

REPRESENTATION OF MARTINGALES, QUADRATIC VARIATION AND APPLICATIONS*

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Abstract. In this paper, we present two related results. First, we shall obtain a sufficient condition under which a second order sample-continuous martingale can be represented as a stochastic integral in terms of a Brownian motion. Secondly, we shall show that if X and Y are sample-continuous local martingales (not necessarily with respect to the same family of σ -algebras) and if either $X + Y$ or $X - Y$ is almost surely of bounded variation, then the quadratic variations of the two martingales are equal. This rather simple result has some surprising consequences.

1. Introduction. Let $\{X_t, t \geq 0\}$ be a sample-continuous second order martingale. Then $\{X_t^2, t \geq 0\}$ is a sample-continuous first order submartingale and the conditions for Meyer's decomposition [1] are always satisfied; thus we can write

$$(1) \quad X_t^2 = M_t + A_t, \quad t \geq 0,$$

where M is a martingale, A is an increasing process, and both are sample-continuous. The decomposition is unique if we set $M_0 = X_0^2$. If X is a local martingale [2], (1) remains valid except now M is a local martingale. Following Kunita and Watanabe [2], we shall adopt the suggestive notation $\langle X \rangle_t$ for the increasing process A_t .

In this paper, we present two related results. First, we shall obtain a sufficient condition under which a second order sample-continuous martingale can be represented as a stochastic integral in terms of a Brownian motion. Second, we shall show that if X and Y are sample-continuous local martingales (not necessarily with respect to the same family of σ -algebras) and if either $X + Y$ or $X - Y$ is almost surely of bounded variation, then $\langle X \rangle_t = \langle Y \rangle_t$. This rather simple result has some surprising consequences.

2. Martingales and stochastic integrals. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, and let $\{\mathcal{A}_t, t \geq 0\}$ be an increasing family of sub- σ -algebras. A process $\{X_t, t \geq 0\}$ is said to be *adapted* to $\{\mathcal{A}_t\}$ if, for each t , X_t is \mathcal{A}_t -measurable. We say that $\{X_t, \mathcal{A}_t\}$ is a *martingale* if X is adapted to $\{\mathcal{A}_t\}$ and for every $t > s$,

$$(2) \quad E^{\mathcal{A}_s} X_t = X_s$$

almost surely. If $\{X_t, \mathcal{A}_t\}$ is a sample-continuous second order martingale, then the increasing process $\langle X \rangle_t$ introduced earlier is well-defined and $E\langle X \rangle_t < \infty$.

A nonnegative random variable τ is said to be a *stopping time* of $\{\mathcal{A}_t\}$ if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{A}_t$ for every t . A process $\{X_t, \mathcal{A}_t\}$ is said to be a *local martingale* if there exists an increasing sequence of stopping times $\{\tau_n\}$ such that $\tau_n \uparrow \infty$

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almost surely and $\{X_{\min(t, \tau_n)}, \mathcal{A}_t\}$ is a second order martingale. Let $t \wedge s$ denote $\min(t, s)$ and set

$$X_{n,t} = X_{t \wedge \tau_n}.$$

Kunita and Watanabe have proved [2] that if $\{X_t, \mathcal{A}_t\}$ is a sample-continuous local martingale, then there exists a sample-continuous increasing process $\langle X \rangle_t$, such that

$$\langle X \rangle_{t \wedge \tau_n} = \langle X_n \rangle_t.$$

Of course, now we need not have $E\langle X \rangle_t < \infty$.

If $\{W_t, \mathcal{A}_t\}$ is a sample-continuous second order martingale such that for $t > s$,

$$(3) \quad E^{\mathcal{A}_s}(W_t - W_s)^2 = t - s,$$

then W is necessarily a Brownian motion and for each s , $\{W_t - W_s, t \geq s\}$ is independent of \mathcal{A}_s [3, p. 384]. We describe this situation by saying that $\{W_t, \mathcal{A}_t\}$ is a Brownian motion. Let $\{W_t, \mathcal{A}_t\}$ be a Brownian motion and let $\{\phi_t, t \geq 0\}$ be a measurable process adapted to $\{\mathcal{A}_t\}_t$ such that

$$(4) \quad \int_0^t E\phi_s^2 ds < \infty$$

for each t . The stochastic integral $\int_0^t \phi_s dW_s$ is well-defined as the quadratic limit of a sequence of sums $\sum_v \phi_{t_v^{(n)}}[W_{t_{v+1}^{(n)}} - W_{t_v^{(n)}}]$, where $\{t_v^{(n)}\}$ is a sequence of partitions of $[0, t]$ such that

$$\max_v (t_{v+1}^{(n)} - t_v^{(n)}) \xrightarrow{n \rightarrow \infty} 0.$$

If we define

$$(5) \quad X_t = \int_0^t \phi_s dW_s$$

and choose a separable version for X , then $\{X_t, \mathcal{A}_t\}$ is a second order sample-continuous martingale, with

$$(6) \quad E^{\mathcal{A}_s}(X_t - X_s)^2 = \int_s^t E^{\mathcal{A}_s} \phi_\tau^2 d\tau, \quad 0 \leq s \leq t.$$

