**The Distribution of Intervals Between Zeros for a Stationary Gaussian Process**

E. Wong

*SIAM Journal on Applied Mathematics*, Vol. 18, No. 1 (Jan., 1970), 67-73.

Stable URL:
http://links.jstor.org/sici?sici=0036-1399%28197001%2918%3A1%3C67%3ATDOIBZ%3E2.0.CO%3B2-O

*SIAM Journal on Applied Mathematics* is currently published by Society for Industrial and Applied Mathematics.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/siam.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.

http://www.jstor.org/
Thu Nov 17 13:41:47 2005

SIAM J. APPL. MATH.
Vol. 18, No. 1, January 1970

# THE DISTRIBUTION OF INTERVALS BETWEEN ZEROS FOR A STATIONARY GAUSSIAN PROCESS*

E. WONG†

**1. Introduction.** This note supplements an earlier paper [1], where the distribution for the interval between two successive zeros was found for a real zero-mean Gaussian process with the following covariance function:

$$\rho(\tau) = Ex(t)x(t + \tau) = \tfrac{3}{2}e^{-|\tau|/\sqrt{3}}(1 - \tfrac{1}{3}e^{-(2/\sqrt{3})|\tau|}). \tag{1}$$

This earlier work was based on a time change for the process after which a formula of McKean [2] was used to derive the desired result. A second look at McKean's paper has revealed that the distribution of the interval between any two zeros (not only successive ones) can be found in a similar way. However, except for the case of two successive zeros, I have not been able to carry out a final integration to reduce the distribution to a closed-form expression.

**2. Computation of the inter-zero interval distribution.** Let $x(t)$ be a zero-mean Gaussian process with a covariance function given by (1). All processes in this paper are assumed to be separable and real-valued. It is easy to verify by direct computation that $x(t)$ can be represented in terms of a standard Brownian motion $W(t)$ as

$$x(t) = \sqrt{3}e^{-\sqrt{3}t}\int_0^{\exp(2/\sqrt{3})t} W(s)\,ds, \tag{2}$$

a standard Brownian motion process being defined as a zero-mean Gaussian process with $EW(t)W(s) = \min(t, s)$. It follows that for $t \geqq t_0$ we can write

$$\begin{aligned} x(t) &= x(t_0)e^{-\sqrt{3}(t-t_0)}[1 + \tfrac{3}{2}g(t - t_0)] \\ &\quad + \sqrt{3}e^{-\sqrt{3}(t-t_0)}\cdot\left[\tfrac{1}{2}\dot{x}(t_0)g(t - t_0) + \int_0^{g(t-t_0)} W(s)\,ds\right], \end{aligned} \tag{3}$$

where $W(s)$ is again a standard Brownian motion (but not the same one as in (2)) and is independent of both $x(t_0)$ and $\dot{x}(t_0)$. We have also set

$$g(t) = \exp(2/\sqrt{3})t - 1. \tag{4}$$

Let $\tau_n(t_0)$, $n = 0, 1, \cdots$, be defined as follows:

$$\begin{aligned} \tau_0(t_0) &= t_0, \\ \tau_{n+1}(t_0) &= \min\{t : t > \tau_n(t_0), x(t) = 0\}. \end{aligned} \tag{5}$$

---

* Received by the editors April 9, 1968, and in revised form October 31, 1968.

† Electronics Research Laboratory, College of Engineering, University of California, Berkeley, California 94720. This research was sponsored by the Joint Services Electronics Program under AF-AFOSR-139-67 and the United States Army Research Office-Durham under Contract DAHCO4-67-C-0046.

