A Likelihood Ratio Formula for Two-Dimensional Random Fields

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***Abstract*—This paper is concerned with the detection of a random signal in white Gaussian noise when both the signal and the noise are two-dimensional random fields. The principal result is the derivation of a recursive formula for the likelihood ratio relating it to certain conditional moments of the signal. It is also shown that, except for some relatively uninteresting cases, a simple exponential formula for the likelihood ratio, such as one has in one dimension, is not possible.**

I. Introduction

ADDITIVE white Gaussian noise has proved to be a useful model for many signal processing problems. On the one hand, it is often an adequate approximation of the underlying physical situation. On the other hand, the assumption of additive white Gaussian noise often allows the analysis to be carried to fruition. This is illustrated by the familiar binary detection problem outlined here.

Let ξ_t, $0 \le t \le T$, be the observed process. Let S_t, $0 \le t \le T$, be a possibly random signal. We wish to test between the two hypotheses

$$H_0: \xi_t \text{ is a zero-mean Gaussian process with covariance function } E_0\xi_t\xi_s = \delta(t-s)$$

$$H_1: \eta_t = \xi_t - S_t \text{ is a zero-mean Gaussian process with covariance function } E_1\eta_t\eta_s = \delta(t-s). \tag{1.1}$$

Under rather general conditions on S_t, the likelihood ratio is given by the well-known formula (see, e.g., [1])

$$L = \exp\left[\int_0^T \hat{S}_t\xi_t\,dt - \tfrac{1}{2}\int_0^T \hat{S}_t^2\,dt\right] \tag{1.2}$$

where $\hat{S}_t = E_1(S_t \mid \xi_\tau, 0 \le \tau \le t)$ and the integral $\int_0^T \hat{S}_t\xi_t\,dt$ is to be interpreted as an Itô integral.

Equation (1.2) shows that the likelihood-ratio detector can be realized by an estimator matched-filter combination provided that the matched filter operation (viz: $\int_0^T \hat{S}_t\xi_t\,dt$) is carefully interpreted as an Itô integral. The likelihood ratio can also be expressed in a recursive form. Let L_t be defined by

$$L_t = E_0(L \mid \xi_\tau, \tau \le t). \tag{1.3}$$

Then an application of the Itô differentiation rule (see, e.g., [7]) shows that L_t satisfies the equation

$$L_t = 1 + \int_0^t L_\tau\hat{S}_\tau\xi_\tau\,d\tau \tag{1.4}$$

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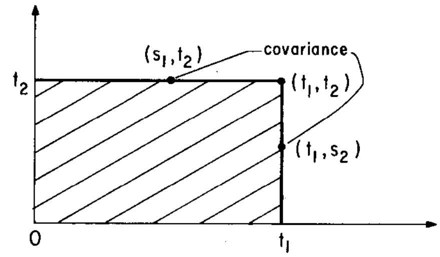

Fig. 1. Parameter values for conditional moments.

which shows that the incremental change in L_t is expressed by

$$dL_t = L_{t+dt} - L_t \cong L_t\hat{S}_t\left(\int_t^{t+dt} \xi_\tau\,d\tau\right)$$

where the right side depends on the current values of L_t and $\hat{S}_t$ and on the new observation $\int_t^{t+dt} \xi_\tau\,d\tau$.

In this paper we shall derive a generalization of (1.4) for two-dimensional random fields. Let $\xi(t_1,t_2)$, $0 \le t_1 \le T_1$, $0 \le t_2 \le T_2$, be the observed field and let S_t, $t \in [0,T_1] \times [0,T_2]$, be the signal field. Again we wish to test between the pair of hypotheses given by (1.1). If we now let L_t be defined by

$$L_t = E_0(L \mid \xi_\tau, \tau \in [0,t_1] \times [0,t_2]) \tag{1.5}$$

then our result will imply that the incremental change

$$dL_t = L_{(t_1+dt_1,t_2+dt_2)} - L_{(t_1+dt_1,t_2)} - L_{(t_1,t_2+dt_2)} + L_{(t_1,t_2)} \tag{1.6}$$

is expressible in terms of L_t

$$\hat{S}_t = E_1(S_t \mid \xi_\tau, \tau \in [0,t_1] \times [0,t_2]) \tag{1.7}$$

and

$$E_1\{S_{(s_1,t_2)}S_{(t_1,s_2)} \mid \xi_\tau, \tau \in [0,t_1] \times [0,t_2]\},\quad s \in [0,t_1] \times [0,t_2]. \tag{1.8}$$

Fig. 1 gives an another interpretation of the previous result. The increment dL_t can be expressed in terms of L_t the conditional mean of S_τ (τ being on the boundary of the rectangle), given the values of ξ in the rectangle, and of the conditional covariance of pairs of values of S, one on each leg of the boundary, given the values of ξ in the rectangle.

