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THE DISTRIBUTION OF INTERVALS BETWEEN ZEROS FOR A STATIONARY GAUSSIAN PROCESS

E. WONG†

1. Introduction. This note supplements an earlier paper [1], where the distribution for the interval between two successive zeros was found for a real zero-mean Gaussian process with the following covariance function:

\[ \rho(\tau) = Ex(t)x(t + \tau) = \frac{3}{2}e^{-|\tau|/\sqrt{3}}(1 - \frac{1}{3}e^{-2|\tau|/\sqrt{3}}). \]

This earlier work was based on a time change for the process after which a formula of McKean [2] was used to derive the desired result. A second look at McKean’s paper has revealed that the distribution of the interval between any two zeros (not only successive ones) can be found in a similar way. However, except for the case of two successive zeros, I have not been able to carry out a final integration to reduce the distribution to a closed-form expression.

2. Computation of the inter-zero interval distribution. Let \( x(t) \) be a zero-mean Gaussian process with a covariance function given by (1). All processes in this paper are assumed to be separable and real-valued. It is easy to verify by direct computation that \( x(t) \) can be represented in terms of a standard Brownian motion \( W(t) \) as

\[ x(t) = \sqrt{3}e^{-\sqrt{3}t} \int_0^{\exp(2/\sqrt{3})t} W(s) \, ds, \]

a standard Brownian motion process being defined as a zero-mean Gaussian process with \( EW(t)W(s) = \min(t, s) \). It follows that for \( t \geq t_0 \) we can write

\[ x(t) = x(t_0)e^{-\sqrt{3}(t - t_0)}[1 + \frac{3}{2}g(t - t_0)] \]

\[ + \sqrt{3}e^{-\sqrt{3}(t - t_0)} \cdot \left[ \frac{1}{2}x(t_0)g(t - t_0) + \int_0^{g(t - t_0)} W(s) \, ds \right], \]

where \( W(s) \) is again a standard Brownian motion (but not the same one as in (2)) and is independent of both \( x(t_0) \) and \( x(t_0) \). We have also set

\[ g(t) = \exp(2/\sqrt{3})t - 1. \]

Let \( \tau_n(t_0), n = 0, 1, \ldots, \) be defined as follows:

\[ \tau_0(t_0) = t_0, \]

\[ \tau_{n+1}(t_0) = \min \{ t : t > \tau_n(t_0), x(t) = 0 \}. \]

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† Electronics Research Laboratory, College of Engineering, University of California, Berkeley, California 94720. This research was sponsored by the Joint Services Electronics Program under AF-AFOSR-139-67 and the United States Army Research Office-Durham under Contract DAHC04-67-C-0046.
Because of the stationarity of \( x(t) \), the probability

\[
\text{Prob} \left( \tau_{m+1}(t_0) < t_0 + t | x(t_0) = 0, \text{h.w.} \right)
\]

\[
= \lim_{s \downarrow 0} \text{Prob} \left( \tau_{m+1}(t_0) < t_0 + t | 0 \leq \frac{x(t_0)}{\hat{x}(t_0)} \leq s \right)
\]

depends only on \( t \) and not on \( t_0 \) and will be denoted by \( F_m(t) \). In (6) the notation "\( x(t_0) = 0, \text{h.w.} \)" stands for "\( x(t_0) = 0 \) in the horizontal-window sense" [3] and is defined by the limit on the right-hand side. It turns out that for the process being considered here the limit in (6) is the same as

\[
\lim_{s \downarrow 0} \text{Prob} \left( \tau_{m+1}(t_0) < t_0 + t | x(s) \text{ has at least one zero for } s \in [t_0 - \alpha, t_0] \right).
\]

However, we do not make use of this fact in this paper.

Now, let \( \text{Prob} \left( \tau_{m+1}(0) < t | x(0) = x_0, \dot{x}(0) = y_0 \right) \) be the conditional probability defined in the ordinary sense. Because of the symmetry of the process \( x(t) \) about zero, we have

\[
\text{Prob} \left( \tau_{m+1}(0) < t | x(0) = -x_0, \dot{x}(0) = -y_0 \right)
\]

\[
= \text{Prob} \left( \tau_{m+1}(0) < t | x(0) = -x(0) = -y_0 \right).
\]

It is also easy to verify that \( x(0) \) and \( \dot{x}(0) \) have a joint density

\[
p(x_0, y_0) = \frac{1}{2\pi} e^{(x_0^2 + y_0^2)/2}.
\]

