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HOMOGENEOUS GAUSS-MARKOV RANDOM FIELDS[1]

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1. Introduction. In this paper we consider real Gaussian random fields which are: (a) homogeneous with respect to the motions of an n-dimensional space of constant curvature, and (b) Markovian in the sense of Lévy [1]. The principal result of this paper is the characterization of such random fields in terms of their covariance functions. We recall that in one dimension a similar question has the very simple answer that the covariance function of a stationary Gauss-Markov process must be an exponential. The answer in the n-dimensional case is nearly as simple, and will be given in this paper.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a fixed probability space, and let $\{x(\omega, z), \omega \varepsilon \Omega, z \varepsilon V_n\}$ be a family of real Gaussian random variables with an n-dimensional parameter space V_n. We shall only consider three cases: (a) $V_n = R^n$, Euclidean space. (b) $V_n = S^n$, sphere. (c) $V_n = H^n$, hyperbolic space. Let $G(V_n)$ be the full group of motions in V_n which preserve distances. Suppose that for any finite set $A = \{z_i\} \subset V_n$, $\{x(\cdot, z_i), z_i \varepsilon A\}$ and $\{x(\cdot, gz_i), z_i \varepsilon A\}$ have the same distribution whenever $g \varepsilon G(V_n)$. Then we say $\{x(\cdot, z), z \varepsilon V_n\}$ is a homogeneous random field.

Markovian property in higher dimensions was introduced by Lévy [1] in connection with Brownian motion. Let ∂D be a smooth closed surface of dimension $n - 1$ in V_n, separating V_n into a bounded part D^-, and a possibly unbounded part D^+. A random field $\{x(z), z \varepsilon V_n\}$ is said to be Markovian of degree $\leqq p + 1$, if for any such ∂D every approximation $\hat{x}(z)$ to $x(z)$ in a neighborhood of ∂D which satisfies

$$|\hat{x}(z) - x(z)| = o(\delta^p) \qquad \delta = \text{distance } (z, \partial D)$$

also has the property that given $\hat{x}(\cdot)$, $x(z)$ and $x(z')$ are independent whenever $z \varepsilon D^-$ and $z' \varepsilon D^+$.

A random field is Markovian of degree p, if it is Markovian of degree $\leqq p$, but not $\leqq p - 1$. In this paper we are primarily concerned with Markovian fields of degree 1. For this special case it is more convenient to define the Markovian property by: given $\{x(z), z \varepsilon \partial D\}$, $x(z), z \varepsilon D^-$, and $x(z), z \varepsilon D^+$, are independent. If $x(z)$ has continuous sample functions, this definition clearly reduces to that of Lévy. This latter definition is more convenient when we have occasion later to consider the possibility of defining Markovian property for generalized random fields.

Since Gaussian distributions are uniquely determined by second order prop-

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erties, whether a Gaussian random field is Markovian or not is completely determined by its covariance function. While it would be nice to give a necessary and sufficient condition on the covariance function for a Gaussian random field to be Markovian, we are able to do this only when the random field is homogeneous.

2. Second-order properties. There is no essential loss of generality in assuming that V_n has curvature 0, 1, and -1 corresponding to R^n, S^n, H^n respectively. With the assumption we can adopt a coordinate system $(\varphi_1, \cdots, \varphi_{n-1}, r)$, $\varphi = (\varphi_1, \cdots, \varphi_{n-1}) \varepsilon S^{n-1}$, $r \varepsilon [0, \infty)$ for R^n, H^n, and $r \varepsilon [0, \pi)$ for S^n. We express the Riemannian metric in the form of the differential arc length ds:

$$ds^2 = dr^2 + g^2(r) \sum_{i=1}^{n-1} \left(\prod_{k=i+1}^{n-1} \sin^2 \varphi_k\right) d\varphi_i^2, \tag{1}$$

where $g(r) = r$, $\sin r$, $\sinh r$ for R^n, S^n and H^n respectively. The length of a sectionally smooth (piecewise differentiable in terms of coordinates) curve is found by integrating ds along the curve. The distance $d(z_1, z_2)$ between two points $z_1, z_2 \varepsilon V_n$ is the infimum of the lengths of all sectionally smooth curves connecting z_1 and z_2. It can be shown that for the three cases being considered, we have

$$\begin{aligned} d((\varphi, r), (\varphi', r')) &= [r^2 + r'^2 - 2rr' \cos \theta(\varphi, \varphi')]^{\frac{1}{2}} \\ &= \cos^{-1} [\cos r \cos r' + \sin r \sin r' \cos \theta(\varphi, \varphi')] \\ &= \cosh^{-1} [\cosh r \cosh r' - \sinh r \sinh r' \cos \theta(\varphi, \varphi')] \end{aligned} \tag{2}$$

for R^n, S^n and H^n respectively, where $\theta(\varphi, \varphi')$ is the spherical distance between φ and φ' on S^{n-1}.