If the signal S and the noise η are jointly Gaussian (under hypothesis H_1) then the conditional covariance will be a deterministic function. In such a case, the conditional mean of S on the boundary given the observation in the rectangle and L_t will suffice to determine dL_t.

II. The Wiener Process

As in one dimension, we define a two-parameter white noise $\{\eta_t, t \in R^2\}$ as a zero-mean random field with

$$E\eta_t\eta_{t'} = \delta(t - t'). \tag{2.1}$$

Just as in the one-dimensional case, the calculus of white noise is made precise by the introduction of a random field

$$W_t = \int_0^{t_1} \int_0^{t_2} \eta_{(s_1,s_2)}\, ds_1\, ds_2, \qquad t \in R_+{}^2 = [0,\infty)^2 \tag{2.2}$$

which is a zero-mean random field with

$$EW_tW_{t'} = \min(t_1,t_1')\min(t_2,t_2'). \tag{2.3}$$

If η is Gaussian, we shall call W a Wiener process.

Actually, it is convenient to introduce a random set function

$$W(A) = \int_A \eta_s\, ds, \qquad A \subset R^2 \tag{2.4}$$

which has a covariance property

$$EW(A)W(B) = \int_{A \cap B} ds. \tag{2.5}$$

If η is a Gaussian white noise, then $W(A)$ is an additive process (or a process with independent areas). That is, if $A_1,A_2,\cdots,A_m$ are disjoint sets, then $W(A_1),W(A_2),\cdots,W(A_m)$ are independent random variables.

For points in R^2, we define a partial ordering $\succ$ by

$$t \succ t' \Leftrightarrow t_1 \geq t_1',\ t_2 \geq t_2'.$$

With respect to $\succ$, the Wiener process is a martingale; i.e.,

$$E(W_t \mid W_s, s \prec t') = W_{t'}, \qquad t' \prec t. \tag{2.6}$$

To prove (2.6), let A_t denote the rectangle $[0,t_1] \times [0,t_2]$ and $\bar{A}_t$ denote its complement. Then, for $t \succ t'$ we have

$$W_t = W(A_t) = W(A_{t'}) + W(A_t \cap \bar{A}_{t'}).$$

Since $A_{t'}$ and $A_t \cap \bar{A}_{t'}$ are disjoint, $W(A_{t'})$ and $W(A_t \cap \bar{A}_{t'})$ are independent, so that

$$E(W_t \mid W_s, s \prec t') = W(A_{t'}) + EW(A_t \cap \bar{A}_{t'}) = W_{t'}.$$

Multiparameter Wiener processes are not new and have been studied by a number of authors [2]–[4]. They provide a natural framework for dealing with white Gaussian noise in a precise way. For example, the two hypotheses in (1.1) can now be restated for two-parameter random fields as follows.

Let $T = [0,T_1] \times [0,T_2]$ and let $X_t, S_t, t \in T$, be a pair of random fields. We wish to test between the pair of hypotheses

H_0: X_t is a Wiener process

$$H_1: N_t = X_t - \int_{A_t} S_\tau\, d\tau \text{ is a Wiener process.} \tag{2.7}$$

Throughout this paper A_t will always denote the rectangle $[0,t_1] \times [0,t_2]$.

III. Stochastic Integrals

Since both (1.2) and (1.3) involve stochastic integrals, it is not surprising that we need a generalization of the Itô integral to the two-dimensional parameter case. This can be done as follows [5], [6].

