On the Statistical Theory of Optimum Demodulation*

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Summary—The multidimensional demodulation problem is considered from the point of view of statistical estimation theory and *a posteriori* most probable signal estimates are derived. Correlated signals and noises are treated. This formulation yields a set of two matrix integral equations which must be solved for the optimum estimates.

For amplitude modulation, the problem reduces to that of finding a set of time varying filters which are, again, solutions to a matrix integral equation. Special cases such as two-receiver systems, quadrature modulation, and single-sideband have particularly simple representations and are considered in some detail.

An interesting problem in statistical communication theory is the "optimum" estimation of modulated intelligence in the presence of additive noise. For linear forms of modulation, the problem is essentially that of linear nonstationary filtering, and application of the minimum mean-squared error criterion leads to a reasonably simple integral equation.[1,2] Similarly, for nonlinear modulations, *e.g.*, FM, PM, etc., minimum mean-squared error nonlinear filtering theory can be applied.[3] However, even with simplifying restrictions,[4] the resulting mathematics is formidable and not usually amendable to explicit solutions. The methods of statistical estimation theory have been used to obtain *a posteriori most probable* estimates of generally modulated Gaussian signals in Gaussian noise.[5] This treatment results in two integral equations which specify the optimum receiver.

An extension of such estimation techniques to the multidimensional case is considered here. This extension treats the reception of more than one waveform, the estimation of more than one signal, and the case where signals and noises are correlated.

Formulation

The use of *a posteriori most probable* estimation is discussed in detail in the literature.[5-8] It suffices to state here that, given the received waveforms, those signals are chosen as estimates which have the greatest conditional likelihood of occurrence.

Let the received waveforms be

$$\bar{r}(u) = \bar{m}[\bar{a}(u), u] + \bar{n}(u), \qquad t - T \leq u \leq t, \tag{1}$$

where $\bar{r}(u)$, $\bar{m}[\bar{a}(u), u]$, $\bar{a}(u)$, and $\bar{n}(u)$ are column vectors; *e.g.*,

$$\bar{r}(u) = \begin{Bmatrix} r_1(u) \\ r_2(u) \\ \cdot \\ \cdot \\ \cdot \\ r_q(u) \end{Bmatrix}. \tag{2}$$

Here, $\bar{a}(u)$ represents the modulating signals and $\bar{n}(u)$, additive noises. The components of both $\bar{a}(u)$ and $\bar{n}(u)$ are assumed to be correlated Gaussian time series with zero means. The vector $\bar{m}[\bar{a}(u), u]$ is a general modulation function whose form depends on the modulating scheme. It is assumed that this modulation function is differentiable with respect to the elements $a_i(u)$.

In general, the noise vector $\bar{n}(u)$ will have q components, as will $\bar{m}[\bar{a}(u), u]$. The signal vector $\bar{a}(u)$ will be taken to have k components where k and q are not necessarily related.

The problem is to find the set of $a_i(u)$, denoted by $a_i^*(u)$, such that the conditional probability $p(\bar{a}/\bar{r})$ is a maximum. Let the joint probability $p(\bar{a}, \bar{n}, \bar{r})$ be written

$$p(\bar{a}, \bar{n}, \bar{r}) = p[(\bar{a}, \bar{n})/\bar{r}]p(\bar{r}) = p[\bar{r}/(\bar{a}, \bar{n})]p(\bar{a}, \bar{n}) \tag{3}$$

where $p(\bar{a}, \bar{n}, \bar{r})$ is the probability of the simultaneous realizations of $\bar{r}(u)$, $\bar{a}(u)$ and $\bar{n}(u)$ in the interval $t - T \leq u \leq t$, and a similar definition holds for the other terms. Eq. (3) may be rewritten

$$p[(\bar{a}, \bar{n})/\bar{r}] = \frac{p[\bar{r}/(\bar{a}, \bar{n})]p(\bar{a}, \bar{n})}{p(\bar{r})}. \tag{4}$$

If it is noted that

$$p[\bar{r}/(\bar{a}, \bar{n})] = \delta[\bar{n} - (\bar{r} - \bar{m})], \tag{5}$$

where $\delta(x)$ is the Dirac delta-function, then

$$p(\bar{a}/\bar{r}) = \frac{p[\bar{a}, (\bar{r} - \bar{m})]}{p(\bar{r})}. \tag{6}$$

Eq. (6) was obtained by integrating both sides of (4) with respect to $\bar{n}$ and using the relationship of (1). For a given set of received waveforms, $p(\bar{r})$ is a constant; therefore,

$$p(\bar{a}/\bar{r}) = k_1 p[\bar{a}, (\bar{r} - \bar{m})]. \tag{7}$$

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[1] R. C. Booton, Jr., "An optimization theory for time-varying linear systems with nonstationary statistical inputs," Proc. IRE, vol. 40, pp. 977–981; August, 1952.