Consider the full group G of one-to-one differentiable mappings of V_n onto itself which preserve distances. G acts transitively on V_n, i.e., taking any point into any other point. Hence, if we let K be the maximal subgroup leaving $(\cdot, 0)$ invariant, then V_n can be identified with the homogeneous coset space G/K. For a homogeneous Gaussian random field $\{x(\cdot, z), z \varepsilon V_n\}$, we have $Ex(\cdot, z) = Ex(\cdot, gz)$ for all $g \varepsilon G$. Hence, $Ex(\cdot, z) =$ constant which we shall assume to be zero. Similarly, whenever $g \varepsilon G, Ex(\cdot, z)x(\cdot, z_0) = Ex(\cdot, gz)x(\cdot, gz_0)$. Since G acts transitively on V_n, there always exists g taking z_0 into $(\cdot, 0)$ and z into $(0, d(z, z_0))$. Thus, $E x(\cdot, z)x(\cdot, z_0)$ depends only on $d(z, z_0)$ and we can write

$$Ex(\cdot, z)x(\cdot, z_0) = R(d(z, z_0)). \tag{3}$$

Analogous to Bochner's theorem in one dimension, the class of continuous covariance functions of the form of (3) can be put into a one-to-one correspondence with the class of all bounded non-decreasing functions defined on $[0, \infty)$ in the case of R^n and H^n, and the class of all non-negative functions defined on the integers in the case of S^n. This is done via a spectral representation for $R(\cdot)$.

A function $R(r)$, $r \varepsilon [0, \infty)$, is said to be a *positive-definite* function on V_n, if for arbitrary complex constants $a_1, a_2, \cdots, a_k$ and arbitrary elements z_1, z_2,

$\cdots, z_k$ in V_n, $R(\cdot)$ satisfies the inequality

(4) $$\textstyle\sum_{i,j}^{k} a_i \bar{a}_j R(d(z_i, z_j)) \geqq 0.$$

As is well-known, $R(\cdot)$ is the covariance function of a homogeneous random field on V_n if and only if it is a positive-definite function on V_n. It is also well-known, that a random field $\{x(\cdot, z), z \in V_n\}$ is continuous in quadratic mean if and only if its covariance function: $R(d(z, z_0))$ is continuous on $V_n \times V_n$. Let $\psi(r)$, $r \in [0, \infty)$, be a continuous positive-definite function on V_n. Suppose $\psi(0) = 1$ and ψ is an eigenfunction of the operator

(5) $$\Delta_0 = \{g^{n-1}(r)\}^{-1} \frac{d}{dr}\left[g^{n-1}(r) \frac{d}{dr}\right],$$

then ψ is called a *spherical function*. Let $\mathfrak{M}$ denote the set of all spherical functions. Let L denote the set of all complex valued functions f which satisfy

(6) $$\textstyle\int_0^\infty g^{n-1}(r)|f(r)|\, dr < \infty.$$

Suppose $\mathfrak{M}$ is given the weakest topology for which the Fourier transform

(7) $$\hat{f}(\psi) = \textstyle\int_0^\infty \psi(r) f(r) g^{n-1}(r)\, dr, \qquad \psi \in \mathfrak{M}$$

is continuous for every $f \in L$. Then it can be shown [2, 3] that every continuous positive-definite function on V_n is of the form

(8) $$R(r) = \textstyle\int_{\mathfrak{M}} \psi(r) \sigma(d\psi),$$

where σ is a finite Borel measure on $\mathfrak{M}$.

