinuity if for every increasing sequence of s.t.s \(S_k, \mathcal{F}_{lim}S_k = \cup S_k \mathcal{F}_k\).

**Proposition 3.1.** The family \(\mathcal{F}_t\) is free of times of discontinuity.

**Proof.** By Lemma 2.2 and Assumption (ii) a s.t. \(T\) is totally inaccessible if and only if its graph \([T]\) is contained in the union \(\bigcup_n [T_n]\) of the graphs of \(T_n\), whereas by Lemma 2.3, \(T\) is predictable if \([T] \cap \bigcup_n [T_n] = \emptyset\). The assertion follows from [14, §III, Thm. 51, p. 62]. □

It will be useful to recall some definitions at this time. This will be followed by some remarks and a reproduction of some known results which will be used in the discussion to follow.

A process \(y_t\) is said to be adapted (to the family \(\mathcal{F}_t\)) if \(y_t\) is \(\mathcal{F}_t\)-measurable for all \(t\). Two processes \(y_t\) and \(y'_t\) are said to be indistinguishable, and are written \(y_t \equiv y'_t\), if for almost all \(\omega, y_t(\omega) = y'_t(\omega)\) for all \(t \in \mathbb{R}^+\).

Let \(m_t\) be a martingale with respect to \((\Omega, \mathcal{F}_t, P)\). It is said to be uniformly integrable (u.i.), and one writes \(m_t \in \mathcal{M}^1\), if \(\{m_t|t \in \mathbb{R}^+\}\) is a u.i. set of r.v.s. It is said to be square integrable (s.i.), and one writes \(m_t \in \mathcal{M}^2\), if \(\sup \{E|t|t \in \mathbb{R}^+\} < \infty\).

Let \(m_t\) be a process. It is said to be a locally integrable martingale \([\text{locally square integrable martingale}]\), and one writes \(m_t \in \mathcal{M}^1_{\text{loc}}\), if there is an increasing sequence of s.t.s \(S_k\) with \(S_k \to \infty\) a.s. such that for each \(k\),

\[
m_{t \wedge S_k}I_{\{S_k > 0\}} \in \mathcal{M}^1[m_{t \wedge S_k}I_{\{S_k > 0\}} \in \mathcal{M}^2].
\]

An adapted process \(a_t\) is said to be an increasing process if \(a_0 = 0\) and if its sample paths are nondecreasing and right-continuous. It is said to be integrable, and one writes \(a_t \in \mathcal{A}^+\) if \(\sup \{E|t|t \in \mathbb{R}^+\} < \infty\). \(\mathcal{A}^+\) is defined in a manner analogous to the previous definition. Finally let \(\mathcal{A} = \mathcal{A}^+ - \mathcal{A}^+ = \{a_t - a_0|a_t \in \mathcal{A}^+, a_0 \in \mathcal{A}^+\}\) and \(\mathcal{A}_{\text{loc}} = \mathcal{A}_{\text{loc}} - \mathcal{A}^+\).

It will be assumed throughout that all the local martingales have sample paths which are right-continuous and have left-hand limits. It is known that since the \(\sigma\)-fields \(\mathcal{F}_t\) are complete and since by Corollary 2.5, \(\mathcal{F}_t = \mathcal{F}_t\) for all \(t \in \mathbb{R}^+\), therefore one can always choose a modification of a local martingale so that its sample paths have the above mentioned property (see [24, §VI, Thm. 4]). Two modifications with this property are indistinguishable.

It can be immediately verified that \(\mathcal{M}^2 \subset \mathcal{M}^1\) and so \(\mathcal{M}^2_{\text{loc}} \subset \mathcal{M}^1_{\text{loc}}\), and if \(m_t \in \mathcal{M}^1\) has continuous sample paths, then \(m_t \in \mathcal{M}^2_{\text{loc}}\). However if the sample paths of \(m_t \in \mathcal{M}^1\) are not continuous, then \(m_t\) may not belong to \(\mathcal{M}^2_{\text{loc}}\). Thus in dealing with discontinuous martingales one may be unable to use the Hilbert space structure of square integrable r.v.s.

The next result follows from Proposition 3.1 and [22, Thm. 1.1].

**Theorem 3.1.** Let \(m_t\) and \(m'_t\) be in \(\mathcal{M}^2_{\text{loc}}\). Then there exists a unique, continuous process \(\langle m, m' \rangle_t \in \mathcal{A}\) such that \(m_t m'_t - \langle m, m' \rangle_t \in \mathcal{M}^1_{\text{loc}}\).

\(^4\) \([T] = \{(\omega, T(\omega))|\omega \in \Omega\} \subset \Omega \times [0, \infty].\)

\(^5\) Throughout “unique” means unique up to indistinguishability.
DEFINITION 3.1. Let $B \in \mathcal{F}$. Let

$$P(B, t) = \sum_{s \leq t} I_{\{x_s \neq x_t\}} I_{[x_t \in B]}$$

be the number of jumps of $x$ which occur prior to $t$ and which end in the set $B$.

PROPOSITION 3.2. There is a unique continuous process $\overline{P}(B, t) \in \mathcal{M}^+_{\text{loc}}$ such that the process $Q(B, t) = P(B, t) - \overline{P}(B, t)$ is in $\mathcal{M}^2_{\text{loc}}$.

Proof. Let $P_n(B, t) = P(B, t \wedge T_n)$. Then $P_n(B, t) \leq n$ so that it is square integrable. Furthermore the jumps of $P_n(B, t)$ occur at the s.t.s $T_n$, $1 \leq i \leq n$, and these s.t.s are totally inaccessible by assumption. It follows from [24, VIII, Thm. 31] that there is a unique, continuous, integrable, increasing process $\overline{P}_n(B, t)$ such that $Q_n(B, t) = P_n(B, t) - \overline{P}_n(B, t) \in \mathcal{M}^2$. From this last relation and the uniqueness of $\overline{P}_n$ one can conclude that $\overline{P}_{n+1}(B, t \wedge T_n) = \overline{P}_n(B, t), Q_{n+1}(B, t \wedge T_n) = Q_n(B, t)$. Hence the processes $\overline{P}, Q$ defined by

$$\overline{P}(B, t \wedge T_n) = \overline{P}_n(B, t), \quad Q(B, t \wedge T_n) = Q_n(B, t)$$

satisfy the assertion. \qed

Remark. If the conditional distribution of the jump times $T_{n+1}$ and the jumps $x_{T_{n+1}}$ given $\mathcal{F}_{T_n}$ is available, then, following the results of [35], [36] and using Lemma 2.1, an explicit characterization of the processes $\overline{P}(B, t)$ can be obtained. Specifically, for each $B \in \mathcal{F}$ and integer $n$ let

$$F_n(B, t) = P(T_{n+1} - T_n \leq t, x_{T_{n+1}} \in B|\mathcal{F}_{T_n})$$

Then

$$\overline{P}(B, t) = \sum_{T_i \leq t} \left[ \int_{0}^{T_i} \frac{F_i(B, ds)}{1 - F_i(Z, s)} \right] + \int_{0}^{t} \frac{F_n(B, ds)}{1 - F_n(Z, s^-)}.$$

It follows from this result that $\overline{P}(B, t)$ is continuous (absolutely continuous) in $t$ if for each $n$, $F_n(B, t)$ is continuous (absolutely continuous) in $t$. (Compare Theorem 2.1.)