If, instead of (4), ϕ merely satisfies

$$(7) \quad \int_0^t \phi_s^2 ds < \infty, \quad \text{a.s.},$$

then $\int_0^t \phi_s dW_s$ can be defined as follows: Let $\tau_n(\omega)$ be defined by

$$(8) \quad \tau_n(\omega) = \begin{cases} \inf \left\{ t : \int_0^t \phi_s^2(\omega) ds \geq n \right\} \\ \infty \quad \text{if } \int_0^t \phi_s^2(\omega) ds < n \text{ for all } t, \end{cases}$$

and set

$$(9) \quad \phi_{n,s}(\omega) = \begin{cases} \phi_s(\omega), & s \leq \tau_n(\omega), \\ 0, & s > \tau_n(\omega). \end{cases}$$

For each n , $\int_0^t \phi_{n,s} dW_s$ is well-defined. It can be shown that $\int_0^t \phi_{n,s} dW_s$ converges in probability as $n \rightarrow \infty$, and we define $\int_0^t \phi_s dW_s$ as this limit. Now, the process $X_t = \int_0^t \phi_s dW_s$ need no longer be second order or a martingale, but it is still sample-continuous if a separable version is chosen. Moreover, for each n , $\{X_{t \wedge \tau_n}, \mathcal{A}_t\}$ is a second order martingale. By definition, X is a local martingale.

If X is a stochastic integral of the form

$$(10) \quad X_t = X_0 + \int_0^t \phi_s dW_s$$

and f is a twice continuously differentiable function of a real variable, then Ito's differentiation formula [4] yields

$$(11) \quad f(X_t) = f(X_0) + \int_0^t f'(X_s)\phi_s dW_s + \frac{1}{2} \int_0^t f''(X_s)\phi_s^2 ds.$$

In particular,

$$(12) \quad X_t^2 - X_0^2 = 2 \int_0^t X_s \phi_s dW_s + \int_0^t \phi_s^2 ds.$$

We can now identify $X_0^2 + 2 \int_0^t X_s \phi_s dW_s$ as the local martingale term in the decomposition (1) of X_t^2 , and thus the increasing process is given by

$$(13) \quad \langle X \rangle_t = \int_0^t \phi_s^2 ds.$$

If X is of the form (10), then we can define the stochastic integral $\int_0^t \psi_s dX_s$ by

$$\int_0^t \psi_s dX_s = \int_0^t \psi_s \phi_s dW_s,$$

provided that $\int_0^t \psi_s^2 \phi_s^2 ds < \infty$ almost surely. If X is a local martingale, not necessarily of the form (10), the stochastic integral $\int_0^t \phi_s dX_s$ can still be defined, provided that $\int_0^t \phi_s^2 d\langle X \rangle_s < \infty$ almost surely [2]. Finally, if $X = Y + Z$ where Y is a local martingale and $\{Z_t, t \geq 0\}$ is a process with sample functions almost surely of bounded variation, then we can define $\int_0^t \psi_s dX_s$ by

$$\int_0^t \psi_s dX_s = \int_0^t \psi_s dY_s + \int_0^t \psi_s dZ_s,$$

provided that the first integral exists as a stochastic integral and the second as a Stieltjes integral. For the case where Y and Z are sample continuous and f is a

twice continuously differentiable function, Kunita and Watanabe has extended Ito's differentiation rule to read

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle Y \rangle_s.$$

In particular, we note that

$$X_t^2 - X_0^2 - 2 \int_0^t X_s dX_s = \langle Y \rangle_t$$

is independent of Z and serves to define $\langle X \rangle_t$.

3. Representation of martingales. Not every sample-continuous second order martingale can be represented as a stochastic integral in the form of (10). It is clear from (13) that for such a representation to be possible the increasing process $\langle X \rangle(\omega, t)$ must be an absolutely continuous function of t (w.r.t. the Lebesgue measure) for almost all ω . As Fisk has observed [5], this condition is also sufficient by virtue of a theorem of Doob [3, p. 449], but it may be necessary to enlarge the underlying probability space by the adjunction of a Brownian motion. Specifically, Doob proved the following theorem.