[2] R. C. Booton, Jr., and M. H. Goldstein, Jr., "The design and optimization of synchronous demodulators," 1957 IRE Wescon Convention Record, pt. 2, pp. 154–170.

[3] L. A. Zadeh, "Optimum nonlinear filters," *J. Appl. Phys.*, vol. 24, pp. 396–404; April, 1953.

[4] Such as restricting the nonlinear filter to be a one-convolution filter.

[5] D. C. Youla, "The use of maximum likelihood in estimating continuously modulated intelligence which has been corrupted by noise," IRE Trans. on Information Theory, vol. IT-3, pp. 90–105; March, 1954.

[6] P. M. Woodward and I. L. Davies, "A theory of radar information," *Phil. Mag.*, ser. 7, vol. 41, pp. 1001–1017; October, 1950.

[7] P. M. Woodward and I. L. Davies, "Information theory and inverse probability in telecommunication," *Proc. IEE*, vol. 99, pp. 37–44; March, 1952.

[8] F. W. Lehan and R. J. Parks, "Optimum demodulation," 1953 IRE National Convention Record, pt. 8, pp. 101–103.

It is desired to maximize this expression with respect to the elements of $\bar{a}$.

Analysis

Define $\bar{x}(u)$ to be a column vector

$$\bar{x}(u) = \begin{Bmatrix} \bar{a}(u) \\ \bar{n}(u) \end{Bmatrix} \tag{8}$$

and the associated covariance function matrix $\mathbf{R}(u, v)$ with elements

$$R_{ij}(u, v) \triangleq E\{x_i(u)x_j(v)\} \tag{9}$$

where $E\{\ \}$ indicates the expectation of the bracketted quantity. It is apparent that $\mathbf{R}(u, v)$ can be partitioned as

$$\mathbf{R}(u, v) = \begin{Bmatrix} \mathbf{R}_{aa}(u, v) & \mathbf{R}_{an}(u, v) \\ \mathbf{R}_{na}(u, v) & \mathbf{R}_{nn}(u, v) \end{Bmatrix}. \tag{10}$$

It is convenient to use a multidimensional expansion recently introduced,[9] and to write

$$\bar{x}(u) = \sum_{p=1}^{\infty} \alpha_p \bar{\varphi}_p(u), \qquad t - T \leq u \leq t, \tag{11}$$

where $\bar{\varphi}_p(u)$ is a column vector of $q + k$ components,

$$\bar{\varphi}_p(u) = \begin{Bmatrix} \varphi_p^{(1)}(u) \\ \varphi_p^{(2)}(u) \\ \vdots \\ \varphi_p^{(q+k)}(u) \end{Bmatrix}. \tag{12}$$

If the $\bar{\varphi}_p(u)$ are the vector eigenfunctions of the matrix integral equation

$$\bar{\varphi}(u) = \lambda \int_{t-T}^{t} \mathbf{R}(u, v)\bar{\varphi}(v)\, dv, \qquad t - T \leq u \leq t, \tag{13}$$

then it can be shown[9] that these vectors $\bar{\varphi}_p(u)$ are orthogonal in the sense that, after normalization

$$\int_{t-T}^{t} \bar{\varphi}_p(u) \cdot \bar{\varphi}_s(u)\, du = \delta_{ps}, \tag{14}$$

and that the coefficients α_p are uncorrelated, *i.e.*,

$$E\{\alpha_p \alpha_s\} = \frac{1}{\lambda_p} \delta_{ps}. \tag{15}$$

Since the conditional probability $p(\bar{a}/\bar{r})$ is proportional to the joint probability of the components of $\bar{x}(u)$,

$$p(\bar{a}/\bar{r}) \sim \exp\left(-\tfrac{1}{2} \sum_{p=1}^{\infty} \lambda_p \alpha_p^2\right). \tag{16}$$

