If $c < \frac{1}{2}$, then it is true that

$$\lim_{n\to\infty} n \exp_2 - n^{1-2c}\left[1 - n^{2c}o_1\left(\frac{1}{n^{2c}}\right)\right] = 0. \quad (41)$$

Dividing the logarithm of both sides of (40) by n, taking the limit as $n \to \infty$ and utilizing (41), Property a) is obtained, provided that (39) holds, which we now show.

Using (38), we find that (39) is equivalent to

$$1 - \frac{1}{n^{2c}} + o_1\left(\frac{1}{n^{2c}}\right) < 1 - \frac{\log (4n)}{n},$$

which holds provided that

$$\frac{n^{1-2c}}{\log (4n)}\left[1 + n^{2c}o_1\left(\frac{1}{n^{2c}}\right)\right] > 1. \quad (42)$$

As n increases, the bracketed term on the left side of (42) approaches one. By l'Hôpital's rule, $n^{1-2c}/\log (4n)$ goes to infinity with n if $c < \frac{1}{2}$. For large enough n, (42) and thus (39) must hold. This completes the proof.

Proof of Property b): The proof runs parallel to that of Property a). Let

$$\alpha = \frac{\sqrt{2 \ln 2}}{n^c} \qquad \beta = \frac{\sqrt{2 \ln 2}}{n^v}, \qquad c > 0, \quad v > 0. \quad (43)$$

$H((1 - \beta)/2)$ can be expanded in a Taylor series about $\beta = 0$. The series can be grouped as

$$H\left(\frac{1-\beta}{2}\right) = 1 - \frac{1}{n^{2v}} + o_2(n^{2v}). \quad (44)$$

Apply (44) to (40). Dividing the logarithm of both sides by n^{1-2v} and taking the limit as $n \to \infty$ yields Property b). Q.E.D.

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Recursive Filtering for Two-Dimensional Random Fields

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***Abstract*—A class of recursive filtering problems for random fields with a two-dimensional parameter is considered. After a brief introduction of two-parameter stochastic calculus, a class of Markovian random fields generated by stochastic integral equations is defined and considered. It is then shown that the problem of estimating such a Markovian field in additive white Gaussian noise can be reduced to a recursive formalism. If the random field is itself Gaussian, the recursive formalism reduces to a finite set of stochastic integral equations involving the conditional mean and covariance.**

I. Introduction

Two-dimensional random fields are families of random variables parametrized by a two-dimensional parameter. As in the one-dimensional case, a model of additive white Gaussian noise is often a reasonable approximation to the physical situation. Intuitively, such an assumption should lead to a simple analysis that can be carried to fruition. Starting from this premise, we began in [1] with a study of detecting a random signal in white Gaussian noise and obtained a stochastic integral equation for the likelihood ratio in terms of some conditional moments of the signal. This correspondence is a sequel to [1], in which we shall consider the problem of estimating a random signal in additive white Gaussian noise.

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Let $R_+{}^2$ denote $[0,\infty) \times [0,\infty)$. For a point $t = (t_1,t_2) \in R_+{}^2$, we denote the rectangle $[0,t_1) \times [0,t_2)$ by A_t. Let ξ_t, $t \in R_+{}^2$, be the observed field and assume that it is expressible as

$$\xi_t = Z_t + \eta_t$$

where η is Gaussian with zero mean and $E\eta_t\eta_s = \delta(t - s)$. We will show that if Z is Markov, in a suitable sense to be defined, then the conditional distribution of Z on the boundary of A_t given $\{\xi_s, s \in A_t\}$ can be evaluated recursively, again in a suitable sense to be explained. Similar results are well known to hold in the one-dimensional case (see e.g., [2], [3]).

II. Stochastic Integral Equations and Markovian Fields

A white Gaussian noise η can be dealt with in a precise way by considering its integral

$$W_t = \int_{A_t} \eta_s \, ds, \qquad t \in R_+{}^2 \quad (2.1)$$

where A_t denotes the rectangle $[0,t_1] \times [0,t_2]$. The process W will be called a Wiener process. It is also useful to introduce a set function (also see [1])

$$W(A) = \int_A \eta_s \, ds, \qquad ds = ds_1 \, ds_2 \quad (2.2)$$

for any Borel set A in $R_+{}^2$. Because of the white noise property, we have

$$EW(A) = 0, \qquad EW(A)W(B) = \text{area } (A \cap B). \quad (2.3)$$

Hence $\{W(A_i), i = 1,2,\cdots,n\}$ are mutually independent if A_i are disjoint. It also follows that

$$EW_tW_{t'} = \text{area}\,(A_t \cap A_{t'}) = \min\,(t_1,t_{1'})\min\,(t_2,t_{2'}). \quad (2.4)$$

Now consider a stochastic integral equation

$$X_t = X_0 + \int_{A_t} m(X_s,s)\,ds + \int_{A_t} \sigma(X_s,s)W(ds) \quad (2.5)$$

where the stochastic integral was defined in [4] (see also [1]). Existence and uniqueness of a solution X can be established in exactly the same way as in one dimension, provided that m and σ satisfy a linear growth condition and a uniform Lipschitz condition [5].