For the three cases R^n, S^n, H^n, the spherical functions can be found explicitly by solving the differential equation

(9) $$\{g^{n-1}(r)\}^{-1} \frac{d}{dr}\left[g^{n-1}(r) \frac{d\psi(r)}{dr}\right] = \lambda\psi(r)$$

for positive-definite functions which satisfy $\psi(0) = 1$. The function $g(r)$ is given by r, $\sin r$, $\sinh r$, according as $V_n = R^n$, S^n and H^n respectively. The spherical functions for these cases are listed below.

(a) R^n: $\lambda = -\nu^2, \quad 0 \leqq \nu < \infty$

(10) $$\psi(r) = \frac{J_{(n-2)/2}(\nu r)}{(\nu r)^{(n-2)/2}} = K_n \int_0^\pi \exp(i\nu r \cos\theta) \sin^{n-2}\theta\, d\theta.$$

(b) S^n: $\lambda = -k(k + n - 1), \quad k = 0, 1, 2, \cdots$

(11) $$\psi(r) = C_k^{\frac{1}{2}(n-1)}(\cos r) = K_n \textstyle\int_0^\pi (\cos r + \sin r \cos\theta)^k \sin^{n-2}\theta\, d\theta.$$

(c) H^n: $-\infty < \lambda \leqq 0, \quad \mu(\lambda) = \frac{1}{2} + \{[\frac{1}{2}(n - 1)]^2 + \lambda\}^{\frac{1}{2}}$

(12) $$\psi(r) = \frac{P_{\mu(\lambda)}^{-\frac{1}{2}(n-2)}(\cosh r)}{(\sinh r)\frac{1}{2}(n - 2)} = K_n \int_0^\pi (\cosh + \sinh r \cos\theta)^{\mu - \frac{1}{2}(n-2)} \sin^{n-2}\theta\, d\theta.$$

In each of these cases, K_n is fixed by setting $K_n \int_0^\pi \sin^{n-2}\theta\, d\theta = 1$. It should be noted that in (12), for the range $-\infty < \lambda < [\frac{1}{2}(n-1)]^2$, it makes no difference whether we take $\mu(\lambda) = -\frac{1}{2} + i\{|\lambda| - [\frac{1}{2}(n-1)]^2\}^{\frac{1}{2}}$ or $\mu(\lambda) = -\frac{1}{2} - i\{|\lambda| - [\frac{1}{2}(n-1)]^2\}^{\frac{1}{2}}$. This follows from the following property of the Legendre functions:

$$P_\mu^{\ \nu} = P_{-\mu-1}^{\nu}.$$

Using these results, we can reduce (8) to a sum over the non-negative integers in the case of S^n, and to a Stieltjes integral over $[0, \infty)$ for R^n and H^n.

Let Δ be the Laplace-Beltrami operator given by

$$\Delta(V_n) = \{g^{n-1}(r)\}^{-1}\frac{\partial}{\partial r}\left[g^{n-1}(r)\frac{\partial}{\partial r}\right] + (g^2(r))^{-1}\Delta(S^{n-1}) \tag{13}$$

where $\Delta(S^{n-1})$ can be recursively generated as follows:

$$\Delta(S^n) = \{\sin^{n-1}(\varphi_n)\}^{-1}\frac{\partial}{\partial\varphi_n}\left[\sin^{n-1}(\varphi_n)\frac{\partial}{\partial\varphi_n}\right] + (\sin^2\varphi_n)^{-1}\Delta(S^{n-1}). \tag{14}$$

It is well known that Δ commutes with any g in G, and every differential operator commuting with G is a polynomial in Δ with constant coefficients. Let $L^2(S^{n-1})$ denote the set of all square-integrable functions on S^{n-1} (with respect to the uniform measure). For each m ($m = 0, 1, 2, \cdots$), any maximal set of linearly independent solutions in $L^2(S^{n-1})$ of the equation

$$\Delta(S^{n-1})h = -m(m+n-2)h \tag{15}$$

forms a basis for a subspace Σ_m which is invariant under $G(S^{n-1})$. Denoting the dimension of Σ_m by d_m, we have

$$1 = d_0 < d_1 \leqq d_2 \leqq \cdots.$$

We choose the basis $\{h_{ml}, l \leqq d_m\}$ to be real and orthonormalized so that

$$\int_{S^{n-1}} h_{ml}(\varphi)h_{pk}(\varphi)\, d0 = \delta_{mp}\delta_{kl}, \quad d0 = \text{uniform measure}, \quad \int_{S^{n-1}} d0 = 1. \tag{16}$$