Two processes $m, m'$ in $\mathcal{M}^2_{\text{loc}}$ are said to be orthogonal if $m, m' \in \mathcal{M}^2_{\text{loc}}$ or equivalently if $\langle m, m' \rangle \equiv 0$.

LEMMA 3.1. Let $B_i \in \mathcal{F}, i = 1, 2$. Then $Q(B_1, t)Q(B_2, t) - \overline{P}(B_1 \cap B_2, t) \in \mathcal{M}^1_{\text{loc}}$, i.e., $\langle Q(B_1, \cdot), Q(B_2, \cdot) \rangle t \equiv \overline{P}(B_1 \cap B_2, t)$. In particular, $Q(B_1, t)$ and $Q(B_2, t)$ are orthogonal if $B_1 \cap B_2 = \emptyset$.

Proof. $Q(B_1, t \wedge T_n) = Q(B_1 \cap B_2, t \wedge T_n) + Q(B_1 - B_2, t \wedge T_n)$

and

$$Q(B_2, t \wedge T_n) = Q(B_1 \cap B_2, t \wedge T_n) + Q(B_2 - B_1, t \wedge T_n),$$

where $B - B' = \{z|z \in B, z \notin B'\}$. The s.i. martingales

$$Q(B_1 \cap B_2, t \wedge T_n), \quad Q(B_1 - B_2, t \wedge T_n) \quad \text{and} \quad Q(B_2 - B_1, t \wedge T_n)$$

have no discontinuities in common so that they are pairwise orthogonal by
The assertion follows then if one can show that for any $B \in \mathcal{F}$,

$$Q^2(B, t \wedge T_n) - \bar{P}(B, t \wedge T_n) \in \mathcal{M}^1.$$

Let $Q(t) = Q(B, t \wedge T_n)$, $P(t) = P(B, t \wedge T_n)$ and $\bar{P}(t) = \bar{P}(B, t \wedge T_n)$. Let $\varepsilon > 0$ and $s < t$ be arbitrary. Let $S_0 \leq S_1 \leq S_2 \leq \cdots$ be a sequence of s.t.s such that $S_0 = s$, $\lim_{k \to \infty} S_k = t$ a.s. and such that $0 \leq \bar{P}(S_k) - \bar{P}(S_{k-1}) \leq \varepsilon$ and $0 \leq P(S_k) - P(S_{k-1}) \leq 1$ a.s. Such a sequence exists since $\bar{P}$ is continuous. Then

$$\sum_{k=1}^{\infty} (Q(S_k) - Q(S_{k-1}))^2 = \sum_{k=1}^{\infty} (P(S_k) - P(S_{k-1}) - \bar{P}(S_k) + \bar{P}(S_{k-1}))^2$$

$$= \sum_{k=1}^{\infty} (P(S_k) - P(S_{k-1}))^2$$

$$- 2 \sum_{k=1}^{\infty} (P(S_k) - P(S_{k-1}))(\bar{P}(S_k) - \bar{P}(S_{k-1}))$$

$$+ \sum_{k=1}^{\infty} (\bar{P}(S_k) - \bar{P}(S_{k-1}))^2.$$

The first term in the last expression is equal to $P(t) - P(s)$ since $P(S_k) - P(S_{k-1})$ is zero or one. Hence

$$\left| E \left\{ \sum_{k=1}^{\infty} (Q(S_k) - Q(S_{k-1}))^2 - (P(t) - P(s)) \right\} \right|$$

$$\leq 2\varepsilon E\{P(t) - P(s)\} + \varepsilon E\{\bar{P}(t) - \bar{P}(s)\}.$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$E \left\{ \sum_{k=1}^{\infty} (Q(S_k) - Q(S_{k-1}))^2 - (P(t) - P(s)) \right\} = 0.$$

Now $Q \in \mathcal{M}^2$ so that $E\{(Q(S_k) - Q(S_{k-1}))^2 \} = E\{Q^2(S_k) - Q^2(S_{k-1})\}$. Also $P - \bar{P} \in \mathcal{M}^1$ so that $E\{P(t) - P(s)\} = E\{\bar{P}(t) - \bar{P}(s)\}$. Substituting these relations in (3.2) one obtains

$$E \left\{ \sum_{k=1}^{\infty} (Q^2(S_k) - Q^2(S_{k-1})) - (\bar{P}(t) - \bar{P}(s)) \right\} = E\{Q^2(t) - Q^2(s)

- (\bar{P}(t) - \bar{P}(s))\} = 0,$$

which is the same as (3.1). $\square$

For fixed $t$, $Q(B, t)$, $P(B, t)$ and $\bar{P}(B, t)$ can be regarded as set functions on $\mathcal{F}$. In order to define stochastic integrals and Lebesgue–Stieltjes integrals with respect to these set functions it is necessary to show that they are countably additive.

**Lemma 3.2.** Let $B_k$, $k \geq 1$, be a decreasing sequence in $\mathcal{F}$ such that $\bigcap B_k = \emptyset$. Then for almost all $\omega \in \Omega$, $Q(B_k, t) \to 0$, $P(B_k, t) \to 0$, $\bar{P}(B_k, t) \to 0$ for all $t \in R_+$ as $k \to \infty$. Furthermore for all $t \in R_+$ and $n \geq 0$, $EQ^2(B_k, t \wedge T_n) \to 0$ as $k \to \infty$. 

[24, § VIII, Thm. 31]
Proof. Fix \( t \in \mathbb{R}_+ \). The nonnegative r.v.s \( P(B_k, t) \) and \( \tilde{P}(B_k, t) \) decrease as \( k \) increases so that they converge to some r.v.s \( P(t) \) and \( \tilde{P}(t) \) respectively. Hence \( Q(B_k, t) = P(B_k, t) - \tilde{P}(B_k, t) \) converges to \( Q(t) = P(t) - \tilde{P}(t) \). From the definition of \( P(B_k, t) \) it is clear that \( P(t) = 0 \) a.s. and from Lemma 3.1 it follows that \( Q_t \in \mathcal{H}_t^{2, \text{loc}} \). Thus \( Q(t) = -\tilde{P}(t) \in \mathcal{H}_t^{2, \text{loc}} \). But \( \tilde{P}(t) \) is an increasing process and \( \tilde{P}(0) = 0 \) so that this is possible only if \( Q(t) = -\tilde{P}(t) = 0 \) a.s. Thus \( P(t) = P(t) = Q(t) = 0 \) for \( \omega \) not belonging to a null set \( N \in \mathcal{F} \). The monotonicity of the sample functions of \( P, \tilde{P} \) implies that \( P(s) = \tilde{P}(s) = 0 \), hence \( Q(s) = 0 \) for \( \omega \notin N \) and \( s \leq t \). To prove the remaining assertion it is enough to note that by Lemma 3.1 and by what has just been shown,

\[
EQ^2(B_k, t \wedge T_n) = E\tilde{P}(B_k, t \wedge T_n) \to 0 \quad \text{as } k \to \infty.
\]

The following definition relates to the different classes of integrands for which a satisfactory theory of integration is available.