THEOREM 3.1 (Doob). *Let $\{X_t, \mathcal{A}_t, 0 \leq t \leq T\}$ be a sample-continuous second order martingale. Suppose that there exists a nonnegative measurable process $\{\psi_t, 0 \leq t \leq T\}$ adapted to $\{\mathcal{A}_t\}$ such that for $t > s$,*

$$(14) \quad E^{\mathcal{A}_s}(X_t - X_s)^2 = \int_s^t E^{\mathcal{A}_s} \psi_\tau d\tau.$$

If the set $\{(\omega, t) : \psi(\omega, t) = 0\}$ has zero $d\mathcal{P} dt$ measure, then there exists a Brownian motion $\{W_t, \mathcal{A}_t, 0 \leq t \leq T\}$ such that

$$(15) \quad X_t = X_0 + \int_0^t \psi_s^{1/2} dW_s$$

with probability 1. Without the hypothesis that ψ vanishes almost nowhere, representation (15) is still valid with the adjunction of a Brownian motion to the probability space.

The condition that $\langle X \rangle$ be almost surely continuous with respect to the Lebesgue measure is both somewhat stringent and difficult to verify. Perhaps, it is more natural to consider representations of a more general form

$$(16) \quad X_t(\omega) = X_0(\omega) + \int_0^t \phi_s(\omega) dW_{F(s)}(\omega),$$

where W is a Brownian motion and F is a continuous increasing function.

THEOREM 3.2. *Let $\{X_t, \mathcal{A}_t, 0 \leq t \leq T\}$ be a sample-continuous local martingale. We assume that $\{\mathcal{A}_t\}$ is right-continuous (i.e. $\bigcap_{s>t} \mathcal{A}_s = \mathcal{A}_t$) and each \mathcal{A}_t is completed. A representation of the form (16) exists if and only if $\langle X \rangle$ is absolutely continuous with respect to F with probability 1.*

Proof. We only need to prove the theorem for the case $F(t) = t$, because $\{X_t, \mathcal{A}_t\}$ can be transformed into a sample-continuous local martingale $\{\tilde{X}_t, \tilde{\mathcal{A}}_t\}$ with $E\tilde{X}_t^2 = t$ by defining

$$\begin{aligned} F^{-1}(t) &= \inf \{s : F(s) = t\}, \\ \tilde{X}_t &= X_{F^{-1}(t)}, \\ \tilde{\mathcal{A}}_t &= \mathcal{A}_{F^{-1}(t)}. \end{aligned}$$

Even though F^{-1} may be discontinuous, $\{\tilde{\mathcal{A}}_t\}$ is right-continuous and \tilde{X} is still sample-continuous, because $F(t) = F(s)$ implies $X_t = X_s$ almost surely. Since $X_t = \tilde{X}_{F(t)}$ with probability 1, a representation

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \tilde{\phi}_s dW_s,$$

where $\{W_t, \tilde{\mathcal{A}}_t\}$ is a Brownian motion, implies a representation

$$\begin{aligned} X_t &= \tilde{X}_{F(t)} = X_0 + \int_0^{F(t)} \tilde{\phi}_s dW_s \\ &= X_0 + \int_0^t \phi_s dW_{F(s)}, \end{aligned}$$

which is just (16).

To prove Theorem 3.2 for the case $F(t) = t$, we first note that necessity follows from (13). To prove sufficiency, we assume that $\langle X \rangle$ is absolutely continuous with probability 1 and write

$$(17) \quad \langle X \rangle_t(\omega) = \int_0^t \psi_s(\omega) ds,$$

where ψ can always be chosen to be a measurable process because $\langle X \rangle$ is a measurable process. For each t , ψ_t is measurable with respect to $\cap_{s>t} \mathcal{A}_s$, which is equal to \mathcal{A}_t by assumption.

We now follow Doob [3, p. 449] and define

$$(18) \quad g_s(\omega) = \begin{cases} \psi_s^{-1/2}(\omega) & \text{if } \psi_s(\omega) > 0, \\ 0 & \text{if } \psi_s(\omega) = 0. \end{cases}$$

Since $\int_0^t g_s^2 d\langle X \rangle_s \leq t < \infty$, the integral $\int_0^t g_s dX_s = W_t$ is well-defined as a local martingale. If $\psi_s(\omega) > 0$ for almost all (ω, s) , then $\langle W \rangle_t = t$ and it follows from Theorem 2.3 of [2] that W is a Brownian motion. If not, we adjoin an independent Brownian motion B to the underlying probability space and define

$$W_t = \int_0^t g_s dX_s + \int_0^t [1 - g_s \sqrt{\psi_s}] dB_s.$$

Then, W is a Brownian motion. In either case, we have

$$X_t = X_0 + \int_0^t \sqrt{\psi_s} dW_s$$

and the proof is complete.