Therefore, in order to maximize $p(\bar{a}/\bar{r})$, it is sufficient to minimize the quantity $\sum_{p=1}^{\infty} \lambda_p \alpha_p^2$. It is shown in the Appendix that

$$\sum_{p=1}^{\infty} \lambda_p \alpha_p^2 = \int_{t-T}^{t} \int_{t-T}^{t} \bar{x}(u) \cdot \mathbf{Q}(u, v)\bar{x}(v)\, du\, dv, \tag{17}$$

[9] This expansion has been used by L. A. Zadeh and one of us in connection with other work not yet published.

where the matrix $\mathbf{Q}(u, v)$ satisfies the integral equation

$$\int_{t-T}^{t} \mathbf{R}(u, v)\mathbf{Q}(v, w)\, dv = \delta(u - w)1, \qquad t - T \leq u, w \leq t, \tag{18}$$

1 being a unit matrix. It is apparent from (83) that the matrix $\mathbf{Q}(u, v)$ may be partitioned as

$$\mathbf{Q}(u, v) = \begin{bmatrix} \mathbf{Q}_{aa}(u, v) & \mathbf{Q}_{an}(u, v) \\ \mathbf{Q}_{na}(u, v) & \mathbf{Q}_{nn}(u, v) \end{bmatrix}. \tag{19}$$

It is now easy to minimize (16) with respect to the $a_i(u)$. By the familiar techniques of the calculus of variation, the following is obtained:

$$\begin{aligned} \int_{t-T}^{t} [\mathbf{Q}_{aa}(u, v) - \mathbf{M}(\bar{a}^*, u)\mathbf{Q}_{na}(u, v)]\bar{a}^*(v)\, dv \\ = \int_{t-T}^{t} [\mathbf{M}(\bar{a}^*, u)\mathbf{Q}_{nn}(u, v) - \mathbf{Q}_{an}(u, v)] \\ \cdot [\bar{r}(v) - \bar{m}(\bar{a}^*, v)]\, dv, \qquad t - T \leq u \leq t, \end{aligned} \tag{20}$$

where the modulation matrix $\mathbf{M}(\bar{a}^*, u)$ has the elements

$$M_{ij}(\bar{a}^*, u) = \left.\frac{\partial m_j(\bar{a}, u)}{\partial a_i(u)}\right|_{\bar{a}=\bar{a}^*}. \tag{21}$$

Eq. (20) together with (18) is sufficient for the solution of the $a_i^*(u)$ in terms of the received waveforms $\bar{r}(u)$.

In the special case where the noises are uncorrelated with the signals, (21) and (18) can be used to obtain

$$\bar{a}^*(u) = \int_{t-T}^{t} \mathbf{R}_{aa}(u, v)\mathbf{M}(\bar{a}^*, v)\bar{g}(v)\, dv, \ t - T \leq u \leq t, \tag{22}$$

and

$$\bar{r}(u) - \bar{m}(\bar{a}^*, u) = \int_{t-T}^{t} \mathbf{R}_{nn}(u, v)\bar{g}(v)\, dv, \qquad t - T \leq u \leq t, \tag{23}$$

where $\bar{g}(v)$ has been written for the expression

$$\int_{t-T}^{t} \mathbf{Q}_{nn}(v, w)[\bar{r}(w) - \bar{m}(\bar{a}^*, w)]\, dw.$$

In the one-dimensional case, (22) and (23) reduce to those obtained by Youla.[5]

In principle, the *a posteriori most probable* demodulator has been found. It is only necessary to specify the form of modulation and the covariance functions of the signals and noises. In practice, the solutions to the equations may be prohibitively difficult depending on the form of modulation.

Amplitude Modulation

General forms of amplitude modulation produce relatively simple expressions for the specifying equations and will be considered in some detail. In these cases, the modulation matrix $\mathbf{M}$ is not a function of the signals $a_i(t)$ and can be written as $\mathbf{M}(u)$. Then, the received waveforms are

$$\bar{r}(u) = \tilde{\mathbf{M}}(u)\bar{a}(u) + \bar{n}(u), \qquad t - T \leq u \leq t, \tag{24}$$

where $\tilde{\mathbf{M}}$ is the transpose of $\mathbf{M}$.