Let Δ be a rectangle $[t_1, t_1 + \delta_1] \times [t_2, t_2 + \delta_2]$. We can write

$$X_{(t_1+\delta_1,t_2+\delta_2)} - X_{(t_1+\delta_1,t_2)} - X_{(t_1,t_2+\delta_2)} + X_{(t_1,t_2)} = \int_\Delta m(X_s,s)\,ds + \int_\Delta \sigma(X_s,s)W(ds). \quad (2.6)$$

Now consider a point t and the rectangle A_t. For any point τ outside of A_t, X_τ can be expressed in terms of $\{X_s, s \in \partial A_t\}$ and $\{W(\Delta), \Delta \subset \bar{A}_t\}$, where ∂A_t is the boundary

$$\partial A_t = \{(s_1,t_2), s_1 \le t_1\} \cup \{(t_1,s_2), s_2 \le t_2\}$$

and $\bar{A}_t$ is the complement of A_t. To see this, consider three cases

1) $\tau \succ t$, i.e., $\tau_1 \ge t_1, \tau_2 \ge t_2$
2) $\tau_1 \ge t_1, \tau_2 < t_2$
3) $\tau_1 < t_1, \tau_2 \ge t_2$.

In case 1) we can write from (2.5)

$$X_\tau - X_t = \int_{A_\tau \cap \bar{A}_t} m(X_s,s)\,ds + \int_{A_\tau \cap \bar{A}_t} \sigma(X_s,s)W(ds).$$

For case 2) we can write

$$X_\tau - X_{(t_1,\tau_2)} = \int_{A_\tau \cap \bar{A}_t} m(X_s,s)\,ds + \int_{A_\tau \cap \bar{A}_t} \sigma(X_s,s)W(ds).$$

Indeed, for all three cases we can define $t \wedge \tau = (\min\,(\tau_1,t_1), \min\,(\tau_2,t_2))$ and write

$$X_\tau - X_{t\wedge\tau} = \int_{A_\tau \cap \bar{A}_t} m(X_s,s)\,ds + \int_{A_\tau \cap \bar{A}_t} \sigma(X_s,s)W(ds), \qquad \tau \in \bar{A}_t. \quad (2.7)$$

Now take a point $t' \succ t$ and consider (2.7), for $\tau \in A_{t'} \cap A_t$. Then (2.7) can be viewed as a stochastic integral equation for X_τ, $\tau \in A_t \cap \bar{A}_{t'}$, with $\{X_s, s \in \partial A_t\}$ as an "initial" condition. Hence, for any $\tau \in A_{t'} \cap \bar{A}_t$, X_τ can be expressed in terms of the values of X on ∂A_t and $\{W(\Delta), \Delta \subset A_{t'} \cap \bar{A}_t\}$. It follows from the independence property of $W(\Delta)$ that for $s \in A_t$ and $\tau \in \bar{A}_t$, X_s and X_τ are conditionally independent given X on ∂A_t. We shall call a random field satisfying this property a Markovian field.

We should note that this definition of a Markovian field is a special case of the definition due to Lévy [6] and McKean [7]. The difference is that we require conditional independence of "future" and "past" given the "present" only when the "present" is the boundary of a rectangle, while Lévy and McKean allowed more general boundaries.

The Markov property can also be stated in terms of propagation from boundary to boundary. Suppose $\tau \succ t$ and X is Markov. Given $\{X_s, s \in A_t\}$, the distribution of $\{X_s, s \in \partial A_\tau\}$ depends only on $\{X_s, s \in \partial A_t\}$.

III. Change of Probability

The filtering problem outlined in the introduction can now be restated more precisely. Let $T = [0,a_1] \times [0,a_2]$ and let X_t, $t \in T$, be the observed field. X is assumed to have the form

$$X_t = \int_{A_t} Z_s\,ds + W_t, \qquad t \in T \quad (3.1)$$

where (Z,W) are independent processes. W is a Wiener process, and Z is Markov in the sense defined in Section II. It will be shown that a recursive formula can be derived for the conditional distribution of $\{Z_s, s \in \partial A_t\}$ given $\{X_s, s \in A_t\}$. We begin with a change in probability first considered in [1]. This is again a generalization of a known one-dimensional result (see e.g., [3]).