Theorem 3.2 is basically the same as Theorem 2.1 of [5], except for the introduction of the increasing function F and the generalization to local martingales. We now come to the main result of the section, namely, a sufficient condition for the representation (16) that can be verified in terms of two-dimensional distributions of a martingale X .

THEOREM 3.3. *Let $\{X_t, \mathcal{A}_t, 0 \leq t \leq T\}$ be a sample-continuous second order martingale and let $F(t) = E(X_t - X_0)^2 = E\langle X \rangle_t$. Suppose that there exist finite positive constants α and β such that*

$$(19) \quad \sup_{0 < F(t) - F(s) \leq \beta} \frac{E|X_t - X_s|^{2+2\alpha}}{[F(t) - F(s)]^{1+\alpha}} < \infty.$$

Then $\langle X \rangle$ is almost surely absolutely continuous with respect to the F -measure, and X has a representation of the form of (16).

Proof. By virtue of the Lebesgue decomposition, we can always write

$$(20) \quad \langle X \rangle_t(\omega) = \int_0^t \psi_s(\omega) dF(s) + \mu_t(\omega),$$

where μ is almost surely singular with respect to F . Now, $E\psi_s = 1$ implies

$$E\mu_t = - \int_0^t dF(t) + E\langle X \rangle_t = 0,$$

which in turn implies that $\mu \equiv 0$ almost surely since μ is nonnegative and sample-continuous. Therefore, we only need to prove that (19) implies $E\psi_s = 1, 0 \leq s \leq T$.

Let $T_n = \{t_v^{(n)}, v = 0, 1, \dots, n\}$ be a sequence of nested (i.e., $T_{n+1} \supset T_n$) partitions of the interval $[0, T]$ such that

$$\max_v [F(t_{v+1}^{(n)}) - F(t_v^{(n)})] \xrightarrow{n \rightarrow \infty} 0.$$

Define $\psi_{n,t}, 0 \leq t \leq T$, as follows:

$$(21) \quad \psi_{n,t} = \frac{\langle X \rangle_{t_{v+1}^{(n)}} - \langle X \rangle_{t_v^{(n)}}}{F(t_{v+1}^{(n)}) - F(t_v^{(n)})}, \quad t_v^{(n)} \leq t < t_{v+1}^{(n)}.$$

It is well known (see, e.g., [3, pp. 346–347]) that for each $\omega, \psi_{n,t}$ converges for almost all t (F -measure) to the Radon-Nikodym derivative of the absolutely continuous component of $\langle X \rangle$ with respect to F . That is, $\psi_{n,t} \rightarrow \psi_t$ for almost all (ω, t) . Since it is obvious that $E\psi_{n,t} = 1$, the desired result $E\psi_t = 1$ will follow if for each $t, \{\psi_{n,t}\}$ is a uniformly integrable family of random variables.

Now, it is known [6] that for any $p > 1/2$ there exists a constant κ_p such that

$$E|\langle X \rangle_t - \langle X \rangle_s|^p \leq \kappa_p E|X_t - X_s|^{2p}.$$

Therefore, if we let N be the smallest n such that

$$\max_v [F(t_{v+1}^{(n)}) - F(t_v^{(n)})] \leq \beta,$$

then

$$\sup_{n \geq N} E\psi_{n,t}^{1+\alpha} \leq \kappa_{1+\alpha} \sup_{0 < F(t) - F(s) \leq \beta} \left\{ \frac{E|X_t - X_s|^{2+2\alpha}}{[F(t) - F(s)]^{1+\alpha}} \right\} < \infty,$$

so that $\{\psi_{n,t}\}$ is a uniformly integrable family of random variables. This, together with Theorem 3.2, completes the proof.

Theorem 3.2 is reminiscent of Kolmogorov's condition for sample continuity and has similar advantages, the primary one being that it can be verified in terms of the two-dimensional distributions of X .

4. Quadratic variation. Let $T_n = \{t_v^{(n)}\}$ be a nested sequence of partitions of $[0, T]$ such that $\max_v (t_v^{(n)} - t_{v+1}^{(n)}) \xrightarrow{n \rightarrow \infty} 0$. Let X be a sample-continuous second order martingale, and let $t \wedge s$ denote $\min(t, s)$. Fisk [5] has shown that the sequence of sums

$$(22) \quad Q_n(X, t) = \sum_v [X_{t \wedge t_v^{(n)}} - X_{t \wedge t_{v+1}^{(n)}}]^2$$

converges to $\langle X \rangle_t$ in L^1 -mean, i.e.,

$$(23) \quad E|Q_n(X, t) - \langle X \rangle_t| \xrightarrow{n \rightarrow \infty} 0.$$