If, furthermore, the noises are uncorrelated with the signals, manipulation of (22) and (23) yields

$$\bar{a}^*(u) = \int_{t-T}^{t} \mathbf{W}(u, v)\bar{r}(v)\, dv, \qquad t - T \leq u \leq t, \tag{25}$$

and

$$\int_{t-T}^{t} \mathbf{W}(u, v)[\tilde{\mathbf{M}}(v)\mathbf{R}_a(v, w)\mathbf{M}(w) + \mathbf{R}_n(v, w)]\, dv = \mathbf{R}_a(u, w)\mathbf{M}(w), \qquad t - T \leq u, w \leq t, \tag{26}$$

where $\mathbf{W}(u, v)$ is a weighting function matrix determined from (26). Eq. (26) can also be derived as the specifying equation for the minimum mean-squared error nonstationary filter.

Eqs. (25) and (26) may be used to investigate a number of special cases of interest.

Case 1—Multireceiver Systems

When the noise level at the receiver itself is large compared to that of the transmission link, it is advantageous to consider multireceiver systems. Their advantage lies in the fact that the noises in the various inputs are uncorrelated, while the signals are either highly correlated or the same. Applications for these types of systems occur, for example, in the field of radio astronomy.[10,11]

Let us consider a two-receiver system where the received waveforms are

$$r_1(u) = M(u)a(u) + n_1(u), \qquad t - T \leq u \leq t \tag{27}$$

and

$$r_2(u) = M(u)a(u) + n_2(u), \qquad t - T \leq u \leq t. \tag{28}$$

Then (25) and (26) become

$$a^*(u) = \int_{t-T}^{t} W_1(u, v)r_1(v)\, dv + \int_{t-T}^{t} W_2(u, v)r_2(v)\, dv, \tag{29}$$

and

$$\int_{t-T}^{t} W_1(u, v)[M(v)M(w)R_a(v, w) + R_{n11}(v, w)]\, dv + \int_{t-T}^{t} W_2(u, v)M(v)M(w)R_a(v, w)\, dv = R_a(u, w)M(w). \tag{30}$$

$$\int_{t-T}^{t} W_2(u, v)[M(v)M(w)R_a(v, w) + R_{n22}(v, w)]\, dv + \int_{t-T}^{t} W_1(u, v)M(v)M(w)R_a(v, w)\, dv = R_a(u, w)M(w). \tag{31}$$

It is interesting to note that if $R_{n11} = R_{n22} \triangleq R_n$, then the symmetry of (30) and (31) implies that

$$W_1(u, v) = W_2(u, v) \triangleq W(u, v), \tag{32}$$

and (30) and (31) reduce to

$$\int_{t-T}^{t} W(u, v)[2M(v)M(w)R_a(v, w) + R_n(v, w)]\, dv = R_a(u, w)M(w), \tag{33}$$

while the corresponding equation for one dimension is

$$\int_{t-T}^{t} W(u, v)[M(v)M(w)R_a(v, w) + R_n(v, w)]\, dv = R_a(u, w)M(w). \tag{34}$$

A compasison of (33) and (34) indicates the advantage of a two-receiver system. Effectively, the signal level relative to noise is doubled.

Case 2—Multiplex Systems

Various multiplex modulation schemes are used in communication. They have the common characteristic that more than one signal is transmitted simultaneously on a time or frequency sharing basis.[12]

Quadrature Modulation

One of the most familiar examples of multiplexing is quadrature modulation, which, in the formulation discussed here, has a particularly simple representation.