Let \( \mathcal{H} \) denote the set of all processes \( h(t) = h(\omega, t) \) of the form

\[
h(t) = h_0 I_{(0, t_1]} + h_1 I_{(t_1, t_2]} + \cdots + h_k I_{(t_k, t_{k+1}[,1]},
\]

where \( h_i \) is a bounded r.v. measurable with respect to \( \mathcal{F}_{t_i} \) and \( 0 \leq t_0 \leq \cdots \leq t_{k+1} < \infty \). Let \( \mathcal{P}_0 \) denote the set of all functions \( f(z, t) = f(z, \omega, t) \) of the form

\[
f(z, \omega, t) = \sum_{i=0}^k \phi_i(z) h_i(\omega, t),
\]

where \( \phi_i \) is a bounded function measurable with respect to \( \mathcal{F} \) and \( h_i \in \mathcal{H} \).

DEFINITION 3.2. A function \( f(z, t) = f(z, \omega, t) \) is said to be predictable if there exists a sequence \( f_k \) in \( \mathcal{P}_0 \) such that

\[
\lim_{k \to \infty} f_k(z, \omega, t) = f(z, \omega, t) \quad \text{for all } (z, \omega, t) \in \mathcal{F} \times \Omega \times \mathbb{R}_+.
\]

Let \( \mathcal{P} \) denote the set of all predictable functions and let \( \mathcal{F}^\mathcal{P} \) be the sub-\( \sigma \)-field of \( \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{B} \) generated by \( \mathcal{P} \).

If \( f(z, t) = f(z, \omega, t) \) is measurable with respect to \( \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{B} \) and if for all fixed \((z, \omega)\), \( f(z, \omega, t) \) is left-continuous in \( t \), then \( f \in \mathcal{P} \).

DEFINITION 3.3.

\[
L^2(\mathcal{P}) = \left\{ f \in \mathcal{P} : \|f\|_2 = \sqrt{E \int_{\mathbb{R}_+} |f(z, t)|^2 \tilde{P}(dz, dt) < \infty} \right\},
\]

\[
L^1(\mathcal{P}) = \left\{ f \in \mathcal{P} : \|f\|_1 = E \int_{\mathbb{R}_+} |f(z, t)| \tilde{P}(dz, dt) < \infty \right\}.
\]

Similarly

\[
L^1(\mathcal{P}) = \left\{ f \in \mathcal{P} : \|f\|_1 = E \int_{\mathbb{R}_+} |f(z, t)| P(dz, dt) < \infty \right\}.
\]

\( L^2_{\text{loc}}(\mathcal{P}) \) is the set of all \( f \in \mathcal{P} \) for which there exists a sequence of s.t.s \( S_k \uparrow \infty \) a.s. such that \( f I_{t \leq S_k} \in L^2(\mathcal{P}) \) for all \( k \). \( L^1_{\text{loc}}(\mathcal{P}) \) and \( L^1_{\text{loc}}(\mathcal{P}) \) are defined in an analogous manner. The integrals in this definition are to be interpreted as Lebesgue–Stieltjes
integrals. Finally let $L^1(Q) = L^1(P) \cap L^1(\bar{P})$, $L^1_{\text{loc}}(Q) = L^1_{\text{loc}}(P) \cap L^1_{\text{loc}}(\bar{P})$. If $f(z, t) \in L^1(Q)$, then the integral

$$\int_{Z} \int_{R^+} f(z, t)P(dz, dt) - \int_{Z} \int_{R^+} f(z, t)\bar{P}(dz, dt)$$

is denoted

$$\int_{Z} \int_{R^+} f(z, t)Q(dz, dt).$$

Lemma 3.3. To each $f \in L^2(\bar{P})$ there corresponds a unique process $(f \circ Q)_t \in \mathcal{M}^2$, called the “stochastic integral of $f$ with respect to $Q$” with the following properties:

(i) if $f(z, \omega, t) = I_{B}(z)A(\omega)I_{(t_0, t_1)}(t) \in L^2(\bar{P})$, where $B \in \mathcal{F}$ and $A \in \mathcal{F}_{t_0}$, then

$$(f \circ Q)_t = \left\{ \begin{array}{ll}
I_{A}(\omega)[Q(B, t \land t_1) - Q(B, t \land t_0)] & \text{for } t > t_0, \\
0 & \text{for } t \leq t_0.
\end{array} \right.$$  

(ii) if $f, g$ are in $L^2(\bar{P})$ and $\alpha, \beta$ are in $\mathbb{R}$, then

$$(\alpha f + \beta g) \circ Q = \alpha (f \circ Q) + \beta (g \circ Q).$$

Furthermore the stochastic integral satisfies the following relations:

(3.3) \[ \langle f \circ Q, g \circ Q \rangle_t = \int_{Z} \int_{R^+} f(z, s)g(z, s)I_{(0, t)}(s)\bar{P}(dz, ds), \]

and in particular,

(3.4) \[ E(f \circ Q)^2 = (\|f\|_2^2)^2. \]

Proof. The proof follows quite closely that of [22, Prop. 5.1]. Let

$$f^j = \sum_{i=0}^{k} \alpha_i I_{A_i}(z)I_{(t_i, t_{i+1})}(t), \quad j = 1, 2,$$

be simple functions in $L^2(\bar{P})$ with $A_i \in \mathcal{F}$, $A_i \in \mathcal{F}_{t_i}$ and $0 = t_0 < t_1 < \cdots < t_{k+1} < \infty$. Then from (i), (ii) and Lemma 3.1, it can be verified directly that

$$(f^1 \circ Q)_t(f^2 \circ Q)_t - \int_{Z} \int_{R^+} f^1(z, s)f^2(z, s)I_{(0, t)}(s)\bar{P}(dz, ds) \in \mathcal{M}^1$$

so that (3.3) and (3.4) hold for all simple functions in $L^2(\bar{P})$. Since such simple functions are dense in $L^2(\bar{P})$, (3.4) implies that there is a unique extension of the map $f \rightarrow (f \circ Q)$ to all of $L^2(\bar{P})$. Evidently (3.3) and (3.4) will hold for the extension.

Lemma 3.4. Let $m_t \in \mathcal{M}^2$ have continuous sample paths. Then $m_t \equiv m_0$.