Now, let $\psi(\lambda, r)$ be a spherical function corresponding to the eigenvalue λ. By definition, we have $\Delta_0\psi = \lambda\psi$. Since ψ does not depend on φ, we also have $\Delta\psi = \lambda\psi$. Because Δ commutes with every motion in G, $\psi(\lambda, d(z, z'))$ as a function of either z or z' is an eigenfunction of Δ corresponding to eigenvalue λ. These considerations together with the symmetry of $\psi(\lambda, d(z, z'))$ in z and z' suggest that ψ can be written as

$$\psi(\lambda, d(z, z')) = \sum_{m=0}^{\infty}\sum_{l=1}^{d_m}\alpha_{ml}h_{ml}(\varphi)h_{ml}(\varphi')\psi_m(\lambda, r)\psi_m(\lambda, r') \tag{17}$$

where ψ_m satisfy

$$[\Delta_0 - \{m(m+n-2)\}\{g^2(r)\}^{-1}]\psi_m = \lambda\psi_m \tag{18}$$

and α_{ml} are constants. It turns out that with h_{ml} normalized as in (16) the constants α_{ml} do not depend on l and hence can be absorbed into ψ_m. Thus, (17) takes on the form

$$\psi(\lambda, d(z, z')) = \sum_{m=0}^{\infty} \sum_{l=1}^{d_m} h_{ml}(\varphi) h_{ml}(\varphi') \psi_m(\lambda, r) \psi_m(\lambda, r'). \tag{19}$$

Because ψ is positive-definite, the functions ψ_m are necessarily real valued. It should be noted that for $V_n = S^n$ the sum in (19) is actually a finite sum. For that case, we have $\lambda = -k(k + n - 1)$, and then $\psi_m(\lambda, r)$ is non-zero only for $m = 0, 1, \cdots, k$. It should also be noted that because $\psi(\lambda, 0) = 1$, $\sum_{l=1}^{d_m} h_{ml}^2(\varphi) = d_m$ and $\psi_m(\lambda, 0) = 0$ for $m \geqq 1$, we have

$$\psi_0(\lambda, r) = \psi(\lambda, r). \tag{20}$$

If we denote by Λ the set of eigenvalues corresponding to spherical functions, then (8) can be rewritten in a more convenient form as follows:

$$R(r) = \int_\Lambda \psi(\lambda, r) F(d\lambda) \tag{21}$$

where F is a finite Borel measure on Λ.

3. Homogeneous Gauss-Markov Fields. Equations (19) and (21) show that a homogeneous Gaussian field $\{x(\cdot, \varphi, r), (\varphi, r) \varepsilon V_n\}$ has a representation

$$x(\cdot, \varphi, r) = \sum_{m=0}^{\infty} \sum_{l \leqq d_m} h_{ml}(\varphi) x_{ml}(\cdot, r) \tag{22}$$

where $\{x_{ml}(\cdot, r)\}$ are independent Gaussian one-dimensional processes, and

$$E x_{ml}(\cdot, r) x_{pk}(\cdot, r') = \delta_{mp} \delta_{lk} \int_\Lambda \psi_m(\lambda, r) \psi_m(\lambda, r') F(d\lambda). \tag{23}$$

LEMMA 1. *Let $x(\cdot, r, \varphi)$ be a homogeneous Gauss-Markov random field. Then $\{x_{ml}(\cdot, r)\}$ defined by*

$$x_{ml}(\cdot, r) = \int_{S^{n-1}} h_{ml}(\varphi) x(\cdot, \varphi, r)\, d0 \tag{24}$$

is a set of independent Gauss-Markov processes in one dimension, and there exist functions $f_m(r)$, $g_m(r)$ such that

$$\begin{aligned} E x_{ml}(r) x_{ml}(r') &= \int_\Lambda \psi_m(\lambda, r) \psi_m(\lambda, r') F(d\lambda) \\ &= f_m(\max(r, r')) g_m(\min(r, r')). \end{aligned} \tag{25}$$