Because of the stationarity of $x(t)$, the probability

$$\text{Prob}\,(\tau_{m+1}(t_0) < t_0 + t | x(t_0) = 0, \text{h.w.}) \tag{6}$$
$$= \lim_{\alpha \downarrow 0} \text{Prob}\left(\tau_{m+1}(t_0) < t_0 + t | 0 \leqq \frac{x(t_0)}{\dot{x}(t_0)} \leqq \alpha\right)$$

depends only on $t$ and not on $t_0$ and will be denoted by $F_m(t)$. In (6) the notation "$x(t_0) = 0$, h.w." stands for "$x(t_0) = 0$ in the horizontal-window sense" [3] and is defined by the limit on the right-hand side. It turns out that for the process being considered here the limit in (6) is the same as

$$\lim_{\alpha \downarrow 0} \text{Prob}\,(\tau_{m+1}(t_0) < t_0 + t | x(s) \text{ has at least one zero for } s \text{ in } [t_0 - \alpha, t_0]).$$

However, we do not make use of this fact in this paper.

Now, let $\text{Prob}\,(\tau_{m+1}(0) < t | x(0) = x_0, \dot{x}(0) = y_0)$ be the conditional probability defined in the ordinary sense. Because of the symmetry of the process $x(t)$ about zero, we have

$$\text{Prob}\,(\tau_{m+1}(0) < t | x(0) = -x_0, \dot{x}(0) = -y_0)$$
$$= \text{Prob}\,(\tau_{m+1}(0) < t | x(0) = -\dot{x}(0) = -y_0).$$

It is also easy to verify that $x(0)$ and $\dot{x}(0)$ have a joint density

$$p(x_0, y_0) = \frac{1}{2\pi} e^{(x_0^2 + y_0^2)/2}.$$

Using these facts, we can evaluate $F_m(t)$ as follows:

$$F_m(t) = \lim_{\alpha \downarrow 0} \text{Prob}\left(\tau_{m+1}(0) < t | 0 \leqq \frac{x(0)}{\dot{x}(0)} \leqq \alpha\right)$$
$$= \lim_{\alpha \downarrow 0} \left[\frac{\int_0^\infty dy_0 \int_0^{\alpha y_0} dx_0 e^{-(x_0^2 + y_0^2)/2}\, \text{Prob}\,(\tau_{m+1}(0) < t | x(0) = x_0, \dot{x}(0) = y_0)}{\int_0^\infty dy_0 \int_0^{\alpha y_0} dx_0 e^{-(x_0^2 + y_0^2)/2}}\right]$$
$$= \lim_{\alpha \downarrow 0} \int_0^\infty d\eta_0 \eta_0 e^{-\eta_0^2/2} \tag{7}$$
$$\cdot \left[\frac{1}{\tan^{-1} \alpha} \int_0^{\tan^{-1} \alpha} d\theta\, \text{Prob}\,(\tau_{m+1}(0) < t | x(0) = \eta_0 \sin\theta, x(0) = \eta_0 \cos\theta)\right]$$
$$= \int_0^\infty d\eta_0 \eta_0 e^{-\eta_0^2/2}\, \text{Prob}\,(\tau_{m+1}(0) < t | x(0) = 0, \dot{x}(0) = \eta_0),$$

where we have made use of dominated convergence and the continuity of $\text{Prob}\,(\tau_{m+1}(0) < t | x(0) = x_0, \dot{x}(0) = y_0)$ in $x_0$ and $y_0$. We note that the conditional probability in the last expression of (7) is in the ordinary sense and not in the horizontal-window sense. The principal concern of this paper is the computation of the density function $P_m(t) = \dot{F}_m(t)$.

Let $\{W(t), t \geqq 0\}$ be a standard Brownian motion, and let $\sigma_n(\eta_0), n = 0, 1, \cdots$, be defined as follows:

$$
(8) \qquad \begin{aligned} &\sigma_0(\eta_0) = 0, \qquad \eta_0 \geqq 0, \\ &\sigma_{n+1}(\eta_0) = \min\left\{t : t > \sigma_n(\eta_0), t\eta_0 + \int_0^t W(s)\, ds = 0\right\}, \qquad \eta_0 \geqq 0. \end{aligned}
$$

From (3) and a comparison of (5) and (8), it is seen that

$$
(9) \qquad \begin{aligned} &\mathrm{Prob}\,(\tau_{m+1}(0) < t | x(0) = 0, \dot{x}(0) = \eta_0) \\ &\quad = \mathrm{Prob}\left(\sigma_{m+1}\left(\frac{\eta_0}{2}\right) < g(t)\right), \qquad \eta_0 \geqq 0. \end{aligned}
$$