For this reason, $\langle X \rangle_t$ is said to be the quadratic variation of X on $[0, t]$. Now, suppose that $\{Z_t, 0 \leq t \leq T\}$ is a sample-continuous process, the sample functions of which are almost surely of bounded variation. Then there exists an almost surely finite random variable A such that

$$\sup_n \sum_v |Z_{t_v^{(n)}} - Z_{t_{v+1}^{(n)}}| \leq A.$$

Therefore,

$$Q_n(Z, T) = \sum_v |Z_{t_v^{(n)}} - Z_{t_{v+1}^{(n)}}|^2 \leq A \max_v |Z_{t_v^{(n)}} - Z_{t_{v+1}^{(n)}}| \xrightarrow[n \rightarrow \infty]{a.s.} 0$$

so that Z has zero quadratic variation on $[0, T]$.

THEOREM 4.1. *Let $\{\mathcal{A}_t\}$ and $\{\tilde{\mathcal{A}}_t\}$ be two increasing families of σ -algebras and let $\{X_t, \mathcal{A}_t\}$ and $\{\tilde{X}_t, \tilde{\mathcal{A}}_t\}$ be sample-continuous local martingales. If $X + \tilde{X}$ or $X - \tilde{X}$ is of bounded variation, then $\langle X \rangle_t = \langle \tilde{X} \rangle_t$ almost surely for every t .*

Proof. First, suppose that X and \tilde{X} are second order martingales. Let B denote the process of bounded variation given by $X - \tilde{X}$ or $X + \tilde{X}$. From the inequality

$$\left(\sqrt{\sum_k a_k^2} - \sqrt{\sum_k b_k^2} \right)^2 \leq \sum_k (a_k \pm b_k)^2 \leq \left(\sqrt{\sum_k a_k^2} + \sqrt{\sum_k b_k^2} \right)^2,$$

we have

$$\left(\sqrt{Q_n(\tilde{X}, t)} - \sqrt{Q_n(B, t)} \right)^2 \leq Q_n(X, t) \leq \left(\sqrt{Q_n(\tilde{X}, t)} + \sqrt{Q_n(B, t)} \right)^2.$$

Since $Q_n(\tilde{X}, t)$, $Q_n(X, t)$ and $Q_n(B, t)$ converge respectively to $\langle \tilde{X} \rangle_t$, $\langle X \rangle_t$ and 0, we have $\langle X \rangle_t = \langle \tilde{X} \rangle_t$ almost surely for all $t \in [0, T]$.

If X and \tilde{X} are local martingales, then there exist sequences of stopping times $\{\tau_m\}$ and $\{\tilde{\tau}_m\}$ increasing to ∞ , so that $X_{m,t} = X_{\tau_m \wedge t}$ and $\tilde{X}_{m,t} = \tilde{X}_{\tilde{\tau}_m \wedge t}$ are second order martingales for each m . Let

$$\Omega_{m,t} = \{\omega : t < \tau_m \wedge \tilde{\tau}_m\}.$$

Since $\tau_m \uparrow \infty$ and $\tilde{B}_m \uparrow \infty$, we have $\Omega_{m,t} \uparrow \Omega$. For $\omega \in \Omega_{m,t}$, we have

$$\begin{aligned} X_{m,s}(\omega) &= X_s(\omega), & 0 \leq s \leq t, \\ \tilde{X}_{m,s}(\omega) &= \tilde{X}_s(\omega), & 0 \leq s \leq t. \end{aligned}$$

Therefore, for $\omega \in \Omega_{m,t}$, either $X_m(\omega) + \tilde{X}_m(\omega)$ or $X_m(\omega) - \tilde{X}_m(\omega)$ is of bounded variation on $[0, t]$, and it follows from the same inequality as before that

$$\langle X_m \rangle_t(\omega) = \langle \tilde{X}_m \rangle_t(\omega), \quad \omega \in \Omega_{m,t},$$

or

$$\langle X \rangle_t(\omega) = \langle \tilde{X} \rangle_t(\omega), \quad \omega \in \Omega_{m,t}.$$

The proof is completed by letting $m \uparrow \infty$.

We should note that in Theorem 4.1 we do not assume that X and \tilde{X} are local martingales with respect to the same family of σ -algebras. If they are, then $X + \tilde{X}$ and $X - \tilde{X}$ are both local martingales. If one of them is also of bounded variation, say $X + \tilde{X}$, then $X_t + \tilde{X}_t = X_0 + \tilde{X}_0$ with probability 1 for all $t \in [0, T]$, in which case the result of Theorem 4.1 trivially follows. The more interesting cases arise when neither $X + \tilde{X}$ nor $X - \tilde{X}$ is a local martingale.