Let the received waveform be

$$r(u) = \cos \omega_0 u a_1(u) + \sin \omega_0 u a_2(u) + n(u), t - T \leq u \leq t. \tag{36}$$

Then, (25) and (26) become

$$a_1^*(u) = \int_{t-T}^{t} W_1(u, v)r(v)\, dv, \tag{37}$$

$$a_2^*(u) = \int_{t-T}^{t} W_2(u, v)r(v)\, dv, \tag{38}$$

and

$$\int_{t-T}^{t} W_1(u, v)[\cos \omega_0 v R_{a11}(v, w) \cos \omega_0 w + \cos \omega_0 v R_{a12}(v, w) \sin \omega_0 w + \sin \omega_0 v R_{a21}(v, w) \cos \omega_0 w + \sin \omega_0 v R_{a22}(v, w) \sin \omega_0 w + R_n(v, w)]\, dv = R_{a11}(u, w) \cos \omega_0 w + R_{a12}(u, w) \sin \omega_0 w, \tag{39}$$

$$\int_{t-T}^{t} W_2(u, v)[\cos \omega_0 v R_{a11}(v, w) \cos \omega_0 w + \cos \omega_0 v R_{a12}(v, w) \sin \omega_0 w + \sin \omega_0 v R_{a21}(v, w) \cos \omega_0 w + \sin \omega_0 v R_{a22}(v, w) \sin \omega_0 w + R_n(v, w)]\, dv = R_{a21}(u, w) \cos \omega_0 w + R_{a22}(u, w) \sin \omega_0 w. \tag{40}$$

[10] R. H. Dicke, "The measurement of thermal radiation at microwave frequencies," *Rev. Sci. Instr.*, vol. 17, pp. 268–275; July, 1946.

[11] S. J. Goldstein, "A comparison of two radiometer circuits," Proc. IRE, vol. 43, pp. 1663–1666; November, 1955.

[12] H. S. Black, "Modulation Theory," D. Van Nostrand Co. Inc., New York, N. Y.; 1953.

It should be noted that the kernels of (39) and (40) are identical.

Single Sideband

Although single-sideband amplitude modulation is basically a one-dimensional problem, it can be conveniently treated as a special case of quadrature modulation. In this case,

$$r(u) = \cos \omega_0 u a(u) + \sin \omega_0 u \hat{a}(u) + n(u), \qquad t - T \le u \le t, \tag{41}$$

where $\hat{a}(u)$, the Hilbert transform of $a(u)$, is defined[13] by

$$\hat{a}(u) \triangleq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a(v)}{u - v}\, dv. \tag{42}$$

In the case when the signal and noise are stationary, the representation simplifies even further. If we define $R(u) \triangleq E\{a(t)\, a(t+u)\}$, then the following relationships are easily derived:

$$E\{a(t)\hat{a}(t+u)\} = \hat{R}(u), \tag{43}$$

$$E\{\hat{a}(t)a(t+u)\} = -\hat{R}(u), \quad \text{and} \tag{44}$$

$$E\{\hat{a}(t)\hat{a}(t+u)\} = R(u). \tag{45}$$

Using these expressions, we find that (39) reduces to

$$\int_{t-T}^{t} W_1(u, v)[R_a(w - v) \cos \omega_0(w - v) + \hat{R}_a(w - v) \sin \omega_0(w - v) + R_n(w - v)]\, dv = R_a(w - u) \cos \omega_0 w + \hat{R}_a(w - u) \sin \omega_0 w. \tag{46}$$

In this case, $W_2(u, v)$ is of no interest since it gives the estimate of $\hat{a}(u)$. It should be noted that $\hat{R}_a(u)$ is an odd function of u, and therefore, the kernel of (46) remains symmetric with respect to the variables u and v, as it must.

A Quadrature Modulation Example

In the special case where $a_1(u)$, $a_2(u)$ and $n(u)$ are stationary, and where $a_1(u)$ and $a_2(u)$ are uncorrelated and have the same autocorrelation function, (39) and (40) reduce to

$$\int_{t-T}^{t} W_1(u, v)[R_a(w - v) \cos \omega_0(w - v) + R_n(w - v)]\, dv = R_a(w - u) \cos \omega_0 w \tag{47a}$$

and

$$\int_{t-T}^{t} W_2(u, v)[R_a(w - v) \cos \omega_0(w - v) + R_n(w - v)]\, dv = R_a(w - u) \sin \omega_0 w, \tag{47b}$$

where

$$R_a(w - v) = R_{a_{11}}(v, w) = R_{a_{22}}(v, w). \tag{48}$$

[13] E. C. Titchmarch, "Introduction to the Theory of Fourier Integrals," Oxford University Press, London, England; 1937.