Let $\{W_t, t \in R_+{}^2\}$ be a Wiener process and $\{\phi_t, t \in R_+{}^2\}$ be a random field such that as follows:

(a) for any $t \in R_+{}^2$, $\{\phi_s, W_s, s \in A_t\}$ is independent of $\{W(\Delta), \Delta \subset \bar{A}_t\}$

$$\text{(b)} \quad \int_{R_+{}^2} E\phi_t^2\, dt < \infty. \tag{3.1}$$

We interpret (a) to mean that "future" increments of W are independent of the "past" of W and ϕ. First, take a rectangle A of finite area and subdivide it by a sequence of partitions $\Pi_n = \{\Delta_{\nu,n}\}$ such that $\max_\nu$ area $(\Delta_{\nu,n}) \xrightarrow[n\to\infty]{} 0$. We define

$$\int_A \phi_t W(dt) = \lim_{n\to\infty} \text{in q.m.} \sum_\nu \phi_{t_{\nu,n}} W(\Delta_{\nu,n}) \tag{3.2}$$

where lim in q.m. is the limit in the quadratic mean and $t_{\nu,n}$ denotes the lower left corner of $\Delta_{\nu,n}$; i.e.,

$$t_{\nu,n} = (\inf t_1, \inf t_2), \qquad t \in \Delta_{\nu,n}.$$

It is clear that (3.2) makes the stochastic integral a forward increment integral. To complete our definition set

$$\int_{R_+{}^2} \phi_t W(dt) = \lim_{m\to\infty} \text{in q.m.} \int_{[0,m]^2} \phi_t W(dt). \tag{3.3}$$

The stochastic integral so defined has a number of important properties which are direct generalizations of one-dimensional counterparts.

Linearity:

$$\int_{R_+{}^2} (a\phi_t + b\psi_t)W(dt) = a\int_{R_+{}^2} \phi_t W(dt) + b\int_{R_+{}^2} \psi_t W(dt). \tag{3.4}$$

Martingale:

If $X_t = \int_{A_t} \phi_s W(ds)$, then $\{X_t, t \in R_+{}^2\}$ is a martingale. It is sample continuous if a separable version is chosen. (3.5)

If $X_t = \int_{A_t} \phi_s W(ds)$,

$$\text{then } Y_t = X_t^2 - \int_{A_t} \phi_s^2\, ds \text{ is a martingale.} \tag{3.6}$$

We note that (3.5) implies that $EX_t = 0$ and (3.6) implies that

$$EX_t^2 = \int_{A_t} \phi_s^2\, ds$$

which together with linearity imply that

$$E\left(\int_{A_t} \phi_s W(ds)\right)\left(\int_{A_t} \psi_s W(ds)\right) = \int_{A_t} E\phi_s\psi_s\, ds. \tag{3.7}$$

So far there are no surprises. For a one-dimensional Wiener process W_t, $t \in [0,\infty)$, $W_t^2 - t$ is a martingale which can be expressed (by Itô's differentiation rule) as $W_t^2 - t = 2\int_0^t W_s\, dW_s$. For $t \in R_+^2$, although $W_t^2 - t_1 t_2$ is a martingale, it cannot be expressed as a stochastic integral $\int_{A_t} \phi_s W(ds)$ for any ϕ. In this sense the integrals defined by (3.2) and (3.3) are incomplete. We need stochastic integrals of a second type which we shall denote by [6]

$$\left[\int_{R_+^2} \int_{R_+^2}\right] \psi(t,t')W(dt)W(dt')$$

where ψ satisfies the two following conditions similar to (3.1).

a) For any $t \in R_+^2$, $\{\psi(s,s'), W(s); s, s' \in A_t\}$ is independent of $\{W(\Delta), \Delta \subset \bar{A}_t\}$.

b)

$$\int_{R_+^2} \int_{R_+^2} E\psi^2(t,t')\, dt\, dt' < \infty.$$

Briefly, the definition is given as follows. Again, first consider a finite rectangle and subdivide it by a sequence of rectangular partitions $\{\Delta_{\nu,n}\}$. Denote the lower left corner of $\Delta_{\nu,n}$ by $t_{\nu,n}$. We set

$$\left[\iint_{A \times A}\right] \psi(t,t')W(dt)W(dt') = \lim_{n\to\infty} \text{in q.m.} \sum_\nu \sum_\mu \psi(t_{\nu,n},t_{\mu,n})W(\Delta_{\nu,n})W(\Delta_{\mu,n}) \quad (3.8)$$

where the double sum is taken over only those pairs (ν,μ) for which $t_{\nu,n}$ and $t_{\mu,n}$ are unordered. Extending the integral from a finite rectangle A to R_+^2 can be done in the usual manner.