PROOF. We need only to prove that $x_{ml}(r)$ are Markov, i.e., that whenever $r > r' > r_0$, $x_{ml}(r)$ and $x_{ml}(r_0)$ are independent given $x_{ml}(r')$. Since for different m and l, $x_{ml}(r)$ are independent processes, we need only to prove that $x_{ml}(r)$ and $x_{ml}(r_0)$ are independent given $x_{pk}(r')$ for all p, k. But given $x_{pk}(r')$ for all p, k is the same as given $x(\varphi', r')$ for all $\varphi' \varepsilon S^{n-1}$. Thus, what needs to be proved is the independence of $x_{ml}(r)$ and $x_{ml}(r_0)$ given $\{x(\varphi', r'), \varphi' \varepsilon S^{n-1}\}$. But from the definition of a Markovian random field, whenever $r > r' > r_0$, $x(\varphi, r)$ and $x(\varphi_0, r_0)$ are independent given $\{x(\varphi', r'), \varphi' \varepsilon S^{n-1}\}$. The proof of the Markovian nature of $x_{ml}(r)$ is completed by noting (24). Finally, the form given by (25) is the required form for the covariance function of a one-dimensional Gauss-Markov process [4].

We are now in a position to state a necessary and sufficient condition on the covariance function for a homogeneous Gaussian random field to be Markovian.

THEOREM 1. *Let $\{x(\cdot, z), z \in V_n\}$ be a homogeneous Gaussian random field with a continuous covariance function. Then, for $x(\cdot, z)$ to be Markovian, it is necessary and sufficient that*

$$Ex(\cdot, z)x(\cdot, z') = C\psi(d(z, z')) \tag{26}$$

where ψ is a spherical function on V_n and $C = Ex^2(\cdot, z)$ is a positive constant.

PROOF. Necessity. From (25) we have

$$\int_\Lambda \psi_0(\lambda, r)\psi_0(\lambda, r')F(d\lambda) = f_0(r)g_0(r'), \qquad r > r'. \tag{27}$$

For a fixed $r > r'$ it is easy to show that $\int_\Lambda \lambda\psi_0(\lambda, r)\psi_0(\lambda, r')F(d\lambda)$ is a convergent integral. Whence

$$g_0(r')\Delta_0 f_0(r) = \int_\Lambda \lambda\psi_0(\lambda, r)\psi_0(\lambda, r')F(d\lambda) = f_0(r)\Delta_0' g_0(r') \tag{28}$$

whenever $r > r'$. This means that

$$\{g_0(r')\}^{-1}\Delta_0' g_0(r') = \{f_0(r)\}^{-1}\Delta_0 f_0(r) = \text{constant} \tag{29}$$

or

$$\Delta_0 f_0(r) = \text{constant}\, f_0(r), \qquad r > 0. \tag{30}$$

From (20), (21) and (27) we have

$$Ex(\cdot, z)x(\cdot, z') = R(d(z, z')) = g_0(0)f_0(d(z, z')). \tag{31}$$

If we set

$$\psi(d(z, z')) = \frac{f_0(d(z, z'))}{f_0(0)} = \frac{Ex(\cdot, z)x(\cdot, z')}{Ex^2(\cdot, z)} \tag{32}$$

then $\psi(0) = 1$ and ψ is a continuous positive-definite function on V_n. Furthermore, because of (30), ψ is also an eigenfunction of Δ_0. Thus, ψ is a spherical function, and necessity is proved.

Sufficiency is rather trivial, because a Gaussian random field, whose covariance function is a spherical function, is degenerate in the following sense: for any smooth closed surface ∂D

$$x(\cdot, z) = E\{x(\cdot, z) \mid x(\cdot, z'), z' \in \partial D\} \tag{33}$$

with probability 1 for all $z \in V_n$. To prove this, we note that if $Ex(\cdot, z)x(\cdot, z') = C\psi(\lambda_0, d(z, z'))$ then $x(\cdot, z)$ has the form

$$x(\omega, \varphi, r) = \sum_{m,l} x_{ml}(\omega)h_{ml}(\varphi)\psi_m(\lambda_0, r) \tag{34}$$

which yields

$$\Delta x(\omega, z) = \lambda_0 x(\omega, z). \tag{35}$$

Given a smooth closed $n - 1$ surface ∂D, (35) can be treated as an interior or an exterior Dirichlet problem with boundary data on ∂D, which results in (33).

4. Generalized Markovian fields. In this section we shall show that it is possible to define the Markovian property for certain generalized random fields, and give a necessary and sufficient condition for a homogeneous Gaussian generalized random field on R^n to be Markovian. This generalizes Theorem 1 for R^n. Non-degenerate examples of homogeneous generalized Gauss-Markov fields do exist, and represent natural generalizations of the Ornstein-Uhlenbeck process in one dimension.