Note that the integral equations (47a) and (47b) have kernels which are functions of the difference of two variables and can be solved easily by standard techniques.[14]

In order to obtain an indication of the forms of the optimum demodulators, we shall give an example with explicit solutions. Let

$$R_a(u) = A_0 \frac{\alpha}{2} e^{-\alpha|u|}, \tag{49}$$

$$R_n(u) = N_0\, \delta(u), \tag{50}$$

and the received waveform be given for the range $-\infty$ to t.

Then, with a change in variables, (37) and (47a) become

$$a_1^*(t) = \int_0^{\infty} W_1'(t, u) r(t - u)\, du \tag{51}$$

and

$$\int_0^{\infty} W_1'(t, u)\left[\frac{A_0\alpha}{2} e^{-\alpha|v-u|} \cos \omega_0(v - u) + N_0\, \delta(v - u)\right] du = \frac{A_0\alpha}{2} e^{-\alpha v} \cos \omega_0(t - v). \tag{52}$$

Eq. (52) can be solved in the usual way[14] and $W_1'(t, u)$ is found to have the form

$$W_1'(t, u) = h_{11}(u) \cos \omega_0(t - u) + h_{12}(u) \sin \omega_0(t - u). \tag{53}$$

In other words, the receiver can be represented as shown in Fig. 1. This is a form of synchronous receiver with

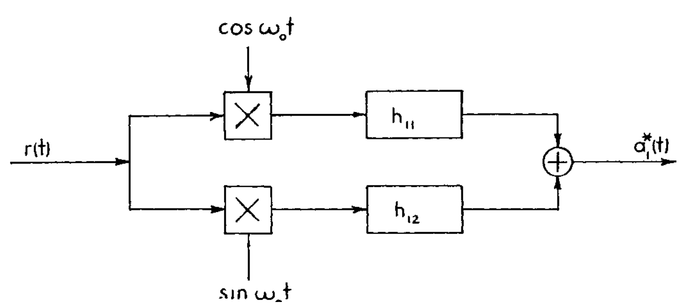

Fig. 1—An optimum quadrature demodulator.

specific stationary filters. For convenience, we define the constants

$$\epsilon \triangleq \frac{A_0}{2N_0} \tag{54}$$

and

$$\beta \triangleq \frac{\alpha}{\omega_0}, \tag{55}$$

and consider two cases.

[14] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's theory of prediction," *J. Appl. Phys.*, vol. 21, pp. 645-655; July, 1950.

1) For the case where $\beta^2 \leq 4(1+\epsilon)/\epsilon^2$, the filters are given by

$$h_{11}(u) = \omega_0 e^{-a\alpha u}[K_1 \cos \omega_0(1-b)u - K_2 \cos \omega_0(1+b)u + K_3 \sin \omega_0(1-b)u - K_4 \sin \omega_0(1+b)u] \quad (56)$$

and

$$h_{12}(u) = \omega_0 e^{-a\alpha u}[K_3 \cos \omega_0(1-b)u - K_4 \cos \omega_0(1+b)u - K_1 \sin \omega_0(1-b)u + K_2 \sin \omega_0(1+b)u], \quad (57)$$

with constants

$$K_1 = \frac{\beta(a-1)[\beta^2(a-1)^2 + (1+b)^2]}{2b}, \quad (58)$$

$$K_2 = \frac{\beta(a-1)[\beta^2(a-1)^2 + (1-b)^2]}{2b}, \quad (59)$$

$$K_3 = \frac{(1-b)[\beta^2(a-1)^2 + (1-b)^2]}{2b}, \quad (60)$$

$$K_4 = \frac{(1+b)[\beta^2(a-1)^2 + (1-b)^2]}{2b}, \quad (61)$$

and

$$a = \frac{1}{\beta}\{\tfrac{1}{2}[(\beta^2+1)^2 + 2\epsilon\beta^2(\beta^2+1)]^{1/2} + (1+\epsilon)\beta^2 - 1\}^{1/2} \quad (62)$$

and

$$b = \{\tfrac{1}{2}[(\beta^2+1)^2 + 2\epsilon\beta^2(\beta^2+1)]^{1/2} - (1+\epsilon)\beta^2 + 1\}^{1/2}. \quad (63)$$