Stochastic integrals of the second kind have a number of interesting properties, some of which are given in the following. More details can be found in [6].

Linearity:

$$\left[\iint_{A \times A}\right] [a\psi(t,t') + b\phi(t,t')]W(dt)W(dt') = a\left[\iint_{A \times A}\right] \psi(t,t')W(dt)W(dt') + b\left[\iint_{A \times A}\right] \phi(t,t')W(dt)W(dt'). \quad (3.9)$$

Martingale:

$$X_t = \left[\iint_{A_t \times A_t}\right] \psi(s,s')W(ds)W(ds') \text{ is a martingale.} \quad (3.10)$$

The integrand $\psi(t,t')$ can be set equal to zero at all ordered pairs (t,t') without changing the integral. (3.11)

The integral is symmetric, so that $\psi(t,t')$ can be replaced by $\frac{1}{2}[\psi(t,t') + \psi(t',t)]$ without changing the integral. (3.12)

It was shown in [6] that any square-integrable functional of a two-parameter Wiener process can be represented in terms of stochastic integrals of the first and second types. This was proved by using an important differentiation rule which will be stated as follows.

Let $\{X_t, t \in R_+^2\}$ be a martingale defined by

$$X_t = \int_{A_t} \phi_s W(ds) \quad (3.13)$$

and define

$$V_t = \int_{A_t} \phi_s^2\, ds. \quad (3.14)$$

Suppose that $f(x,t)$ is a function satisfying

$$\tfrac{1}{2}f''(x,t)\nabla V_t + \nabla f(x,t) = 0 \quad (3.15)$$

where ∇ denotes gradient with respect to t and $f'' = (\partial^2 f/\partial x^2)$. Then, $f(X_t,t)$ is a martingale and can be expressed as

$$f(X_t,t) - f(0,(t_1,0)) - f(0,(0,t_2)) + f(0,(0,0)) = \int_{A_t} f'(X_s,s)\phi_s W(ds) + \frac{1}{2}\left[\iint_{A_t \times A_t}\right] f''(X_{s\vee s'}, s \vee s')\phi_s\phi_{s'}W(ds)W(ds') \quad (3.16)$$

where $f' = (\partial f/\partial x)$, $f'' = (\partial^2 f/\partial x^2)$ and $s \vee s' = (\max(s_1, s_1'), \max(s_2, s_2'))$.

Functions satisfying (3.15) may be called harmonic functions. One important example of a harmonic function is $x^2 - V_t$, for which (3.16) yields

$$X_t^2 - V_t = 2\int_{A_t} X_s\phi_s W(ds) + \left[\iint_{A_t \times A_t}\right] \phi_s\phi_{s'}W(ds)W(ds'). \quad (3.17)$$

Another important example is $\exp(x - \frac{1}{2}V_t)$. If we set

$$\Lambda_t = \exp(X_t - \tfrac{1}{2}V_t) \quad (3.18)$$

then

$$\Lambda_t = 1 + \int_{A_t} \Lambda_s\phi_s W(ds) + \frac{1}{2}\left[\iint_{A_t \times A_t}\right] \phi_s\phi_{s'}\Lambda_{s\vee s'}W(ds)W(ds'). \quad (3.19)$$

It is clear now that (3.19) is the basis for a generalization to (1.4).

IV. A Likelihood Ratio Formula

Let T be a rectangle, say $[0,a_1] \times [0,a_2]$. Let $\{X_t, S_t, t \in T\}$ be a pair of processes with X representing the observed process and S the signal. Consider the two hypotheses

H_0: X_t is a Wiener process

$$H_1: W_t = X_t - \int_{A_t} S_\tau\, d\tau \text{ is a Wiener process.} \quad (4.1)$$

Let $\mathcal{P}_0$ and $\mathcal{P}_1$ denote the probability measures under these two hypotheses, and $\mathcal{P}_0^x$, $\mathcal{P}_1^x$ denote their restrictions to the σ-field of events generated by X. We are interested in conditions which ensure that $\mathcal{P}_1^x$ is absolutely continuous with respect to $\mathcal{P}_0^x$ and in finding the likelihood ratio

$$L_t = E_0\left(\frac{d\mathcal{P}_1^x}{d\mathcal{P}_0^x}\,\middle|\, X_\tau,\ \tau \in A_t\right). \tag{4.2}$$

We shall assume conditions which, though not unreasonable, are much stronger than necessary in order to simplify the derivation of the main result.