Using these facts, we can evaluate \( F_m(t) \) as follows:

\[
F_m(t) = \lim_{s \downarrow 0} \text{Prob} \left( \tau_{m+1}(0) < t | 0 \leq \frac{x(0)}{\hat{x}(0)} \leq s \right)
\]

\[
= \lim_{s \downarrow 0} \left[ \int_0^\infty d\eta_0 \int_0^{\eta_0} dx_0 e^{-(x_0^2 + y_0^2)/2} \text{Prob} \left( \tau_{m+1}(0) < t | x(0) = x_0, \dot{x}(0) = y_0 \right) \right]
\]

\[
\int_0^\infty d\eta_0 \int_0^{\eta_0} dx_0 e^{-(x_0^2 + y_0^2)/2}
\]

\[
= \lim_{s \downarrow 0} \int_0^\infty d\eta_0 \eta_0 e^{-\eta_0^2/2}
\]

\[
\left\{ \frac{1}{\tan^{-1} s} \int_0^{\tan^{-1} s} d\theta \text{Prob} \left( \tau_{m+1}(0) < t | x(0) = \eta_0 \sin \theta, x(0) = \eta_0 \cos \theta \right) \right\}
\]

\[
= \int_0^\infty d\eta_0 \eta_0 e^{-\eta_0^2/2} \text{Prob} \left( \tau_{m+1}(0) < t | x(0) = 0, \dot{x}(0) = \eta_0 \right),
\]

where we have made use of dominated convergence and the continuity of \( \text{Prob} \left( \tau_{m+1}(0) < t | x(0) = x_0, \dot{x}(0) = y_0 \right) \) in \( x_0 \) and \( y_0 \). We note that the conditional probability in the last expression of (7) is in the ordinary sense and not in the horizontal-window sense. The principal concern of this paper is the computation of the density function \( P_m(t) = F_m(t) \).
Let \( \{W(t), t \geq 0\} \) be a standard Brownian motion, and let \( \sigma_d(\eta_0), n = 0, 1, \cdots \), be defined as follows:

\[
\begin{align*}
\sigma_0(\eta_0) &= 0, \quad \eta_0 \geq 0, \\
\sigma_{n+1}(\eta_0) &= \min \left\{ t : t > \sigma_d(\eta_0), t\eta_0 + \int_0^t W(s) \, ds = 0 \right\}, \quad \eta_0 \geq 0.
\end{align*}
\]

From (3) and a comparison of (5) and (8), it is seen that

\[
\text{Prob} \left( \tau_{m+1}(0) < t \mid x(0) = 0, \dot{x}(0) = \eta_0 \right) = \text{Prob} \left( \sigma_{m+1} \left( \frac{\eta_0}{2} \right) < g(t) \right), \quad \eta_0 \geq 0.
\]

Since \( \text{Prob} \left( \tau_{m+1}(0) < t \mid x(0) = 0, \dot{x}(0) = \eta_0 \right) \) is symmetric in \( \eta_0 \), (7) becomes

\[
F_m(t) = \int_0^\infty \eta_0 e^{-\eta_0^2/2} \text{Prob} \left( \sigma_{m+1} \left( \frac{\eta_0}{2} \right) < g(t) \right) d\eta_0.
\]

Thus, the problem of evaluating \( F_m(t) \) and its derivative \( P_m(t) \) is reduced to the problem of finding the distribution of \( \sigma_{m+1} \). Let \( \sigma_d(\eta_0) \) be as defined and let

\[
h_d(\eta_0) = (-1)^n \left[ \eta_0 + W(\sigma_d(\eta_0)) \right], \quad \eta_0 \geq 0.
\]

The quantity \( h_d(\eta_0) \) is the magnitude of the slope of \( \eta_0 t + \int_0^t W(s) \, ds \) at its \( n \)th zero.

The joint distribution of \( \left( \sigma_d(\eta_0), h_d(\eta_0) \right) \) was found by McKean in [2]. Let \( f(t, a) \) denote the density function for the distribution of \( \left( \sigma_d(1), h_d(1) \right) \), i.e.,

\[
\text{Prob} \left\{ \sigma_d(1) \in dt, h_d(1) \in da \right\} = f(t, a) \, dt \, da.
\]