Proof. By replacing the martingale $m_t$ by $m_t - m_0$ it can be assumed that $m_0 = 0$. It will be shown that $m_t \equiv 0$. Suppose $m_{r_{n-1}} = 0$ for some $n \geq 1$ so that in fact $m_t \land T_{n-1} = E\{m_{T_n \land T_{n-1}} \mid \mathcal{F}_{T_{n-1}} \} = 0$ for all $t$ and consider the continuous martingale $\mu_t = m_t \land T_n$. By Corollary 2.2 there exists a function $h$, measurable in its arguments, such that $\mu_t \equiv h(t, x_{T_0 \land t}, T_0 \land t, \cdots, x_{T_{n-1} \land t}, T_{n-1} \land t)$. The process $\mu_t = h(t, x_{T_0 \land t}, \cdots, x_{T_{n-1} \land t}, T_{n-1} \land t, x_{T_{n-1} \land t}, t)$ is then measurable with respect
to $\mathcal{F}_{r_{n-1}}$. Since for $t < T_n$, $x_{T_n \wedge t} = x_{T_n - 1 \wedge t}$ and $t = T_n \wedge t$, it follows that $\mu_t = \mu_t^0$ for $t < T_n$, and so by continuity of $\mu_t$, $\mu_t = \mu_t^0$ for $t \leq T_n$. For $\alpha \in R_+$ define $S_{\alpha}$ by

$$S_{\alpha}(\omega) = \sup \{s \leq \alpha | \mu_s^0(\omega) \geq 0\}.$$  

Then since $\mu_s^0 = \mu_s = 0$ for $s \leq T_{n-1}$ it follows that $S_{\alpha} \geq T_{n-1}$, and since $S_{\alpha}$ is measurable with respect to $\mathcal{F}_{n-1}$, therefore $S_{\alpha}$ is a s.t. for every $\alpha$. Now let

$$T_{\alpha}(\omega) = \sup \{s \leq \alpha \wedge T_n(\omega) | \mu_s^0(\omega) \geq 0\}.$$  

It will be shown that $T_{\alpha}$ is a s.t. Fix $t$. If $\alpha \leq t$, then $\{T_{\alpha} \leq t\} = \Omega \in \mathcal{F}_t$ since $T_n \leq \alpha$. Suppose then that $\alpha > t$. Now

$$(3.5) \quad \{T_{\alpha} \leq t\} = (\{T_x \leq t\} \cap \{T_n \leq t\}) \cup (\{T_x \leq t\} \cap \{T_n > t\}).$$

Since $T_x < T_n$, therefore $\{T_n \leq t\} \subset \{T_x \leq t\}$, so that the first set on the right in (3.5) is equal to $\{T_n \leq t\}$ which is in $\mathcal{F}_t$ since $T_n$ is a s.t. It will be shown now that

$$(3.5a) \quad \{T_x \leq t\} \cap \{T_n > t\} = \{S_x \leq t\} \cap \{T_n > t\}.$$  

Since $S_{\alpha} \geq T_x$, the set on the left is at least as large as the one on the right. Suppose $\omega \in \{T_x \leq t\} \cap \{T_n > t\}$. Then $\mu_s^0(\omega) < 0$ for $s \in [t, \alpha]$ and $t < T_n(\omega)$, so that $S_x(\omega) \leq t$, which proves (3.5a).

Thus $\{T_{\alpha}(\omega) | \alpha \in R_+\}$ is a family of s.t.s and furthermore the sample paths $T_{\alpha}(\omega)$ are nondecreasing functions of $\alpha$. By the optional sampling theorem [17, Thm. 11.8, p. 376] applied to the martingale $\mu_t^0$, the process $\eta_s^0(\omega) = \mu_{T_s(\omega)}(\omega), \alpha \in R_+$, is a martingale. Also, since $T_x < T_n$, therefore $\mu_{T_s} = \mu_{T_x}$. Hence $\eta_s \geq 0$. But $\eta_0 = 0$, so that one must have $\eta_s = 0$. In turn this can happen only if $\mu_s = 0$ which together with $\mu_0 = 0$ implies $\mu_s = 0$. The lemma is proved.

THEOREM 3.2. Let $m_s \in M_{loc}$ have continuous sample paths. Then $m_s \equiv m_0$.

Proof. The s.t.s $S_\alpha(\omega) = \inf \{t | m_t(\omega) > k\}$ converge to $\infty$ and

$$m_{\alpha \wedge S_\alpha} I_{(S_\alpha > 0)} \equiv m_0 \quad \square$$

Thus there are no nontrivial continuous martingales. On the other hand if $m_s$ is a martingale, then its discontinuities occur at the jump times $T_n$ of the process $x_s$ as shown below.

LEMMA 3.5. Let $S$ be a predictable s.t. and let $m_s \in M_2$. Then $\Delta m_s = m_s - m_{s^-} = 0$ a.s.

Proof. By [24, § VIII, Thm. 29] the process $\Delta m_s I_{(s \geq S)}$ is a martingale. By [25, Prop. 7, p. 159], $E\{\Delta M_s | \mathcal{F}_s\} = 0$ a.s. But by Proposition 3.1 and [14, § III, Thm. 51], $\mathcal{F}_s^- = \mathcal{F}_s$ so that $\Delta M_s = 0$ a.s. $\square$

The next result gives the first martingale representation theorem. It should be compared with [22, Thm. 4.2 and Prop. 5.2].

THEOREM 3.3. Let $m_s \in M_2$. Then $m_s - m_0 \in \{f \circ Q | f \in L^2(\tilde{P})\}$.

Proof. It can be assumed without losing generality that $m_0 = 0$. The space $M_0^2 = \{m_s \in M_2 | m_0 = 0\}$ is a Hilbert space under the norm $\|m\|^2 = Em_2^2$ by [16, Thm. 1], and by Lemma 3.3 the set $N = \{f \circ Q | f \in L^2(\tilde{P})\}$ is a closed linear subspace of $M_0^2$. Furthermore $N$ is closed under stopping, i.e., if $(f \circ Q)_{t \wedge T} \in N$ and $T$ is a s.t., then $(f \circ Q)_{t \wedge T} \in N$. This is clear because $(f \circ Q)_{t \wedge T} = (f_T \circ Q)_{t \wedge T}$,
where \( f_t(t) = f_t I_{t < T} \). Thus by [27, Thm. 2 and the remark following Def. 4] the theorem is proved if it can be shown that \( m_t \equiv 0 \) when it is orthogonal to \( f \circ Q \) for every \( f \in L^2(\tilde{P}) \). Let \( m_t \) be such a martingale. By [16, Thm. 4], \( m_t \) can be decomposed uniquely as

\[
m_t = m_t^c + m_t^d,
\]

where \( m_t^c \in \mathcal{M}_0^2 \) is continuous and \( m_t^d \in \mathcal{M}_0^2 \) is orthogonal to every continuous martingale. By Theorem 3.2, \( m_t^c \equiv 0 \). By Lemmas 2.2 and 3.5, the discontinuities of \( m_t^d \) occur during the stopping times \( T_n, n \geq 1 \). Therefore, by [16, Thm. 4] again, \( m_t = m_t^d \) can be further decomposed as

\[
m_t = \sum_{n=1}^{\infty} (M_n I_{t \geq T_n} - a_n(t)) = \sum_{n=1}^{\infty} \mu_{nt} \text{ say,}
\]

where \( M_n = \Delta m_{T_n} = m_{T_n} - m_{T_n^-} \), \( a_n(t) \in \mathcal{A} \) has continuous sample paths, and \( \mu_{nt} \in \mathcal{M}_0^2 \). Furthermore the martingale \( \mu_n \) is orthogonal to every martingale which has no discontinuities at \( T_n \).