Theorem 1: Suppose that under hypothesis H_1 S and W are independent processes and $\int_T S_t^2\, dt < \infty$ with probability 1. Then $\mathcal{P}_1^x$ is mutually absolutely continuous with respect to $\mathcal{P}_0^x$ and L_t satisfies the equation

$$L_t = 1 + \int_{A_t} L_\tau E_1(S_\tau \mid X_s,\ s \in A_\tau) X(d\tau) + \frac{1}{2}\left[\iint_{A_t \times A_t}\right] L_{\tau \vee \tau'} E_1(S_\tau S_{\tau'} \mid X_s,\ s \in A_{\tau\vee\tau'}) X(d\tau) X(d\tau') \tag{4.3}$$

where $\tau \vee \tau' = (\max(\tau_1,\tau_1'), \max(\tau_2,\tau_2'))$.

Proof: First, we need to generalize the stochastic integral

$$\int_{R_+^2} \phi_t W(dt)$$

under the condition

$$\int_{R_+^2} \phi_t^2\, dt < \infty \text{ with probability 1}$$

instead of the condition

$$\int_{R_+^2} E\phi_t^2\, dt < \infty.$$

This is easily done by defining $\int_{R_+^2} \phi_t W(dt)$ as the limit in probability of a sequence $\int_{R_+^2} \phi_{n,t} W(dt)$ where $\{\phi_n\}$ is a sequence of truncations of ϕ such that for each n,

$$\int_{R_+^2} \phi_{n,t}^2\, dt \le n.$$

Next, consider a transformation of the probability measure $\mathcal{P}_1$ by the formula

$$\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}_1} = \exp\left(-\int_T S_\tau W(d\tau) - \frac{1}{2}\int_T S_\tau^2\, d\tau\right).$$

Because W and S are independent under $\mathcal{P}_1$, given $\{S_\tau,\ \tau \in T\}$ the stochastic integral $\int_T S_\tau W(d\tau)$ is a Gaussian variable with zero mean and variance $\int_T S_\tau^2\, d\tau$. Hence

$$E_1 \exp\left(-\int_T S_\tau W(d\tau)\right) = \exp\left(\frac{1}{2}\int_T S_\tau^2\, d\tau\right)$$

and $\tilde{\mathcal{P}}$ must be a probability measure. It is easy to show that (S,X) are distributed exactly under $\tilde{\mathcal{P}}$ as (S,W) under $\mathcal{P}_1$ (cf. [7, pp. 232–233]). Therefore, under $\tilde{\mathcal{P}}$ X is a Wiener process. That is, $\tilde{\mathcal{P}}^x = \mathcal{P}_0^x$. It also means that

$$\left[\exp\left(-\int_T S_\tau W(d\tau) - \frac{1}{2}\int_T S_\tau^2\, d\tau\right)\right]^{-1} = \exp\left(\int_T S_\tau X(d\tau) - \frac{1}{2}\int_T S^2\, d\tau\right)$$

is finite with $\tilde{\mathcal{P}}$ probability 1 and $\tilde{\mathcal{P}}$ must be equivalent to $\mathcal{P}_1$. Therefore, L_t can be calculated by

$$L_t = \tilde{E}\left(\frac{d\mathcal{P}_1}{d\tilde{\mathcal{P}}}\,\middle|\, X_s,\ s \in A_t\right)$$

where

$$\frac{d\mathcal{P}_1}{d\tilde{\mathcal{P}}} = \exp\left(\int_T S_\tau X(d\tau) - \frac{1}{2}\int_T S_\tau^2\, d\tau\right). \tag{4.4}$$

Now, define

$$\Lambda_t = \exp\left(\int_{A_t} S_\tau X(d\tau) - \frac{1}{2}\int_{A_t} S_\tau^2\, d\tau\right), \qquad t \in T. \tag{4.5}$$

Then we know from (3.19) that with $\tilde{\mathcal{P}}$ probability 1

$$\Lambda_t = 1 + \int_{A_t} \Lambda_\tau S_\tau X(d\tau) + \frac{1}{2}\left[\iint_{A_t \times A_t}\right] \Lambda_{\tau\vee\tau'} S_\tau S_{\tau'} X(d\tau) X(d\tau'). \tag{4.6}$$