An interesting application of Theorem 4.1 is in connection with quasi-martingales [7]. A process $\{X_t, 0 \leq t \leq T\}$ is said to be a quasi-martingale with respect to $\{\mathcal{A}_t\}$ if there exist $\{B_t, 0 \leq t \leq T\}$ and $\{M_t, 0 \leq t \leq T\}$ both adapted to $\{\mathcal{A}_t\}$ such that $X = M + B$, B is of bounded variation, and $\{M_t, \mathcal{A}_t\}$ is a martingale. We shall be interested only in those cases where both M and B are sample-continuous and where the total variation of B has a finite expectation. Under these assumptions, if $\{X_t, \mathcal{A}_t\}$ is a quasi-martingale, then $\{X_t, \mathcal{A}_{xt}\}$ is always a quasi-martingale, where \mathcal{A}_{xt} denotes the σ -algebra generated by $\{X_s, 0 \leq s \leq t\}$. More generally, let $\{\mathcal{A}_t\}$ be any increasing family of σ -algebras such that $\mathcal{A}_{xt} \subseteq \tilde{\mathcal{A}}_t \subseteq \mathcal{A}_t$ for every t . Then, $\{X_t, \tilde{\mathcal{A}}_t\}$ is a quasi-martingale.¹ That is, there exist \tilde{B} and \tilde{M} adapted to $\{\tilde{\mathcal{A}}_t\}$ such that B is of bounded variation. $\{\tilde{M}_t, \tilde{\mathcal{A}}_t\}$ is a martingale, both \tilde{M} and \tilde{B} are sample-continuous, and the total variation of \tilde{B} has finite expectation. It follows from Theorem 4.1 that $\langle M \rangle_t = \langle \tilde{M} \rangle_t$ for every t .

An important class of quasi-martingales is made up of Ito processes, which are processes having the representation

$$(24) \quad X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dW_s, \quad 0 \leq t \leq T,$$

where ψ and ϕ are measurable processes adapted to an increasing family of σ -algebras $\{\mathcal{A}_t\}$, $\{W_t, \mathcal{A}_t\}$ is a Brownian motion, and X_0 is \mathcal{A}_0 -measurable. In addition, we assume

$$(25) \quad \int_0^T E|\psi_s| ds < \infty,$$

$$(26) \quad \int_0^T E\phi_s^2 ds < \infty.$$

¹ This fact is easily proved by verifying the two conditions of Theorem 3.3 in [7].

It is clear that $\{X_t, \mathcal{A}_t\}$ is a quasi-martingale. Thus, $\{X_t, \mathcal{A}_{xt}\}$ is also a quasi-martingale and the representation of X as a quasi-martingale with respect to $\{\mathcal{A}_{xt}\}$ is given in the following theorem.

THEOREM 4.2. Let $X_t, 0 \leq t \leq T$, be an Ito process satisfying (24), (25) and

$$(27) \quad \int_0^T \phi_s^2 ds < \infty \quad \text{almost surely.}$$

Then there exists a representation of X in the form

$$(28) \quad X_t = X_0 + \int_0^t E^{\tilde{\mathcal{A}}_s} \psi_s ds + \int_0^t |\psi_s| d\tilde{W}_s,$$

where $\{\tilde{\mathcal{A}}_t\}$ is any increasing family of σ -algebras satisfying

$$(29) \quad \mathcal{A}_{xt} \subseteq \tilde{\mathcal{A}}_t \subseteq \mathcal{A}_t \quad \text{for each } t$$

and \tilde{W} is a Brownian motion.

Remark. If we take $\tilde{\mathcal{A}}_t = \mathcal{A}_{xt}$, then we have

$$X_t = X_0 + \int_0^t E^{\mathcal{A}_{xs}} \psi_s ds + \int_0^t |\phi_s| d\tilde{W}_s,$$

where both integrands $E^{\mathcal{A}_{xs}} \psi_s$ and $|\phi_s|$ are \mathcal{A}_{xs} -measurable functions for almost all s . The latter stems from the fact that

$$(30) \quad |\phi_t| = \left[\frac{d}{dt} \langle X \rangle_t \right]^{1/2},$$

where d/dt stands for the Radon-Nikodym derivative and $\langle X \rangle_t$ is defined by

$$(31) \quad \langle X \rangle_t = X_t^2 - X_0^2 - 2 \int_0^t X_s dX_s.$$

Proof. First, suppose that, in place of (27), the stronger condition (26) is satisfied. Then, $\{X_t, \tilde{\mathcal{A}}_t\}$ is a quasi-martingale and we can write

$$X_t = X_0 + \tilde{B}_t + \tilde{\mathcal{M}}_t, \quad 0 \leq t \leq T,$$

where \tilde{B} is of bounded variation and $\{\tilde{M}_t, \tilde{\mathcal{A}}_t\}$ is a martingale. Since

$$\tilde{M}_t - \int_0^t \phi_s dW_s = \int_0^t \psi_s ds - \tilde{B}_t$$

and $M_t = \int_0^t \phi_s dW_s$ is a martingale under condition (26), we have from Theorem 4.1:

$$\langle \tilde{M} \rangle_t = \langle M \rangle_t = \int_0^t \phi_s^2 ds, \quad 0 \leq t \leq T;$$