Since $\mathrm{Prob}\,(\tau_{m+1}(0) < t | x(0) = 0, \dot{x}(0) = \eta_0)$ is symmetric in $\eta_0$, (7) becomes

$$
(10) \qquad F_m(t) = \int_0^\infty \eta_0 e^{-\eta_0^2/2}\, \mathrm{Prob}\left(\sigma_{m+1}\left(\frac{\eta_0}{2}\right) < g(t)\right) d\eta_0.
$$

Thus, the problem of evaluating $F_m(t)$ and its derivative $P_m(t)$ is reduced to the problem of finding the distribution of $\sigma_{m+1}$. Let $\sigma_n(\eta_0)$ be as defined and let

$$
(11) \qquad h_n(\eta_0) = (-1)^n[\eta_0 + W(\sigma_n(\eta_0))], \qquad \eta_0 \geqq 0.
$$

The quantity $h_n(\eta_0)$ is the magnitude of the slope of $\eta_0 t + \int_0^t W(s)\, ds$ at its $n$th zero. The joint distribution of $(\sigma_1(\eta_0), h_n(\eta_0))$ was found by McKean in [2]. Let $f(t, a)$ denote the density function for the distribution of $(\sigma_1(1), h_1(1))$, i.e.,

$$
(12) \qquad \mathrm{Prob}\,\{\sigma_1(1) \in dt, h_1(1) \in da\} = f(t, a)\, dt\, da.
$$

McKean derived the formula

$$
(13) \qquad \int_0^\infty f(t, a) e^{-\alpha t}\, dt = \int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha}) K_{i\gamma}(\sqrt{8\alpha a})}{2\cosh(\pi\gamma/3)}\, d0,
$$

where $K_\nu$ is the modified Bessel function and $d0 = 2\pi^{-2}\gamma \sinh \pi\gamma$, and, upon inverting the Laplace transform, obtained

$$
(14) \qquad f(t, a) = \frac{3a}{\pi\sqrt{2}t^2} e^{-(2/t)(1 - a + a^2)} \int_0^{4a/t} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}}\, d\theta.
$$

For a standard Brownian motion $W(t)$, $CW(t/C^2)$ is again a standard Brownian motion. From this scaling property we can show that $(\sigma_n(\eta_0), h_n(\eta_0))$ have the same probability law as $(\eta_0^2\sigma_n(1), \eta_0 h_n(1))$. Therefore, if we denote the joint density function of $(\sigma_n(\eta_0), h_n(\eta_0))$ by $\pi_n(t, \eta|\eta_0)$, we find

$$
(15) \qquad \begin{aligned} \pi_1(t, \eta|\eta_0)\, dt\, d\eta &= \mathrm{Prob}\,(\sigma_1(\eta_0) \in dt, h_1(\eta_0) \in d\eta) \\ &= \frac{dt}{\eta_0^2} \frac{d\eta}{\eta_0} f\left(\frac{t}{\eta_0^2}, \frac{\eta}{\eta_0}\right), \end{aligned}
$$

where $f$ is given by (14).

For a more general $n$, the fact that $\eta_0 t + \int_0^t W(s)$ and its derivative are jointly Markovian leads to the recursive relationship

$$\pi_{n+1}(t, \eta|\eta_0) = \int_0^t \int_0^\infty \pi_1(t - s, \eta|\zeta)\pi_n(s, \zeta|\eta_0)\, ds\, d\zeta. \tag{16}$$

Letting

$$\hat{\pi}_n(\alpha, \eta|\eta_0) = \int_0^\infty e^{-\alpha t}\pi_n(t, \eta|\eta_0)\, dt, \tag{17}$$

we can transform (16) into

$$\hat{\pi}_{n+1}(\alpha, \eta|\eta_0) = \int_0^\infty \hat{\pi}_1(\alpha, \eta|\zeta)\hat{\pi}_n(\alpha, \zeta|\eta_0)\, d\zeta. \tag{18}$$