McKean derived the formula

\[
\int_0^\infty f(t, a) e^{-\eta_0^2} \, dt = \int_0^\infty \frac{K_1(\sqrt{8\alpha})K_1(\sqrt{8\alpha a})}{2 \cosh (\pi\gamma/3)} \, d\alpha,
\]

where \( K_\nu \) is the modified Bessel function and \( d\alpha = 2\pi^{-2}\gamma \sinh \pi\gamma \), and, upon inverting the Laplace transform, obtained

\[
f(t, a) = \frac{3a}{\pi \sqrt{2t^2}} e^{-a^2/(t^2 + a^2)} \int_0^{2a/t} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}} \, d\theta.
\]

For a standard Brownian motion \( W(t) \), \( CW(t/C^2) \) is again a standard Brownian motion. From this scaling property we can show that \( \left( \sigma_d(\eta_0), h_d(\eta_0) \right) \) have the same probability law as \( \left( \eta_0^2 \sigma_d(1), \eta_0 h_d(1) \right) \). Therefore, if we denote the joint density function of \( \left( \sigma_d(\eta_0), h_d(\eta_0) \right) \) by \( \pi_1(t, \eta) \), we find

\[
\pi_1(t, \eta) \, d\eta \, dt \, d\eta = \text{Prob} \left( \sigma_d(\eta_0) \in dt, h_d(\eta_0) \in d\eta \right) = \frac{dt \, d\eta}{\eta_0^2} \frac{\left( t \, \eta \right)}{\eta_0^2},
\]

where \( f \) is given by (14).
For a more general $n$, the fact that $\eta_0(t) + \int_0^t W(s)\, ds$ and its derivative are jointly Markovian leads to the recursive relationship

\begin{equation}
\pi_{n+1}(t, \eta|\eta_0) = \int_0^t \pi_1(t-s, \eta|\eta_0) \pi_n(s, \eta_0) \, ds \, d\zeta.
\end{equation}

Letting

\begin{equation}
\hat{\pi}_n(\tau, \eta|\eta_0) = \int_0^{\tau} e^{-\eta t} \pi_n(t, \eta|\eta_0) \, dt,
\end{equation}

we can transform (16) into

\begin{equation}
\hat{\pi}_{n+1}(\tau, \eta|\eta_0) = \int_0^{\tau} \hat{\pi}_1(\tau, \eta|\eta_0) \hat{\pi}_n(\tau, \zeta|\eta_0) \, d\zeta.
\end{equation}

The function $\hat{\pi}_1(\tau, \eta|\eta_0)$ can be found from (15) and (13) as follows:

\begin{equation}
\hat{\pi}_1(\tau, \eta|\eta_0) = \int_0^{\infty} e^{-\eta t} \pi_1(t, \eta|\eta_0) \, dt
\end{equation}

\begin{align*}
&= \int_0^{\infty} \frac{1}{\eta_0^3} f\left(\frac{t}{\eta_0}, \frac{\eta}{\eta_0}\right) e^{-\eta t} \, dt \\
&= \int_0^{\infty} \frac{1}{\eta_0} f\left(\tau, \frac{\eta}{\eta_0}\right) e^{-\eta \beta \zeta} \, d\tau
\end{align*}

\begin{align*}
&= \frac{1}{\eta_0} \int_0^{\infty} \frac{K_i\left(\sqrt{8\pi\eta}\eta_0\right)K_i\left(\sqrt{8\pi\eta}\right)}{2 \cosh(\pi\beta/3)} \, d\eta_0.
\end{align*}

From the Lebedev transform pair [4, vol. 2, p. 173]

\begin{align*}
g(y) &= \int_0^{\infty} f(x)K_{ix}(y)2\pi^{-2}x \sinh \pi x \, dx, \\
f(x) &= \int_0^{\infty} g(y)K_{ix}(y)^{-1} \, dy,
\end{align*}

we conclude that

\begin{equation}
2\pi^{-1}x \sinh \pi x \int_0^{\infty} K_{ix}(y)K_{ix}(y)^{-1} \, dy = \delta(x - x').
\end{equation}

Hence, by using (19) repeatedly in (18), we get

\begin{equation}
\hat{\pi}_n(\tau, \eta|\eta_0) = \frac{1}{\eta_0} \int_0^{\infty} \frac{K_i\left(\sqrt{8\pi\eta}\eta_0\right)K_i\left(\sqrt{8\pi\eta}\right)}{[2 \cosh(\pi\beta/3)]^n} \, d\eta_0.
\end{equation}
where \( d_0 = 2\pi^{-2}\gamma \sinh \pi \gamma \), which is a surprisingly simple formula. From (10) we have

\[
F_m(t) = \int_0^\infty \eta_0 e^{-\eta_0^2/2} \text{Prob} \left( \frac{\sigma_{m+1}}{2} < g(t) \right) d\eta_0
\]

\[
= \int_0^\infty \eta_0 e^{-\eta_0^2/2} \text{Prob} \left( \frac{\eta_0^2}{4\sigma_{m+1}} < g(t) \right) d\eta_0
\]

\[
= \int_0^\infty d\eta_0 e^{-\eta_0^2/2} \int_0^\infty \int_0^\infty ds \int_0^\infty d\eta \pi_{m+1}(s, \eta|1).
\]