Theorem 3.1 yields the representation

$$\langle \tilde{M} \rangle_t = \int_0^t |\phi_s| d\tilde{W}_s,$$

where \tilde{W} is a Brownian motion.

To get an integral representation for \tilde{B} , let $T_n = \{t_v^{(n)}\}$, $n = 1, 2, \dots$, be a nested sequence of partitions of $[0, T]$ such that $\max_v(t_{v+1}^{(n)} - t_v^{(n)}) \xrightarrow{n \rightarrow \infty} 0$. Define

$$\tilde{B}_{n,t} = \sum_v E[X_{t \wedge t_{v+1}^{(n)}} - X_{t \wedge t_v^{(n)}} | \tilde{\mathcal{A}}_{t_v^{(n)}}].$$

Fisk [7] has shown that

$$E|\tilde{B}_{n,t} - \tilde{B}_t| \xrightarrow{n \rightarrow \infty} 0.$$

From (24) we get

$$E[X_{t_{v+1}^{(n)}} - X_{t_v^{(n)}} | \tilde{\mathcal{A}}_{t_v^{(n)}}] = \int_{t_v^{(n)}}^{t_{v+1}^{(n)}} E(\psi_s | \tilde{\mathcal{A}}_{t_v^{(n)}}) ds.$$

If we denote $f_s = E^{\tilde{\mathcal{A}}_s} \psi_s$, then

$$\begin{aligned} & \sup_{0 \leq t \leq T} E \left| \int_0^t E^{\tilde{\mathcal{A}}_s} \psi_s ds - \tilde{B}_t \right| \\ & \leq \lim_{n \rightarrow \infty} \left\{ E \left| \sum_v \int_{t_v^{(n)}}^{t_{v+1}^{(n)}} [f_s - f_{t_v^{(n)}}] ds \right| \right. \\ & \quad \left. + E \left| \sum_v \int_{T^{(n)}}^{t_{v+1}^{(n)}} E[\psi_s - \psi_{t_v^{(n)}} | \tilde{\mathcal{A}}_{t_v^{(n)}}] ds \right| \right\} \\ & \leq \lim_{n \rightarrow \infty} E \sum_v \int_{t_v^{(n)}}^{t_{v+1}^{(n)}} [|f_s - f_{t_v^{(n)}}| + |\psi_s - \psi_{t_v^{(n)}}|] ds. \end{aligned}$$

For a suitable sequence of partitions $\{t_v^{(n)}\}$, the last limit can always be made zero (Doob [3, pp. 63–65]). Since the first expression is independent of $\{t_v^{(n)}\}$, we must have

$$\sup_{0 \leq t \leq T} E \left| \int_0^t E^{\tilde{\mathcal{A}}_s} \psi_s ds - \tilde{B}_t \right| = 0.$$

If ϕ satisfies (27) and not (26), we define $\{\tau_n\}$ by (8). Since $\langle X \rangle_t = \int_0^t \psi_s^2 ds$, $\{\tau_n\}$ is a sequence of stopping times not only for $\{\mathcal{A}_t\}$ but also for $\{\tilde{\mathcal{A}}_t\}$. Therefore, for each n , $\{X_{t \wedge \tau_n}, \tilde{\mathcal{A}}_t\}$ is a quasi-martingale with a representation

$$X_{t \wedge \tau_n} = X_0 + \int_0^{t \wedge \tau_n} E^{\tilde{\mathcal{A}}_s} \psi_s ds + \tilde{M}_{t \wedge \tau_n}.$$

Hence, $\tilde{M}_t = X_t - X_0 - \int_0^t E^{\tilde{\mathcal{A}}_s} \psi_s ds$ is a local martingale. It follows from Theorem 4.1 and (24) that

$$\langle \tilde{M} \rangle_t = \int_0^t \phi_s^2 ds,$$

and the proof is completed by using Theorem 3.2.

Theorem 4.2 can be generalized to a vector Ito process practically without change. Let X be an n -vector-valued process satisfying

$$(32) \quad X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dW_s,$$

where W is an m -vector process, the components of which are independent Brownian motions and ϕ is an $n \times m$ matrix. Instead of (25) and (27), we now assume

$$(33) \quad \int_0^T E\|\psi_s\| ds < \infty$$

and

$$(34) \quad \int_0^T \|\phi_s\|^2 ds < \infty \quad \text{almost surely,}$$

where $\|\cdot\|$ denotes the Euclidean norm.