The function $\hat{\pi}_1(\alpha, \eta|\zeta)$ can be found from (15) and (13) as follows:

$$\begin{aligned}\hat{\pi}_1(\alpha, \eta|\eta_0) &= \int_0^\infty e^{-\alpha t}\pi_1(t, \eta|\eta_0)\, dt \\ &= \int_0^\infty \frac{1}{\eta_0^3} f\left(\frac{t}{\eta_0^2}, \frac{\eta}{\eta_0}\right) e^{-\alpha t}\, dt \\ &= \int_0^\infty \frac{1}{\eta_0} f\left(\tau, \frac{\eta}{\eta_0}\right) e^{-\alpha\eta_0^2\tau}\, d\tau \\ &= \frac{1}{\eta_0}\int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha\eta_0})K_{i\gamma}(\sqrt{8\alpha\eta})}{2\cosh(\pi\gamma/3)}\, d0.\end{aligned} \tag{19}$$

From the Lebedev transform pair [4, vol. 2, p. 173]

$$g(y) = \int_0^\infty f(x)K_{ix}(y)2\pi^{-2}x \sinh \pi x\, dx,$$

$$f(x) = \int_0^\infty g(y)K_{ix}(y)y^{-1}\, dy,$$

we conclude that

$$2\pi^{-1}x \sinh \pi x \int_0^\infty K_{ix}(y)K_{ix'}(y)y^{-1}\, dy = \delta(x - x'). \tag{20}$$

Hence, by using (19) repeatedly in (18), we get

$$\hat{\pi}_n(\alpha, \eta|\eta_0) = \frac{1}{\eta_0}\int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha\eta_0})K_{i\gamma}(\sqrt{8\alpha\eta})}{[2\cosh(\pi\gamma/3)]^n}\, d0, \tag{21}$$

where $d0 = 2\pi^{-2}\gamma \sinh \pi\gamma$, which is a surprisingly simple formula. From (10) we have

$$F_m(t) = \int_0^\infty \eta_0 e^{-\eta_0^2/2} \operatorname{Prob}\left(\sigma_{m+1}\left(\frac{\eta_0}{2}\right) < g(t)\right) d\eta_0$$

(22)
$$= \int_0^\infty \eta_0 e^{-\eta_0^2/2} \operatorname{Prob}\left(\frac{\eta_0^2}{4}\sigma_{m+1}(1) < g(t)\right) d\eta_0$$

$$= \int_0^\infty d\eta_0 e^{-\eta_0^2/2} \int_0^{4g(t)/\eta_0^2} ds \int_0^\infty d\eta\, \pi_{m+1}(s, \eta|1).$$

Thus, the density $P_m(t) = \dot{F}_m(t)$ can be expressed as

(23)
$$P_m(t) = 4\dot{g}(t) \int_0^\infty \int_0^\infty \frac{1}{\eta_0} e^{-\eta_0^2/2} \pi_{m+1}\left(\frac{4g(t)}{\eta_0^2}, \eta|1\right) d\eta_0\, d\eta.$$

The Laplace transform (21) can now be inverted to yield

(24)
$$\pi_{m+1}(t, n|1) = \frac{1}{2t} e^{-2(1+\eta^2)/t} \int_0^\infty \frac{K_{i\gamma}(4\eta/t)}{(2\cosh \pi\gamma/3)^{m+1}}\, d0.$$

Using (24) in (23) yields

(25)
$$P_m(t) = \tfrac{1}{2}\dot{g}(t) \int_0^\infty \int_0^\infty \int_0^\infty \frac{\eta_0}{g(t)} e^{-\eta_0^2/2} e^{-\eta_0^2(1+\eta^2)/(2g(t))} \cdot \frac{K_{i\gamma}(\eta\eta_0^2/g(t))}{(2\cosh \pi\gamma/3)^{m+1}}\, d\eta_0\, d\eta\, d0$$

$$= \tfrac{1}{2}\dot{g}(t) \int_0^\infty \int_0^\infty \int_0^\infty \eta_0 e^{-\eta_0^2(1+\eta_0^2+g(t))/2} \cdot \frac{K_{i\gamma}(\eta\eta_0^2)}{(2\cosh \pi\gamma/3)^{m+1}}\, d\eta_0\, d\eta\, d0.$$