Thus, the density \( P_m(t) = \hat{F}_m(t) \) can be expressed as

\[
P_m(t) = 4g(t) \int_0^\infty \int_0^\infty \frac{1}{\eta_0} e^{-\eta_0^2/2} \pi_{m+1} \left( \frac{4g(t)}{\eta_0^2}, \eta_1 \right) d\eta_0 d\eta.
\]

The Laplace transform (21) can now be inverted to yield

\[
\pi_{m+1}(t, \eta|1) = \frac{1}{2t} e^{-2(1 + \eta^2)/t} \int_0^\infty \frac{K_{\frac{1}{2}}(4\eta/t)}{(2 \cosh \pi \gamma/3)^{m+1}} d\eta.
\]

Using (24) in (23) yields

\[
P_m(t) = \frac{1}{t} g(t) \int_0^\infty \int_0^\infty \int_0^\infty \frac{\eta_0}{g(t)} e^{-\eta_0^2/2} e^{-\eta_0^2(1 + \eta^2)/(2g(t))}
\]

\[
\cdot \frac{K_{\frac{1}{2}}(\eta_0^2/g(t))}{(2 \cosh \pi \gamma/3)^{m+1}} d\eta_0 d\eta d0
\]

\[
= \frac{1}{t} g(t) \int_0^\infty \int_0^\infty \frac{\eta_0}{g(t)} e^{-\eta_0^2(1 + \eta^2)/(2g(t))}
\]

\[
\cdot \frac{K_{\frac{1}{2}}(\eta_0^2)}{(2 \cosh \pi \gamma/3)^{m+1}} d\eta_0 d\eta d0.
\]

Now, if we use the formula

\[
K_{\frac{1}{2}}(a) = \int_0^\infty e^{-a \cosh u} \cos \gamma u \ du
\]

in (25), we get a fourfold integral with variables of integration \( \eta_0, \eta, \gamma, u \). Integrating first with respect to \( \eta_0 \), then \( u \), we find

\[
P_m(t) = \frac{1}{t} g(t) \int_0^\infty \int_0^\infty \frac{1}{(2 \cosh \pi \gamma/3)^{m+1}}
\]

\[
\cdot \pi \sin \left( \gamma \cosh^{-1} \left( 1 + \eta^2 + g(t)/(2\eta) \right) \right)
\]

\[
\cdot \sinh \pi \gamma \sqrt{[1 + \eta^2 + g(t)]^2 - 4\eta^2}
\]

\[
\left( d0 \equiv \frac{2}{\pi^2} \gamma \sinh \pi \gamma \right)
\]

\[
= \frac{1}{\pi} g(t) \int_0^\infty \int_{\cosh^{-1} \left( 1 + g(t)/(2\eta) \right)}^\infty \frac{\gamma \sin \gamma x}{\sqrt{\sinh^2 x - g(t)/(2 \cosh \pi \gamma/3)^{m+1}}} dx.
\]
The expression (27) can be integrated once more with respect to $\gamma$ to yield

$$P_m(t) = \frac{1}{\pi} \hat{g}(t) \int_{\cosh^{-1} \frac{-1}{g(t)}}^{\infty} \frac{1}{\sqrt{\sinh^2 x - g(t)}} \left( -1 \right) \frac{d}{dx} f_m(x) \, dx,$$

where $f_m(x)$ is given by

$$f_0(x) = \frac{3}{4} \frac{1}{\cosh \frac{3}{2} x},$$

$$f_1(x) = \frac{9}{8\pi} \frac{x}{\sinh \frac{3}{2} x},$$

$$f_m = \frac{1}{m} \left[ \left( \frac{3x}{2\pi} \right)^2 + \left( \frac{m - 1}{2} \right)^2 \right] f_{m-2}(x), \quad m \geq 2.$$

I have not been able to carry out the integration in (28), except for $m = 0$. In [1], $P_0(t)$ was obtained in terms of complete elliptic integrals [1, (25)].