To prove that \( m_t \equiv 0 \) it suffices to show that \( M_n = 0 \) for each \( n \). Fix \( n \) and suppose that \( P\left\{ M_n \neq 0 \right\} > 0 \). Since \( M_n \) is measurable with respect to \( \mathcal{F}_{T_n^-} \), therefore by Corollary 2.3 there must exist sets \( A \in \mathcal{F}_{T_n^-}, B \in \mathcal{G}, \) and \( C \in \mathcal{B}[0, \infty) \) such that

\[
E\{M_A(\omega)I_{A(\omega)I_{t \geq T_n^-}}I_{(T_n \leq t)}\} \neq 0.
\]

Consider the function \( f(z, \omega, t) \) defined by

\[
f(z, \omega, t) = I_{B}(z)I_{A(\omega)I_{t \geq T_n^-}}I_{(t \leq T_n)}.
\]

The function \( g(z, \omega, t) = I_{B}(z)I_{A(\omega)I_{t \geq T_n^-}}I_{(t \leq T_n)} \) has left-continuous paths for fixed \((z, \omega)\) and for each fixed \( z, t \) the set

\[
\{I_{A(\omega)I_{t \geq T_n^-}}I_{(t \leq T_n)} = 1\} = A \cap \{T_{n-1} < t\} \cap \{t \leq T_n\} \in \mathcal{F}_t
\]

since \( A \in \mathcal{F}_{T_n^-} \). Therefore \( g(z, t) \) is adapted, so that \( g \in \mathcal{G} \) and hence \( f = gI_{t\leq T_n^-} \) is also predictable. Also \( |f| \leq 1 \) and \( f(z, t) = 0 \) for \( t > T_n \) so that \( f \in L^2(\tilde{P}) \cap L^1(\tilde{P}) \cap L^1 \). Therefore by Lemma 3.6 below it follows that

\[
\eta_t = (f \circ Q)_t = \int_z \int_{R^+} f(z, s)I_{(0,t]}(s)P(dz, ds) - \int_z \int_{R^+} f(z, s)I_{(0,t)}(s)\tilde{P}(dz, ds)
= I_{A(\omega)I_{(t \geq T_n^-)}}I_{(T_n \leq t)} - a(t),
\]

where \( a(t) \) is a continuous process. Thus the discontinuities of \( \eta_t \) occur at \( T_n \).

Since \( m_t \) is orthogonal to \( \eta_t \) by hypothesis, therefore

\[
0 = \langle m_t, \eta_t \rangle = \sum_{k \neq n} \langle \mu_k, \eta_t \rangle + \langle \mu_n, \eta_t \rangle.
\]

Also \( \langle \mu_k, \eta_t \rangle \equiv 0 \) for \( k \neq n \), hence \( \langle \mu_n, \eta_t \rangle \equiv 0 \) so that \( \mu_n \cdot \eta_t \in \mathcal{M}^1 \). By the Corollary in [16, p. 106] and the Definition in [16, p. 87] it follows that \( \Delta \mu_{nt} \cdot \Delta \eta_{T_n} \cdot I_{t \geq T_n} \) is a martingale so that

\[
E\{M_n(\omega)I_{A(\omega)I_{t \geq T_n^-}}I_{(T_n \leq t)}\} = 0,
\]

which contradicts (3.6). The theorem has been proved. \( \Box \)
Lemma 3.3 provides an obvious extension of the definition of the stochastic integral \((f \circ Q)_t\) to \(f \in L^2_{\text{loc}}(\tilde{P})\) and so Theorem 3.3 extends in the following manner.

**COROLLARY 3.1.** \(\{m, m_{\text{loc}}\} \in \mathcal{M}^2_{\text{loc}}\).

To obtain the representation for martingales in \(\mathcal{M}^1_{\text{loc}}\), two preliminary results are needed.

**LEMMA 3.6.** (i) Let \(f \in \mathcal{P}\). Then \(f \in L^1(P)\) if and only if \(f \in L^1(\tilde{P})\). In fact, \(\|f\|_1 = \|f\|_{\tilde{P}}^*\). In particular, \(L^1(P) = L^1(\tilde{P}) = L^1(Q)\).

(ii) Let \(f \in L^2(\tilde{P})\). Then \(f \in L^1(\tilde{P})\) and

\[
(f \circ Q)_t = \int_Z \int_{R^*} f(z, s)I_{(0, t]}(s)Q(dz, ds).
\]

(iii) If \(f \in L^1(\tilde{P})\), then

\[
m_t = \int_Z \int_{R^*} f(z, s)I_{(0, t]}(s)Q(dz, ds) \in \mathcal{M}^1 \cap \mathcal{A}.
\]

**Proof.** By an argument which is almost identical to the proof of [16, Prop. 3], it can be shown that (3.7) holds for \(f \in L^2(\tilde{P}) \cap L^1(\tilde{P}) \cap L^1(P)\).

Since \(L^2(\tilde{P}) \subset L^1(\tilde{P})\) the second assertion will then follow from the first one. Now let \(\Phi\) consist of all bounded functions \(f(z, t) \in \mathcal{P}\) such that \(f(z, t) \equiv 0\) for \(t \geq T_n\) for some \(n < \infty\). Then certainly \(\Phi \in L^2(\tilde{P}) \cap L^1(\tilde{P}) \cap L^1(P)\). So \(\|f \circ Q\|_1 \leq 2\|f\|_{\tilde{P}}\) for \(f \in \Phi\), and in particular, by (3.7),

\[
0 = E(f \circ Q)_{\omega_{\infty}} = \|f\|_1 - \|f\|_{\tilde{P}}^*.
\]

Thus the identity map, restricted to \(\Phi\), from \(L^1(P)\) to \(L^1(\tilde{P})\) preserves norms. Since \(\Phi\) is dense in \(L^1(P)\) and \(L^1(\tilde{P})\), the first assertion follows. To prove the last assertion, let \(f_k, k \geq 1\), be a sequence in \(L^2(\tilde{P})\) such that \(\|f - f_k\|_1\) converges to zero. Then \(m_k = (f_k \circ Q)_k \in \mathcal{M}^2\) and by (3.7), \(E|m_k - m| \leq 2\|f - f_k\|_1\) converges to zero uniformly in \(t\) so that \(m \in \mathcal{M}^1\).