Now, if we use the formula

(26)
$$K_{i\gamma}(a) = \int_0^\infty e^{-a\cosh u} \cos \gamma u\, du$$

in (25), we get a fourfold integral with variables of integration $\eta_0, \eta, \gamma, u$. Integrating first with respect to $\eta_0$, then $u$, we find

(27)
$$P_m(t) = \tfrac{1}{2}\dot{g}(t) \int_0^\infty d0 \int_0^\infty d\eta \frac{1}{(2\cosh \pi\gamma/3)^{m+1}} \cdot \frac{\pi \sin(\gamma \cosh^{-1}(1+\eta^2+g(t))/(2\eta))}{\sinh \pi\gamma \sqrt{[1+\eta^2+g(t)]^2 - 4\eta^2}} \qquad \left(d0 \equiv \frac{2}{\pi^2}\gamma \sinh \pi\gamma\right)$$

$$= \frac{1}{\pi}\dot{g}(t) \int_0^\infty d\gamma \int_{\cosh^{-1}\sqrt{1+g(t)}}^\infty dx \frac{\gamma \sin \gamma x}{\sqrt{\sinh^2 x - g(t)}(2\cosh \pi\gamma/3)^{m+1}}.$$

The expression (27) can be integrated once more with respect to $\gamma$ to yield

$$P_m(t) = \frac{1}{\pi}\dot{g}(t)\int_{\cosh^{-1}\sqrt{1+g(t)}}^{\infty} \frac{1}{\sqrt{\sinh^2 x - g(t)}}(-1)\frac{d}{dx}f_m(x)\,dx, \tag{28}$$

where $f_m(x)$ is given by

$$\begin{aligned} f_0(x) &= \frac{3}{4}\frac{1}{\cosh \frac{3}{2}x}, \\ f_1(x) &= \frac{9}{8\pi}\frac{x}{\sinh \frac{3}{2}x}, \\ f_m &= \frac{1}{m}\left[\left(\frac{3x}{2\pi}\right)^2 + \left(\frac{m-1}{2}\right)^2\right]f_{m-2}(x), \qquad m \geqq 2. \end{aligned} \tag{29}$$

I have not been able to carry out the integration in (28), except for $m = 0$. In [1], $P_0(t)$ was obtained in terms of complete elliptic integrals [1, (25)].

**3. Computation of $P_m(0)$.** From (4), we have $g(0) = 0$ and $\dot{g}(0) = 2/\sqrt{3}$. Therefore, for $t = 0$, (27) becomes

$$\begin{aligned} P_m(0) &= \frac{2}{\sqrt{3}\pi}\int_0^\infty d\gamma \int_0^\infty dx \frac{\gamma \sin \gamma x}{\sinh x(2\cosh \pi\gamma/3)^{m+1}} \\ &= \frac{1}{\sqrt{3}}\int_0^\infty \left(\frac{e^{\pi\gamma}-1}{e^{\pi\gamma}+1}\right)\frac{\gamma\,d\gamma}{(2\cosh \pi\gamma/3)^{m+1}} \\ &= \frac{1}{\sqrt{3}}\left(\frac{3}{\pi}\right)^2\int_1^\infty \left(\frac{x^3-1}{x^3+1}\right)\frac{\ln x}{x(x+1/x)^{m+1}}\,dx \\ &= \frac{1}{2\sqrt{3}}\left(\frac{3}{\pi}\right)^2\int_0^\infty \left(\frac{x^3-1}{x^3+1}\right)\frac{x^m \ln x}{(x^2+1)^{m+1}}\,dx, \end{aligned} \tag{30}$$

which can be evaluated by contour integration to yield

$$P_m(0) = \frac{1}{4\sqrt{3}}\left(\frac{3}{\pi}\right)^2(2\pi i)\sum \operatorname{Res}\left\{\frac{(z^3-1)z^m \ln z(1-\ln z/(2\pi i))}{(z^3+1)(z^2+1)^{m+1}}\right\}, \tag{31}$$