2) For the case where $\beta^2 > 4(1+\epsilon)/\epsilon^2$, the filters are given by

$$h_{11}(u) = \omega_0 e^{-a'\alpha u}(K_2' \sin \omega_0 u - K_1' \cos \omega_0 u) + \omega_0 e^{-b'\alpha u}(K_3' \cos \omega_0 u - K_4' \sin \omega_0 u) \quad (64)$$

and

$$h_{12}(u) = \omega_0 e^{-a'\alpha u}(K_2' \cos \omega_0 u + K_1' \sin \omega_0 u) - \omega_0 e^{-b'\alpha u}(K_3' \sin \omega_0 u + K_4' \cos \omega_0 u), \quad (65)$$

where

$$a' = \frac{1}{\beta}\{(1+\epsilon)\beta^2 - 1 - \beta[\beta^2\epsilon^2 - 4(1+\epsilon)]^{1/2}\}^{1/2} \quad (66)$$

$$b' = \frac{1}{\beta}\{(1+\epsilon)\beta^2 - 1 + \beta[\beta^2\epsilon^2 - 4(1+\epsilon)]^{1/2}\}^{1/2} \quad (67)$$

and

$$K_1' = \frac{\beta^2(a'-1)^2 + 1}{\beta(b'-a')}, \quad (68)$$

$$K_2' = \frac{(b'-1)}{(b'-a')}[\beta^2(a'-1)^2 + 1], \quad (69)$$

$$K_3' = \frac{\beta^2(b'-1)^2 + 1}{\beta(b'-a')}, \quad \text{and} \quad (70)$$

$$K_4' = \frac{a'-1}{b'-a'}[\beta^2(b'-1)^2 + 1]. \quad (71)$$

It should be noted that, despite their complexity, the filters can be synthesized as lumped-constant R-L-C networks for any given β and ϵ.

It is interesting to consider some limiting cases of the example:

1) $\epsilon \to 0$ (very small signal power),

$$h_{11}(u) \to \omega_0\beta\epsilon e^{-\alpha u}, \quad (72)$$

$$h_{12}(u) \to 0 \quad (73)$$

to the first order in ϵ.

2) $\epsilon \to \infty$ (noise power becomes negligible),

$$h_{11}(u) \to \frac{(k-\alpha)^2 + \omega_0^2}{\omega_0} \sin \omega_0 u e^{-ku} + \delta(u), \quad (74)$$

where

$$k = (\alpha^2 + \omega_0^2)^{1/2} \quad (75)$$

and

$$h_{12}(u) \to \frac{2k(k-\alpha)}{\omega_0} \cos \omega_0 u e^{-ku} + \frac{k-\alpha}{\omega_0}\delta(u). \quad (76)$$

In the same way as before, the optimum estimate $a_2^*(t)$ can be found to be

$$a_2^*(t) = \int_0^\infty W_2'(t, u) r(t-u)\, du \quad (77)$$

with

$$W_2'(t, u) = h_{21}(u) \cos \omega_0(t-u) + h_{22}(u) \sin \omega_0(t-u). \quad (78)$$

It should be noted that $h_{21}(u)$ and $h_{22}(u)$ are simply related to $h_{12}(u)$ and $h_{11}(u)$; in fact,

$$h_{21}(u) = -h_{12}(u), \quad (79)$$

and

$$h_{22}(u) = h_{11}(u). \quad (80)$$

AM Demodulation with Delay

With present techniques, most of the integral equations involved in optimum demodulation are difficult to solve explicitly. However, if a reasonable delay can be tolerated, approximate solutions to a large class of AM problems can be obtained. The demodulator is found to be a syncronous demodulator followed by a type of Wiener filter. Problems of this nature have been treated in some detail[15] for the one-dimensional case. Extensions to multi-dimensional cases are straightforward.[16]

[15] J. B. Thomas, "On the Statistical Design of Demodulation Systems for Signals in Additive Noise," Stanford University Electronics Res. Lab., Stanford, Calif., Tech. Rept. No. 88; August, 1955.

[16] J. B. Thomas, T. R. Williams, J. Wolf and E. Wong, "The demodulation of AM signals in noise," *Proc. 1959 IRE Convention on Military Electronics*, pp. 138–146.