Let α be an n -vector and let prime denote transpose. For every $\alpha \in R^n$, $\{\alpha' X_t\}$ is a scalar-valued Ito process of the form (24), and we can write

$$\alpha' X_t = \int_0^t \alpha' E^{\tilde{\mathcal{A}}_s} \psi_s ds + M_{\alpha,t},$$

where $\{M_{\alpha,t}, \tilde{\mathcal{A}}_t\}$ is a local martingale for any $\{\tilde{\mathcal{A}}_t\}$ satisfying (29) and

$$\langle M_\alpha \rangle_t = \int_0^t \alpha' \phi_s \phi_s' \alpha ds.$$

It follows that $\int_0^t \phi_s \phi_s' ds$ is \mathcal{A}_{xt} -measurable for every t , and the positive semi-definite matrix $\phi_s \phi_s'$ is \mathcal{A}_{xs} -measurable for almost all s . Let $\phi_s \phi_s'$ be diagonalized so that

$$(35) \quad \phi_s \phi_s' = A_s \Lambda_s A_s',$$

where Λ_s is diagonal and the orthogonal matrix A_s is \mathcal{A}_{xt} -measurable for almost all s . Now, if we define

$$Y_t = \int_0^t A_s' dX_s$$

and apply Theorem 4.2 to the components of Y , we get the following theorem.

THEOREM 4.3. *Let X be an n -vector Ito process of the form (32) satisfying (33) and (34). Let A_s and Λ_s be matrices defined by the diagonalization (35). Then, for any increasing family of σ -algebras satisfying (29), there exists an n -vector Brownian motion \tilde{W} so that*

$$(36) \quad X_t = X_0 + \int_0^t E^{\mathcal{A}_{xs}} \psi_s ds + \int_0^t A_s \Lambda_s^{1/2} d\tilde{W}_s.$$

5. Applications. Equation (32) is widely used to model a dynamical system disturbed by Gaussian white noise. Theorem 4.3 has some interesting and surprising consequences for such models. For example, in filtering problems we often interpret X in (32) as the observed process, ψ as the process to be estimated, and the stochastic integral term as the noise. If we identify $\hat{\psi}_t = E^{\mathcal{A}_{xt}} \psi_t$ as the estimator, then Theorem 4.3 implies that

$$\int_0^t (\psi_s - \hat{\psi}_s) ds = - \int_0^t \phi_s dW_s + \int_0^t A_s \Lambda_s^{1/2} d\tilde{W}_s.$$

The stochastic integral $-\int_0^t \phi_s dW_s$ can always be re-represented as $\int_0^t A_s \Lambda_s^{1/2} dV_s$, where V is now an n -dimensional Brownian motion, so that

$$\int_0^t (\psi_s - \hat{\psi}_s) ds = \int_0^t A_s \Lambda_s^{1/2} (dV_s + d\tilde{W}_s).$$

Formally, the estimation error can be expressed as

$$\psi_t - \hat{\psi}_t = A_t \Lambda_t^{1/2} (\dot{V}_t + \dot{\tilde{W}}_t).$$

One may be tempted to say that the estimation error is "white," but that would be misleading. For the special case $\phi(\omega, t) = K$, a constant, we have

$$\psi_t - \hat{\psi}_t = A \Lambda^{1/2} (\dot{V}_t + \dot{\tilde{W}}_t),$$

which is the sum of Gaussian white noise (albeit dependent) processes. This was originally observed by Wonham [8] in a more limited context and generalized by Kailath [9]. We note that the same observation is true if $\phi\phi'$ is a constant even if ϕ is not.

Another observation worthy of note is that for almost all t , $\phi_t\phi_t'$ is \mathcal{A}_{xt} -measurable. Thus, for example, if ψ_t is a function only of t and $\{\phi_s\phi_s', 0 \leq s \leq t\}$, then the estimation error is necessarily zero for almost all t . Such a result would be difficult to prove outside of the context of quadratic variations. A similar observation can be made with respect to singular detection [10].

Finally, as a simple example of possible applications to control problems, consider a scalar equation

$$dX_t = f(X_0^t, U_0^t, t) dk + U_t dW_t,$$

where X represents the state and U the control. Suppose that $U_t \geq 0$ for all t . Then Theorem 4.2 implies that for any optimization problem whatever, the optimizing control can always be implemented in state feedback form. This means that even if we observe the past of the noise process W and use it in constructing the control, performance cannot be improved. Surely, this is an unexpected result.

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