where the summation is taken over the residues at the five poles $z = e^{\pi i/2}, e^{3\pi i/2}, e^{\pi i}, e^{\pi i/3}, e^{5\pi i/3}$. The expression (31) can be further elaborated to give

$$\begin{aligned} P_m(0) = \frac{9}{2\sqrt{3}}\Bigg\{&\frac{(-1)^m}{3\cdot 2^{m+1}} - \frac{10}{27} + \frac{i}{\pi m!}\frac{d^m}{dz^m}\left[\frac{z^m}{(z+i)^{m+1}}H(z)\right]\Bigg|_{z=e^{\pi i/2}} \\ &+ \frac{1}{\pi}\frac{i}{m!}\frac{d^m}{dz^m}\left[\frac{z^m}{(z-i)^{m+1}}H(z)\right]\Bigg|_{z=e^{3\pi i/2}}\Bigg\} \end{aligned} \tag{32}$$

with

$$H(z) = \frac{z^3-1}{z^3+1}\ln z\left[1 - \frac{\ln z}{2\pi i}\right]. \tag{33}$$

I have not been able to reduce (32) further.

For a stationary zero-mean Gaussian process with covariance function of the form

$$\rho(\tau) = 1 - \frac{\tau^2}{2} + \alpha|\tau|^3 + o(|\tau|^3), \tag{34}$$

it is not hard to show that $P_m(0)$ is proportional to $\alpha$. Longuet-Higgins [5] has obtained bounds for $P_m(0)/\alpha$ for $m$ up to 7. For $m = 0, 1, 2$, these bounds read

$$\begin{aligned} 1.1556 &< \frac{1}{\alpha}P_0(0) < 1.158, \\ 0.1971 &< \frac{1}{\alpha}P_1(0) < 0.198, \\ 0.0491 &< \frac{1}{\alpha}P_2(0) < 0.0556. \end{aligned} \tag{35}$$

Now, the covariance function (1) under consideration in this paper has the form of (34) with $\alpha = \frac{2}{3}\sqrt{3}$. Hence, the true values for $P_m(0)/\alpha$ can be evaluated and compared against the bounds in (35). For $m = 0, 1, 2$, (32) can be evaluated to yield

$$\begin{aligned} P_0(0) &= \frac{2}{3\sqrt{3}}\left(\frac{37}{32}\right) = \frac{2}{3\sqrt{3}}(1.15625), \\ P_1(0) &= \frac{2}{3\sqrt{3}}\left(\frac{47}{64} - \frac{108}{64\pi}\right) \cong \frac{2}{3\sqrt{3}}(0.1972), \\ P_2(0) &= \frac{2}{3\sqrt{3}}\left(\frac{121}{128} - \frac{81}{32\pi} - \frac{27}{32\pi^2}\right) \cong \frac{2}{3\sqrt{3}}(0.0541), \end{aligned} \tag{36}$$

which are in agreement with (35).

**Acknowledgment.** My interest in the extension reported here was first aroused by a communication from Dr. M. S. Longuet-Higgins, who raised the possibility of finding $P_m(0)$ along the lines of [1]. I am grateful to Dr. Longuet-Higgins for his suggestion.

## REFERENCES

[1] E. WONG, *Some results concerning the zero-crossings of Gaussian noise*, this Journal, 14 (1966), pp. 1246–1254.

[2] H. P. MCKEAN, JR., *A winding problem for a resonator driven by a white noise*, J. Math. Kyoto Univ., 2 (1963), pp. 227–235.

[3] M. KAC AND D. SLEPIAN, *Large excursions of Gaussian processes*, Ann. Math. Statist., 30 (1959), pp. 1215–1228.

[4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Tables of Integral Transforms*, McGraw-Hill, New York, 1954.

[5] M. S. LONGUET-HIGGINS, *Bounding approximations to the distribution of intervals between zeros of a stationary Gaussian process*, Time Series Analysis, M. Rosenblatt, ed., John Wiley, New York, 1963, pp. 104–115.