Nonlinear Modulations

For nonlinear forms of modulation such as FM and PM, the problem of optimum estimation cannot be reduced to that of finding a time varying filter. In general, one has to consider the solution of (20) for $a^*(u)$. Although this equation is not usually amenable to explicit solution, it is of a form that can be treated by analog techniques. Indeed, if feedback is allowed in the system, it essentially specifies the demodulator. Some work along this line has been initiated.[17]

Problems in Carrier Specification

In this formulation, the phases, amplitudes and frequencies of the carriers are assumed known. In practice, this knowledge must be obtained frequently either by operating on the received waveforms or by transmitting the carriers over a separate channel. Both of these methods involve errors due to noise and thus cause additional errors in the estimation of signals. Such difficulties are common to all synchronous receiver systems.

Appendix

With the use of the orthonormality condition given by (14), the coefficients of expansion α_p can be expressed as

$$\alpha_p = \int_{t-T}^{t} \bar{\varphi}_p(u) \cdot \bar{x}(u)\, du. \tag{81}$$

Therefore, the sum $\sum_{p=1}^{\infty} \lambda_p \alpha_p^2$ is evaluated to be

$$\sum_{p=1}^{\infty} \lambda_p \alpha_p^2 = \sum_{p=1}^{\infty} \lambda_p \int_{t-T}^{t} \int_{t-T}^{t} \left(\sum_{i=1}^{N} \varphi_p^{(i)}(u) x_i(u) \cdot \sum_{j=1}^{N} \varphi_p^{(j)}(v) x_j(v) \right) du\, dv \tag{82}$$

where $N = q + k$. Now, define the matrix $\mathbf{Q}(u, v)$ by the relationship

$$Q_{ij}(u, v) = \sum_{p=1}^{\infty} \lambda_p \varphi_p^{(i)}(u) \varphi_p^{(j)}(v). \tag{83}$$

Then, the sum $\sum_{p=1}^{\infty} \lambda_p \alpha_p^2$ becomes

$$\sum_{p=1}^{\infty} \lambda_p \alpha_p^2 = \int_{t-T}^{t} \int_{t-T}^{t} \bar{x}(u) \cdot \mathbf{Q}(u, v) \bar{x}(v)\, du\, dv. \tag{84}$$

The matrix $\mathbf{Q}(u, v)$ is related to the covariance function matrix $\mathbf{R}(u, v)$. This relationship becomes clear when the integral

$$\sum_{j=1}^{N} \int_{t-T}^{t} R_{ij}(u, v) Q_{jk}(v, w)\, dv$$

is examined. With the substitution of (83) for $Q_{jk}(v, w)$ and the use of (13), this integral becomes

$$\begin{aligned} \sum_{j=1}^{N} \int_{t-T}^{t} R_{ij}(u, v) Q_{jk}(v, w)\, dv &= \sum_{p=1}^{\infty} \lambda_p \varphi_p^{(k)}(w) \int_{t-T}^{t} \sum_{j=1}^{N} R_{ij}(u, v) \varphi_p^{(j)}(v)\, dv \\ &= \sum_{p=1}^{\infty} \varphi_p^{(i)}(u) \varphi_p^{(k)}(w). \end{aligned} \tag{85}$$

The sum on the right-hand side above satisfies the identity

$$\sum_{p=1}^{\infty} \varphi_p^{(i)}(u) \varphi_p^{(k)}(w) \equiv \delta_{ik}\, \delta(u - w). \tag{86}$$

To prove this identity, multiply both sides of (86) by $\varphi_p^{(i)}(u)$, sum over the index i, and integrate with respect to u. With the use of the orthonormality condition, this procedure yields the identity

$$\varphi_p^{(k)}(w) \equiv \varphi_p^{(k)}(w),$$

showing the validity of (85). Therefore, the matrix $\mathbf{Q}(v, w)$ is related to the covariance function matrix $\mathbf{R}(u, v)$ by matrix integral equation

$$\int_{t-T}^{t} \mathbf{R}(u, v) \mathbf{Q}(v, w)\, dv = \delta(u - w) \mathbf{1}, \tag{87}$$

$\mathbf{1}$ being a unit matrix.

[17] R. Jaffe and E. Rechtin, "Design and performance of phase locked circuits capable of near optimum performance over a wide range of input signal and noise levels," IRE Trans. on Information Theory, vol. IT-1, pp. 66–76; March, 1955.