**PROPOSITION 3.3.** Let \(M\) be a \(\mathcal{F}_{T_n}\)-measurable r.v. for some \(n \geq 1\). Suppose \(E|M| < \infty\). Then there is a unique \(f(z, t) \in L^1(\tilde{P})\) such that

\[
M_{t \geq T_n} = \int_Z \int_{R^*} f(z, s)I_{(0, t]}(s)P(dz, ds).
\]

Furthermore \(f(z, s) = 0\) for \(s \leq T_{n-1}\) and \(s > T_n\), and

\[
E|M_{T_n < \infty}| = \|f\|_1.
\]

**Proof.** Since \(M_{t \geq T_n} = M_{(T_n < \infty)](t \geq T_n)}\), it can be assumed that \(M = M_{(T_n < \infty)}\). By Corollary 2.3 there exist r.v.s \(M^k\) of the form

\[
M^k(\omega) = \sum_i z_i I_{(X_{T_n} \in B_i)} I_{A_i(\omega)}(T_{n \in C_i}),
\]

where \(z_i \in R, B_i \in \mathcal{X}, A_i \in \mathcal{F}_{T_{n-1}}\) and \(C_i \in \mathcal{B}[0, \infty)\), such that \(E|M - M^k| \to 0\). If

---

6 It may be worth repeating, to clarify the content of (3.7), that the integral on the right in (3.7) is a Lebesgue–Stieltjes integral whereas that on the left is the stochastic integral as defined in Lemma 3.3.
$f^k$ is defined by
\[
f^k(z, \omega, t) = \sum_i \alpha_i I_{A_i}(z) I_{C_i}(t) I_{T_{n_i} = t < t_{n_i}},
\]
then it is clear that (3.8) and (3.9) hold for $M^k$ and $f^k$. The assertion now follows by taking limits. \( \square \)

**Lemma 3.7.** Let $m_t \in \mathcal{M}^1 \cap \mathcal{A}$. Then there exists $f \in L^1(\mathcal{P})$ such that
\[
m_t - m_0 = \int_{Z \times R^*} f(z, s) I_{(0, t]}(s) Q(dz, ds)
\]
and
\[
E \int_0^\infty |dm_t| = 2\|f\|_1.
\]

**Proof.** $m_t$ has the representation
\[
m_t - m_0 = \sum_{n=1}^\infty (M_n I_{T_n = t} - a_n(t)) = \sum_{n=1}^\infty \mu_n,
\]
where $M_n = \Delta m_{T_n}$, $a_n(t) \in \mathcal{A}$ is continuous, and $\mu_n \in \mathcal{M}$. Since $m_t \in \mathcal{A}$,
\[
\int_0^\infty E |dm_t| > \sum_{n=1}^\infty E |M_n|,
\]
so that by Proposition 3.3, there exist functions $f_n(z, t) \in L^1(\mathcal{P})$ which vanish outside of $\{T_{n-1} \leq t \leq T_n\}$ such that $E|M_n| = \|f_n\|_1$ and
\[
M_n I_{T_n = t} = \int_{Z \times R^*} f_n(z, s) I_{(0, t]}(s) P(dz, ds).
\]
By Lemma 3.6,
\[
\eta_n(t) = a_n(t) - \int_{Z \times R^*} f_n(z, s) I_{(0, t]}(s) \mathcal{P}(dz, ds) \in \mathcal{M}^1.
\]
But $\eta_n(t)$ is continuous so that $\eta_n(t) \equiv 0$ by Theorem 3.2. Therefore (3.10) holds for $f(z, t) = \sum_{n=1}^\infty f_n(z, t)$ and (3.11) follows from Lemma 3.6 and the fact that $f_k(z, t)f_l(z, t) \equiv 0$ for $k \neq n$. \( \square \)

**Theorem 3.4.** $m_t \in \mathcal{M}^1_{\text{loc}}$ if and only if there exists $f \in L^1_{\text{loc}}(\mathcal{P})$ such that
\[
m_t - m_0 = \int_{Z \times R^*} f(z, s) I_{(0, t]}(s) Q(dz, ds).
\]

**Proof.** The sufficiency follows readily from Lemma 3.6 (iii). To prove the necessity one starts by noting that by [16, Lemma 3 and Prop. 4] there exists an increasing sequence of s.t.s $S_k$ converging to $\infty$ such that for each $k$, $m_{t \wedge S_k} - m_0$ has a decomposition
\[
m_{t \wedge S_k} - m_0 = \mu_{t}^k + \eta_{t}^k,
\]
where $\mu_{t}^k \in \mathcal{M}_0^1$ and $\eta_{t}^k \in \mathcal{M}_0^1 \cap \mathcal{A}$. By Lemmas 3.6 (ii) and 3.7, there exists
\( f^k \in L^1(\bar{P}) \) such that
\[
m_{t \wedge S_k} - m_0 = \int_{R^+} \int_{R^+} f^k(z, s) 1_{(0, \infty)}(s) Q(\text{d}z, \text{d}s).
\]

It is clear that \( f^k(z, t) = f^{k+1}(z, t) \) for \( t \leq S_k \). Thus (3.12) holds for \( f \in L^1_{\text{loc}}(\bar{P}) \) defined by \( f(z, t) = f^k(z, t) \) for \( t \leq S_k \). \( \square \)

The results above give a characterization of the classes \( \mathcal{M}^2, \mathcal{M}^2_{\text{loc}}, \mathcal{M}^1 \cap \mathcal{A} \) and \( \mathcal{M}^1_{\text{loc}} \). It seems much more difficult to obtain a useful characterization of the class \( \mathcal{M}^1 \).

The (local) martingales with respect to \((\Omega, \mathcal{F}_t, P)\) have been represented as sums or integrals of the “basic” martingales \( Q(B, t) \). The latter are associated in a one-to-one manner with the counting processes \( P(B, t) \) which count those jumps of the underlying process \( x \) which end in the set \( B \). These jumps are distinguished by their final values. Now it is also possible to distinguish jumps by their values. The corresponding counting processes will be of the form \( p(A, t) \) which counts those jumps of the \( x \) process which have values in the set \( A \). The martingales \( q(A, t) \) associated with the \( p(A, t) \) also form a “basis” for the set of all martingales on \((\Omega, \mathcal{F}_t, P)\) as will be shown below. The alternative representation obtained with this basis can sometimes be more useful since the description of the \( x \) process is, in practice, often given in terms of a statistical characterization of the jumps of \( x \).

For simplicity of notation it will be assumed in the remainder of this section that the \( x \) process starts at time 0 in a fixed state, i.e., \( x_0(\omega) = x_0(\omega') \) for all \( \omega, \omega' \in \Omega \). \(^7\) Next it is assumed that there is given a set \( \Sigma \) of transformations \( \sigma : Z \rightarrow Z \) with the following properties:

(i) \( \Sigma \) contains the jumps of the \( x \) process, i.e., if \( x_s(\omega) \neq x_s(\omega') \) for some \( s \in R^+ \), \( \omega \in \Omega \), then there is a unique \( \sigma \in \Sigma \) such that \( \sigma(x_s(\omega)) = x_s(\omega) \);

(ii) \( \Sigma \) contains a distinguished element \( \sigma_0 \) corresponding to the identity transformation, i.e., \( \sigma_0(z) = z \) for all \( z \in Z \).