Theorem 1: Let $(\Omega,\mathscr{F},\mathscr{P})$ be a probability space and let $(Z_t, W_t, t \in T)$ be a pair of random fields defined on it. We assume that a) Z and W are independent processes; b) W is a Wiener process; c) with probability one, $\int_T Z_s^2\,ds < \infty$. Define a measure $\tilde{\mathscr{P}}$ on $(\Omega,\mathscr{F})$ by

$$\frac{d\tilde{\mathscr{P}}}{d\mathscr{P}} = \exp\left\{-\int_T Z_sW(ds) - \frac{1}{2}\int_T Z_s^2\,ds\right\}. \quad (3.2)$$

Then the following assertions are true:

1) $\tilde{\mathscr{P}}$ is a probability measure mutually absolutely continuous with respect to $\mathscr{P}$.
2) (X,Z) have the same distribution under $\tilde{\mathscr{P}}$ as (W,Z) under $\mathscr{P}$.

Let Λ_t be defined by

$$\Lambda_t = \exp\left[\int_{A_t} Z_sX(ds) - \frac{1}{2}\int_{A_t} Z_s^2\,ds\right]. \quad (3.3)$$

In [1] it is proved that Λ is a martingale and satisfies the integral equation

$$\Lambda_t = 1 + \int_{A_t} Z_s\Lambda_sX(ds) + \frac{1}{2}\left[\iint_{A_t\times A_t}\right] Z_sZ_{s'}\Lambda_{s\vee s'}X(ds)X(ds') \quad (3.4)$$

where the second integral is defined in [4] and $s \vee s' = (\max\,(s_1,s_1'), \max\,(s_2,s_2'))$. Observe that under $\tilde{\mathscr{P}}$, X is a Wiener process. Since Λ is to be considered a random field on $(\Omega,\mathscr{F},\tilde{\mathscr{P}})$, the integrals in (3.6) are well-defined stochastic integrals.

IV. Recursive Filtering Formula

Let Z, X, W be defined as in Theorem 1 and assume that the conditions of Theorem 1 are satisfied. Let f be a functional of $\{Z_s, s \in \partial A_t\}$ such that $E|f| = \tilde{E}|f| < \infty$, and define

$$H_{t,\tau}f = E(f \mid Z_s, s \in A_\tau), \qquad t \succ \tau. \quad (4.1)$$

If Z is a Markovian field then clearly $H_{t,\tau}f$ is a functional of $\{Z_s, s \in \partial A_t\}$. Now define

$$\Pi_tf = \tilde{E}(\Lambda_tf \mid X_s, s \in A_t). \quad (4.2)$$

Note that (see e.g., [3, p. 234])

$$E(f \mid X_s, s \in A_t) = \frac{\tilde{E}(\Lambda_tf \mid X_s, s \in A_t)}{\tilde{E}(\Lambda_t \mid X_s, s \in A_t)} = \frac{\Pi_tf}{\Pi_t(1)}. \quad (4.3)$$

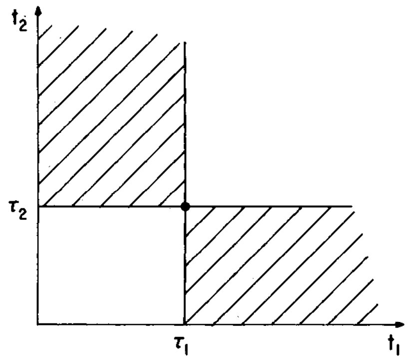

Fig. 1. Parameter values for conditional mean.

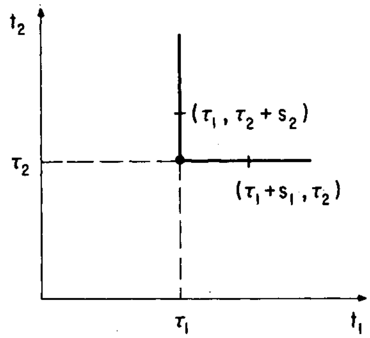

Fig. 2. Parameter values for covariance function.

Therefore, Π_t effectively incorporates the conditional distribution of $\{Z_s, s \in \partial A_t\}$ given $\{X_s, s \in A_t\}$ under $\mathscr{P}$ measure.