Because $T = [0,a_1] \times [0,a_2]$, we have $(d\mathcal{P}_1/d\tilde{\mathcal{P}}) = \Lambda_a$, and $E\Lambda_a = 1$ implies that Λ_t is a martingale, so that

$$L_t = \tilde{E}(\Lambda_t \mid X_s,\ s \in A_t). \tag{4.7}$$

Equation (4.3) is obtained from (4.6) as follows. First

$$\begin{aligned} L_t &= \tilde{E}(\Lambda_t \mid X_s,\ s \in A_t) \\ &= 1 + \int_{A_t} \tilde{E}(\Lambda_\tau S_\tau \mid X_s,\ s \in A_t) X(d\tau) \\ &\quad + \frac{1}{2}\left[\iint_{A_t \times A_t}\right] \tilde{E}(\Lambda_{\tau\vee\tau'} S_\tau S_{\tau'} \mid X_s,\ s \in A_t) X(d\tau) X(d\tau'). \end{aligned} \tag{4.8}$$

Because under $\tilde{\mathcal{P}}$ X is a Wiener process independent of S

$$\tilde{E}(\Lambda_\tau S_\tau \mid X_s,\ s \in A_t) = \tilde{E}(\Lambda_\tau S_\tau \mid X_s,\ s \in A_\tau), \qquad \tau \in A_t$$

and

$$\tilde{E}(\Lambda_{\tau\vee\tau'} S_\tau S_{\tau'} \mid X_s,\ s \in A_t) = \tilde{E}(\Lambda_{\tau\vee\tau'} S_\tau S_\tau \mid X_s,\ s \in A_{\tau\vee\tau'}), \qquad \tau, \tau' \in A_t.$$

Finally, the well-known rule of transforming a conditional expectation under a change in probability [7, pp. 234–235] yields

$$\tilde{E}(\Lambda_\tau S_\tau \mid X_s,\ s \in A_\tau) = L_\tau E_1(S_\tau \mid X_s,\ s \in A_\tau)$$

$$\tilde{E}(\Lambda_{\tau\vee\tau'} S_\tau S_{\tau'} \mid X_s,\ s \in A_{\tau\vee\tau'}) = L_{\tau\vee\tau'} E_1(S_\tau S_{\tau'} \mid X_s,\ s \in A_{\tau\vee\tau'}).$$

Inserting these results in (4.8) completes the derivation of (4.3) and the proof of Theorem 1.

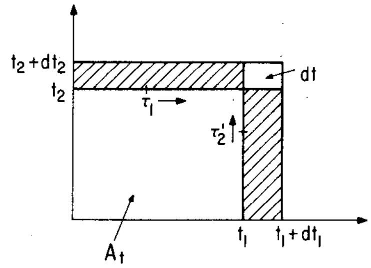

Fig. 2. Parameter values for likelihood ratio formula.

Remark: It is clear that the same proof suffices as long as it is possible to find a $\tilde{\mathscr{P}}$ probability measure on the sample space of (X,S) such that X is a Wiener process under $\tilde{\mathscr{P}}$, $\{X(\Delta), \Delta \subset \bar{A}_t\}$ is independent of $\{X_\tau, S_\tau, \tau \in A_t\}$ for each t, and

$$\frac{d\mathscr{P}_1}{d\tilde{\mathscr{P}}} = \exp\left(\int_T S_\tau X(d\tau) - \frac{1}{2}\int_T S_\tau^2\, d\tau\right).$$

Let

$$dL_t = L_{(t_1+dt_1,t_2+dt_2)} - L_{(t_1+dt_1,t_2)} - L_{(t_1,t_2+dt_2)} + L_{(t_1,t_2)}.$$

Then, roughly speaking, we have from (4.3) the relationship

$$\begin{aligned} dL_t &= L_t E_1(S_t \mid X_s, s \in A_t) X(dt) \\ &\quad + L_t \int_0^{t_1}\int_0^{t_2} E_1[S_{(\tau_1,t_2)} S_{(t_1,\tau_2')} \mid X_s, s \in A_t] \\ &\quad \cdot X(d\tau_1,dt_2) X(dt_1,d\tau_2'). \end{aligned} \tag{4.9}$$

Fig. 2 illustrates the various parameters in (4.9).

Equation (4.9) justifies the assertion that we made in the introduction, and relates L_t to conditional moments up to the second order of S on the boundary of A_t given X within A_t.