3. Computation of $P_m(0)$.

From (4), we have $g(0) = 0$ and $\hat{g}(0) = 2/\sqrt{3}$. Therefore, for $t = 0$, (27) becomes

$$P_m(0) = \frac{2}{\sqrt{3\pi}} \int_0^{\infty} \frac{d\gamma}{\gamma} \int_0^{\infty} dx \frac{\gamma \sin \gamma x}{\sinh x (2 \cosh \pi \gamma/3)^{m+1}}$$

$$= \frac{1}{\sqrt{3}} \int_0^{\infty} \frac{\left( e^{\pi \gamma} - 1 \right)}{\left( e^{\pi \gamma} + 1 \right) (2 \cosh \pi \gamma/3)^{m+1}} \gamma \, d\gamma$$

$$= \frac{1}{\sqrt{3}} \left( \frac{3}{\pi} \right)^2 \int_1^{\infty} \frac{x^3 - 1}{x(x + 1/x)^{m+1}} \frac{\ln x}{x^3 + 1} \, dx$$

$$= \frac{1}{2\sqrt{3}} \left( \frac{3}{\pi} \right)^2 \int_0^{\infty} \frac{x^3 - 1}{x^3 + 1} \frac{x^m \ln x}{(x^2 + 1)^{m+1}} \, dx,$$

which can be evaluated by contour integration to yield

$$P_m(0) = \frac{1}{4\sqrt{3}} \left( \frac{3}{\pi} \right)^2 (2\pi i) \sum \text{Res} \left\{ \frac{(z^3 - 1)z^m \ln z(1 - \ln z/(2\pi i))}{(z^3 + 1)(z^2 + 1)^{m+1}} \right\},$$

where the summation is taken over the residues at the five poles $z = e^{\pi i/2}, e^{3\pi i/2}, e^{\pi i/3}, e^{5\pi i/3}$. The expression (31) can be further elaborated to give

$$P_m(0) = \frac{9}{2\sqrt{3}} \left\{ \frac{(-1)^m}{3 \cdot 2^{m+1}} - \frac{10}{27} + \frac{i}{\pi m} \frac{d^m}{dz^m} \left[ \frac{z^m}{(z + i)^{m+1}} H(z) \right] \right\}_{z = e^{\pi i/2}}$$

$$+ \frac{1}{\pi} \frac{i}{m} \frac{d^m}{dz^m} \left[ \frac{z^m}{(z - i)^{m+1}} H(z) \right] \bigg|_{z = e^{3\pi i/2}} \right\}$$

with

$$H(z) = \frac{z^3 - 1}{z^3 + 1} \ln \left[ 1 - \frac{\ln z}{2\pi i} \right].$$
I have not been able to reduce (32) further.

For a stationary zero-mean Gaussian process with covariance function of the form

\[ \rho(\tau) = 1 - \frac{\tau^2}{2} + \alpha|\tau|^3 + o(|\tau|^3), \]

it is not hard to show that \( P_m(0) \) is proportional to \( \alpha \). Longuet-Higgins [5] has obtained bounds for \( \frac{P_m(0)}{\alpha} \) for \( m \) up to 7. For \( m = 0, 1, 2 \), these bounds read

\[ 1.1556 < \frac{1}{\alpha} P_0(0) < 1.158, \]

\[ 0.1971 < \frac{1}{\alpha} P_1(0) < 0.198, \]

\[ 0.0491 < \frac{1}{\alpha} P_2(0) < 0.0556. \]

Now, the covariance function (1) under consideration in this paper has the form of (34) with \( \alpha = \frac{2}{3\sqrt{3}} \). Hence, the true values for \( \frac{P_m(0)}{\alpha} \) can be evaluated and compared against the bounds in (35). For \( m = 0, 1, 2 \), (32) can be evaluated to yield

\[ P_0(0) = \frac{2}{3\sqrt{3}} \frac{37}{32} = \frac{2}{3\sqrt{3}}(1.15625), \]

\[ P_1(0) = \frac{2}{3\sqrt{3}} \left( \frac{47}{64} - \frac{108}{64\pi} \right) \approx \frac{2}{3\sqrt{3}}(0.1972), \]

\[ P_2(0) = \frac{2}{3\sqrt{3}} \left( \frac{121}{128} - \frac{81}{32\pi} - \frac{27}{32\pi^2} \right) \approx \frac{2}{3\sqrt{3}}(0.0541), \]

which are in agreement with (35).

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**REFERENCES**