To each sample function \( \omega \in \Omega \) of the \( x \) process is associated a function \( \gamma(\omega) : R^+ \rightarrow \Sigma \) defined as follows:

\[
\gamma(\omega) = \begin{cases} 
\sigma_0 & \text{if } t = 0 \text{ or if } x_t(\omega) = x_{t-}(\omega), \\
\sigma & \text{if } x_t(\omega) \neq x_{t-}(\omega), 
\end{cases}
\]

where \( \sigma \in \Sigma \) is the unique element for which \( \sigma(x_{t-}(\omega)) = x_t(\omega) \).

Remark. (i) Given a sample path \( x_t(\omega), 0 \leq s \leq t \), there corresponds in a one-to-one manner a sample path \( \gamma_t(\omega), 0 \leq s \leq t \).

(ii) The functions \( \gamma(\omega) \) are not right continuous. However if \( \gamma_t(\omega) = \sigma_0 \), then \( \gamma_{t-}(\omega) = \sigma_0 \). This observation will be used later in an example.

The following “regularity” assumption appears to be necessary. In practice it is readily verifiable.

Assumption. There is a \( \sigma \)-field \( \Xi \) on \( \Sigma \) such that \( \mathcal{F}_t \) coincides with the \( \sigma \)-field generated by subsets of the form \( \{ \omega | \gamma_s(\omega) \in A \} \), where \( s \leq t \) and \( A \in \Xi \).

\(^*\) It should be noted however that the results below continue to hold in the absence of this simplification.
With the assumptions above it is clear that the processes \( x_t \) and \( \gamma_t \) are equivalent alternative descriptions of the same process. In particular they generate the same \( \sigma \)-fields, so that the two processes have the same martingales. The representation theorems derived earlier for the \( x_t \) process can be applied to the \( \gamma_t \) process but there is a minor point to be cleared up. Recall that it was assumed that the \( x_t \) process was right-continuous whereas \( \gamma_t \) is not. However the assumption of right-continuity was used only to establish the right-continuity of the family \( \mathcal{F}_t \). This continues to hold of course since \( \gamma_t \) and \( x_t \) generate the same \( \sigma \)-fields \( \mathcal{F}_t \). Hence one can apply the representation theorems.

**Definition 3.4.** Let \( A \in \mathcal{F} \). Let
\[
p(A, t) = \sum_{s \leq t} I_{\{x_s = x_{s-1}\}} I_{\gamma_s \in A} = \sum_{s \leq t} I_{\{x_s = x_{s-1}\}} I_{\gamma_s \in A}
\]
be the number of jumps of the \( x_t \) process with "values" in \( A \) and which occur prior to \( t \).

By Proposition 3.2 there is a unique continuous process \( p(A, t) \) such that the process \( q(A, t) = p(A, t) - p(A, 0) \) is in \( \mathcal{M}^2_{\text{loc}} \). In analogy with Definitions 3.2 and 3.3 one can define the subsets of \( \mathcal{P}_x : L^2(\bar{p}), L^2_{\text{loc}}(\bar{p}), L^1(\bar{p}), L^1(p) \) etc.\(^8\) Lemma 3.3 describes the stochastic integrals \( (f \circ q) \) for \( f \in L^2(\bar{p}) \). An application of Theorem 3.3, Corollary 3.1, Lemma 3.7 and Theorem 3.4 yields the following representation theorem.

**Theorem 3.5.** (i) \( m_t \in \mathcal{M}^2(\mathcal{M}^2_{\text{loc}}) \) if and only if \( m_t - m_0 = (f \circ q)_t \) for some \( f \in L^2(\bar{p})(L^2_{\text{loc}}(\bar{p})) \).

(ii) \( m_t \in \mathcal{M}^1 \cap \mathcal{A}(\mathcal{M}^1_{\text{loc}}) \) if and only if \( m_t - m_0 = \int \int f(\sigma, s) I_{(0,\Omega)}(s) q(d\sigma, ds) \) for some \( f \in L^1(\bar{p})(L^1_{\text{loc}}(\bar{p})) \).

**4. An example.** This section consists of a simple example showing how Theorem 3.5 can be applied. The example will be further elaborated in [3]. Let \( Z \) be countable and let \( \mathcal{F} \) consist of all subsets of \( Z \). Let \( x_t \) be a process with values in \( Z \) and satisfying the assumptions listed at the beginning of \( \S \) 3. Suppose that from each state \( z \) the process \( x_t \) can jump to one of \( n \) states. In terms of a state-transition diagram (see Fig. 1) there are \( n \) transitions or links emanating from each state or node. Label these transitions by the symbols \( \sigma_1, \ldots, \sigma_n \). Let \( \Sigma = \{\sigma_0, \ldots, \sigma_n\} \). Thus each \( \sigma \in \Sigma \) corresponds to a transformation in \( Z \). \( \sigma_0 \) is the identity transformation. Let \( \Xi \) be the set of all subsets of \( \Sigma \). The \( x_t \) process defines the process of transitions \( \gamma_t \). Evidently \( \Sigma, \Xi \) satisfy the assumptions made above.

---

\(^8\) \( \mathcal{P}_x \) is the set of predictable functions of \( (\sigma, \omega, t) \in \Sigma \times \Omega \times R_+ \), defined in analogy with Definition 3.2.
Let \( p_i(t) = p_i(\{\sigma_i\}, t) \), \( \tilde{p}_i(t) = \tilde{p}_i(\{\sigma_i\}, t) \) and \( q_i(t) = q_i(\{\sigma_i\}, t) \), \( 0 \leq i \leq n \). From a remark made in the last section, \( \int_{\{x = a\}} I_{(a, x]} = 0 \). Hence \( p_0(t) \equiv 0 \) and so \( q_0(t) \equiv 0 \). Theorem 3.5 simplifies to the following. Here the predictable integrands are functions of \((\omega, t)\) only.

**Theorem 4.1.**

(i) \( m_t \in \mathcal{M}^2(\mathcal{M}^2_{\text{loc}}) \) if and only if \( m_t - m_0 = \sum_{i=1}^n \int_0^t f_i(\omega, s) q_i(ds) \) for some \( f_i \in L^2(\tilde{p}_i)(L^2_{\text{loc}}(\tilde{p}_i)) \), \( 1 \leq i \leq n \).

(ii) \( m_t \in \mathcal{M}^1 \cap \mathcal{M}(\mathcal{M}^1_{\text{loc}}) \) if and only if \( m_t - m_0 = \sum_{i=1}^n \int_{(0, t]} f_i(s)q_i(ds) \) for some \( f_i \in L^1(\tilde{p}_i)(L^1_{\text{loc}}(\tilde{p}_i)) \), \( 1 \leq i \leq n \).

**Example.** Let \( x_t \) be a process taking values in a countable state space and of the type described immediately above. From each state the process can make \( n \) transitions \( \sigma_1, \ldots, \sigma_n \) as sketched in Fig. 1. Let \( p_i(t), \tilde{p}_i(t), q_i(t) \) be as in Theorem 3.6.