Suppose that under H_1 the signal S is Gaussian. Then, since S and W are independent, S and X are jointly Gaussian. Let $\hat{S}_{\tau,t}$ be defined by

$$\hat{S}_{\tau,t} = E_1(S_\tau \mid X_s, s \in A_t) \tag{4.10}$$

and

$$R(\tau,\tau';t) = E_1[(S_\tau - \hat{S}_{\tau,t})(S_{\tau'} - \hat{S}_{\tau',t}) \mid X_s, s \in A_t]. \tag{4.11}$$

Then $R(\tau,\tau';t)$ must be a deterministic function. We can now rewrite (4.3) as

$$\begin{aligned} L_t &= 1 + \int_{A_t} L_\tau \hat{S}_{\tau,\tau} X(d\tau) \\ &\quad + \frac{1}{2}\left[\iint_{A_t\times A_t}\right] L_{\tau\vee\tau'} \hat{S}_{\tau,\tau\vee\tau'} \hat{S}_{\tau',\tau\vee\tau'} X(d\tau) X(d\tau') \\ &\quad + \frac{1}{2}\left[\iint_{A_t\times A_t}\right] L_{\tau\vee\tau'} R(\tau,\tau';\tau\vee\tau') X(d\tau) X(d\tau') \end{aligned} \tag{4.12}$$

and (4.9) can be rewritten as

$$\begin{aligned} dL_t &= L_t\{\hat{S}_{t,t} X(dt) \\ &\quad + \int_0^{t_1}\int_0^{t_2} \hat{S}_{(\tau_1,t_2),t} \hat{S}_{(t_1,\tau_2'),t} X(d\tau_1,dt_2) X(dt_1,d\tau_2) \\ &\quad + \int_0^{t_1}\int_0^{t_2} R[(\tau_1,t_2),(t_1,\tau_2');t] X(d\tau_1,dt_2) X(dt_1,d\tau_2')\}. \end{aligned} \tag{4.13}$$

Since R is deterministic, (4.13) shows that dL_t is completely specified by L_t and $\hat{S}_{\tau,t}$ for τ on the boundary of A_t.

For the Gaussian case an expression for $(d\mathscr{P}_1^x/d\mathscr{P}_0^x)$ can be obtained in terms of multiple Wiener integrals and certain kernels which can be obtained from the covariance function of S [4]. Thus far we have not been able to demonstrate its equivalence to (4.12).

Finally, we note that (4.3) dashes any hope that L_t can be expressed in the form

$$L_t = \exp\left(\int_{A_t} \hat{S}_\tau X(d\tau) - \frac{1}{2}\int_{A_t} \hat{S}_\tau^2\, d\tau\right)$$

because this would require that

$$\begin{aligned} \hat{S}_\tau &= E_1(S_\tau \mid X_s, s \in A_\tau) \\ \hat{S}_{\tau'}\hat{S}_\tau &= E_1(S_{\tau'}S_\tau \mid X_s, s \in A_{\tau\vee\tau'}). \end{aligned}$$

The only examples that we have been able to find which satisfy these conditions are those in which 1) S is deterministic, which is a trivial case; and 2) S is a functional of X, which is excluded by the assumptions of Theorem 1.

References

[1] T. Kailath, "A further note on a general likelihood formula for random signals in Gaussian noise," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 393–396, July 1970.

[2] K. Itô, "Multiple Wiener integral," *J. Math. Soc. Japan*, vol. 3, pp. 157–169, 1951.

[3] J. Yeh, "Wiener measure in a space of functions of two variables," *Amer. Math. Soc. Trans.*, vol. 95, pp. 433–450, 1960.

[4] W. J. Park, "A multiparameter Gaussian process," *Ann. Math. Statist.*, vol. 41, pp. 1582–1595, 1970.

[5] R. Cairoli, "Sur une équation différentielle stochastique," *Compte Rendus Acad. Sc. Paris*, vol. 274, ser. A, pp. 1739–1742, June 12, 1972.

[6] E. Wong and M. Zakai, "Martingales and stochastic integrals for processes with a multidimensional parameter," to be published in *Zeit. Wahrscheinlichkeitstheorie.*

[7] E. Wong, *Stochastic Processes in Information and Dynamical Systems.* New York: McGraw-Hill, 1971.