Theorem 2: Let (X,Z,W) be as defined in Theorem 1 and let the conditions of Theorem 1 be satisfied. In addition, let Z be a Markovian field. Let f be a function of $\{Z_s, s \in \partial A_t\}$ such that $E|f| < \infty$. Then

$$\Pi_t f = Ef + \int_{A_t} \Pi_s(Z_s H_{t,s} f) X(ds) + \frac{1}{2}\left[\iint_{A_t \times A_t}\right] \Pi_{s \vee s'}[Z_s Z_{s'} H_{t,s \vee s'}(f)] X(ds) X(ds'). \quad (4.4)$$

Proof: Equation (4.4) follows from (3.4) by multiplying both sides of (3.4) by f and taking expectation with respect to $\tilde{\mathscr{P}}$ measure. Observe that

$$\tilde{E}\left[\int_{A_t} Z_s \Lambda_s f X(ds) \mid X_\tau, \tau \in A_t\right] = \int_{A_t} \tilde{E}(Z_s \Lambda_s f \mid X_\tau, \tau \in A_t) X(ds).$$

Because under $\tilde{\mathscr{P}}$, X is a Wiener process independent of Z,

$$\begin{aligned} &\tilde{E}(Z_s \Lambda_s f \mid X_\tau, \tau \in A_t) \\ &= \tilde{E}\{Z_s \Lambda_s \tilde{E}(f \mid X_\tau, \tau \in A_t, Z_\tau, \tau \in A_s) \mid X_\tau, \tau \in A_t\} \\ &= \tilde{E}\{Z_s \Lambda_s H_{t,s} f \mid X_\tau, \tau \in A_t\} \\ &= \tilde{E}\{Z_s \Lambda_s H_{t,s} f \mid X_\tau, \tau \in A_s\} \\ &= \Pi_s(Z_s H_{t,s} f). \end{aligned}$$

The last term in (4.4) follows by taking nearly identical steps.

Q.E.D.

Equation (4.4) can now be regarded as an equation for the evolution of Π_t, which is an operator on functions of $\{Z_s, s \in \partial A_t\}$.

V. Gaussian Case

If Z is a Gaussian random field, then Z and X are jointly Gaussian, so that $\{Z_s, s \in \partial A_t\}$ given $\{X_s, s \in A_t\}$ is again a Gaussian process. Therefore, to determine the conditional distribution of $\{Z_s, s \in \partial A_t\}$ given $\{X_s, s \in A_t\}$, we only need to determine

$$\hat{Z}_{\tau,t} = E(Z_\tau \mid X_s, s \in A_t), \qquad \tau \in \partial A_t \quad (5.1)$$

$$R(\tau,\tau';t) = E\{(Z - \hat{Z}_{\tau,t})(Z_{\tau'} - \hat{Z}_{\tau',t}) \mid X_s, s \in A_t\}, \quad \tau, \tau' \in \partial A_t. \quad (5.2)$$

Now, $\tau \in \partial A_t$ means that either $\tau_1 + s_1 = t_1$ and $\tau_2 = t_2$, or $\tau_1 = t_1$ and $\tau_2 + s_2 = t_2$. Therefore, for each τ we need to evaluate $\hat{Z}_{\tau,t}$, for $t = (\tau_1 + s_1, \tau_2)$, $s_1 \geq 0$ and for $t = (\tau_1, \tau_2 + s_2)$, $s_2 \geq 0$. The set of values of t for which $\hat{Z}_{\tau,t}$ must be known is illustrated in Fig. 1.

For the covariance function $R(\tau,\tau';t)$, τ, $\tau' \in \partial A_t$, implies three possibilities: $\tau_1 = \tau_1' = t_1$ and $\max(\tau_2,\tau_2') \leq t_2$, or $\tau_2 = \tau_2' = t_2$ and $\max(\tau_1,\tau_1') \leq t_1$, or τ and τ' are unordered and $\tau \vee \tau' = t$. Fig. 2 illustrates this situation. For a fixed τ, the allowable values of τ' fall on the vertical and horizontal lines intersecting at τ or within the hashed areas, and the allowable t values fall along the vertical and horizontal lines. It is not known whether the covariance function satisfies a Riccati equation as in the one-dimensional case. Preliminary investigations have been discouraging.

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Sample Functions of a Gaussian Process Cannot Be Recovered from Their Zero Crossings

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***Abstract*—The types of ambiguities that preclude the possibility of recovering the sample functions of a Gaussian process from their zero-crossings are illustrated by a simple example.**

Recently a certain uniqueness relationship has been established between the stochastic processes at the input and the output of hard-limiting systems ([1, theorems 1–3]). It is the purpose of this correspondence to point out that these theorems do not

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