Let \( \lambda(t), \rho_1(t), \ldots, \rho_n(t) \) be nonnegative predictable processes such that

\[
\sum_{i=1}^n \rho_i(t) = 1, \tag{4.1}
\]

\[
p_i(t \wedge T_k) - \int_0^{t \wedge T_k} \rho_i(s)\lambda(s) ds \in \mathcal{M}^1, \quad k = 1, 2, \ldots, n. \tag{4.2}
\]

Then the processes \( \lambda(t), \rho_i(t) \) have the following interpretation: since from (4.1) and (4.2),

\[
\left( \sum_{i=1}^n p_i(t \wedge T_k) - \int_0^{t \wedge T_k} \lambda(s) ds \right) \in \mathcal{M}^1 \tag{4.3}
\]

and since \( \sum_{i=1}^n p_i(t) \) is just the total number of jumps of the process occurring prior to \( t \), therefore the probability that the process \( x_t \) makes a transition in the time interval \([t, t+h] \), conditioned on the past \( \mathcal{F}_t \) of the process, is equal to \( \lambda(t)h + o(h) \). Similarly, \( \rho_j(t) \) is the probability that the process makes a transition represented by \( \sigma_j \), conditioned on \( \mathcal{F}_t \) and conditioned on the fact that a transition does occur at \( t \).

Now since the process represented by the indefinite integral in (4.2) has continuous sample paths, it follows quite readily (see, e.g., [25, p. 153]) that the jump times of the process are totally inaccessible. Hence from Theorem 4.1 it can be concluded that every \( m_t \in \mathcal{M}^1_{\text{loc}} \) has a representation

\[
m_t - m_0 = \sum_{i=1}^n \left[ \int_0^t f_i(s) dp_i(s) - \int_0^t f_i(s)\rho_i(s)\lambda(s) ds \right] \tag{4.4}
\]

for some predictable processes \( f_i \in L^1_{\text{loc}}(\rho_i\lambda) \), i.e., for which

\[
\int_0^t f_i(s)\rho_i(s)\lambda(s) ds < \infty \quad \text{a.s. for all } t \in R_+. \]

This result indicates how one can immediately write down the representation results if the process \( x_t \) is described in terms of the "rate" processes \( \lambda \) and the "transition" probabilities \( \rho_i \). It should be kept in mind, however, that it has not been proved that given processes \( \lambda(t) \) and \( \rho_i(t) \) there exists a process \( x_t \) for which (4.2) holds. This question of existence will be pursued in [3]. The next remark
relates to the representation (4.4), which asserts that the $n$ local martingales in (4.2) indeed form a “basis” for the space of all local martingales $\mathcal{M}_1^{loc}$. The question is whether $n$ is the minimum number of martingales in every basis of $\mathcal{M}_1^{loc}$. For the case where $x$ is a Gaussian process the minimum number of martingales has been called the “multiplicity” of the process by Cramer [8], [9]. It turns out that this notion of multiplicity extends in a very natural way to arbitrary processes [13]. From the results of [13] the following sufficient condition can be obtained: Suppose that the processes $\rho_i(s)\lambda(s)$ satisfy

$$\rho_i(s)\lambda(s) > 0 \iff \rho_j(s)\lambda(s) > 0 \quad \text{all } i, j.$$  

Then $n$ is the minimum number of martingales in a representation of $\mathcal{M}_1^{loc}$.

Finally, specialize the example still further and assume that $x$ is a counting process, i.e., $x_0 = 0$, $x_t$ takes integer values and has unit positive jumps. Then $x_t$ is a direct extension of a Poisson process. The state-transition diagram then simplifies to that of Fig. 2 and since $n = 1$ in (4.1), (4.2) and (4.4), therefore $\rho_1(t) \equiv 1$ and can be omitted. Also $p_1(t) \equiv x_1(t)$ and so the representation (4.4) simplifies to (4.5). Every $m_t \in \mathcal{M}_1^{loc}$ can be written as

$$(4.5) \quad m_t - m_0 = \int_0^t f(s) \, dx_s - \int_0^t f(s)\lambda(s) \, ds,$$

where $f$ is a predictable function such that

$$\int_0^t f(s)\lambda(s) \, ds < \infty \quad \text{a.s. for all } t \in R_+.$$  

Fig. 2. Transition diagram for counting process

This representation result has been obtained by very different techniques by several authors [4], [5], [11], [12]. However even here the cited references prove (4.5) for the special case where the probability law of the $x_t$ process is mutually absolutely continuous with respect to the probability law of a standard Poisson process. Hence even for this special case, (4.5) is a strict generalization of the available results.

**Appendix. The increasing processes $\mathcal{P}(A, t)$ and the Lévy system.** This section attempts to give an intuitive interpretation of the increasing processes $\mathcal{P}(B, t)$ and shows the connection with the Lévy system for Hunt processes.

Begin with the observation that for all $B \in \mathcal{Z}$ the measure $P(B, t)$ is absolutely continuous with respect to the measure $\mathcal{P}(Z, t)$, i.e., there exists a predictable function $(\omega, t) \to n(B, \omega, t)$ such that

$$(A.1) \quad \mathcal{P}(B, t) = \int_0^t n(B, \omega, s) P(Z, ds).$$

To see this it is enough to demonstrate that for all predictable functions
\( \phi(\omega, s) = \phi^2(\omega, s) \) (i.e., all indicator functions),

\[
(A.2) \quad E \int_0^{\infty} \phi(\omega, s) \tilde{P}(Z, ds) = 0
\]

implies

\[
(A.3) \quad E \int_0^{\infty} \phi(\omega, s) \tilde{P}(B, ds) = 0.
\]

Suppose \((A.2)\) holds. Then

\[
\left\langle \int_0^t \phi(s) dQ(Z, s), \int_0^t \phi(s) dQ(Z, s) \right\rangle \equiv \int_0^t \phi^2(s) \tilde{P}(Z, ds) \equiv 0,
\]

and so

\[
0 \equiv \left\langle \int_0^t \phi(s) dQ(B, s), \int_0^t \phi(s) dQ(Z, s) \right\rangle
\]

\[= \int_0^t \phi^2(s) \tilde{P}(B \cap Z, ds) \quad \text{(by Lemma 3.1)}
\]

\[= \int_0^t \phi^2(s) \tilde{P}(B, ds),
\]

which proves \((A.3)\).

In exactly the same way as Lemma 3.2 was proved it can be shown that the \(n(B, \omega, s)\) considered as a set function in \(\mathcal{F}\) is countably additive in the sense that if \(B_1, B_2, \cdots\) is a disjoint sequence of sets in \(\mathcal{F}\), then

\[
\tilde{P} \left( \bigcup_i B_i, t \right) \equiv \sum_i \int_0^t n(B_i, s) \tilde{P}(Z, ds).
\]

Hence if one sets \(\tilde{P}(Z, t) \equiv \Lambda(t) \in \mathcal{M}^+_\text{loc}\), then the system \(\{n(B, t, \omega), \Lambda(t)\}\) is analogous to a Lévy system for Hunt processes (see [22]), and has a similar interpretation: the probability of \(x_t\) having a jump in \([t, t + dt]\) is \(d\Lambda(t) + o(dt)\), while \(n(A, t, \omega)\) is the chance that \(x_t \in A\) given \(\mathcal{F}_t\) and given that a jump occurs at \(t\).

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