Let $\mathcal{S}(R^n)$ denote the Schwartz space of real-valued C^∞ functions of rapid descent. That is, $\mathcal{S}$ contains all real-valued functions f on R^n for which there exist finite constants C_{mk} such that

$$(36) \quad \mathrm{Sup}_{z \varepsilon R^n} |z|^m \, |D^k f(z)| \leqq C_{mk},$$

$$z = (z_1, z_2, \cdots, z_n), \qquad k = (k_1, \cdots, k_n), \qquad D^k = \frac{\partial^{k_1+\cdots+k_n}}{\partial z_1^{k_1} \cdots \partial z_n^{k_1}}$$

Convergence in $\mathcal{S}$ of a sequence $\{f_\nu\}$ means

(a) $f_\nu \,\varepsilon\, \mathcal{S}$ for each ν

(b) $\mathrm{Sup}_{z\varepsilon R^n} |z|^m |D^k f_\nu(z)| \leqq C_{mk}$ independent of ν

(c) For each k $\{D^k f_\nu\}$ converges uniformly on every compact set in R^n.

Let $\mathcal{H}$ be a Hilbert space of real Gaussian random variables with zero mean. We shall define a real zero-mean Gaussian generalized random field X to be a continuous linear map of $\mathcal{S}$ into $\mathcal{H}$. An isometry $g: R^n \to R^n$ induces a map $T_g: \mathcal{S} \to \mathcal{S}$ by

$$(T_g f)(z) = f(g^{-1} z).$$

The generalized random field X is said to be homogeneous if for all $g \,\varepsilon\, G(R^n)$ and $f_1, f_2 \,\varepsilon\, \mathcal{S}$

$$(37) \qquad EX(T_g f_1) X(T_g f_2) = EX(f_1) X(f_2) = B(f_1, f_2).$$

We shall call B the covariance bilinear functional of X. A bilinear functional B on $\mathcal{S} \times \mathcal{S}$ is the covariance functional of a homogeneous Gaussian generalized random field X, if and only if

$$(38) \qquad B(f_1, f_2) = \int_0^\infty \sum_{m,l} \hat{f}_{ml}^{(1)}(\lambda) \hat{f}_{ml}^{(2)}(\lambda) F(d\lambda)$$

where F is a non-decreasing function of slow growth on $[0, \infty)$, and

$$(39) \quad \hat{f}_{ml}(\lambda) = \int_{S^{n-1}} d0 \int_0^\infty dr f(\varphi, r) h_{ml}(\varphi) r^{n-1} \psi_m(-\lambda, r), \quad m = 0, 1, \cdots, l \leqq d_m$$

The monotone function F will be called the spectral distribution of X [5]. In terms of $\hat{f}_{ml}(\lambda)$ we can write

$$(40) \qquad X(f) = \sum_{m,l} \int_0^\infty \hat{f}_{ml}(\lambda) \hat{\mathfrak{X}}_{ml}(d\lambda)$$

where $\{\hat{\mathfrak{X}}_{ml}\}$ is a family of independent $\mathcal{H}$-valued Borel measures on $[0, \infty)$ with

$$(41) \qquad E\hat{\mathfrak{X}}_{ml}(\Lambda) \hat{\mathfrak{X}}_{pq}(\Lambda') = \delta_{mp} \delta_{lq} F(\Lambda \cap \Lambda').$$

We note that the sequence $\{X(f_\nu)\}$ converges whenever $\{\hat{f}_\nu\}$ converges in $L^2(dF\, d0)$ norm. Here, $\hat{f}$ is given by

$$\hat{f}(\varphi, \lambda) = \textstyle\sum_{m,l} \hat{f}_{ml}(\lambda) h_{ml}(\varphi). \tag{42}$$

Therefore, if we define $\hat{X}$ by $\hat{X}(\hat{f}) = X(f)$, then $\hat{X}$ can be extended to a continuous linear map of $L^2(dF\, d0)$ into $\mathcal{H}$. In particular, if F is bounded, the corresponding ordinary random field can be recovered by setting

$$\begin{aligned} x(\varphi_0, r_0) &= \hat{X}(\xi_{\varphi_0, r_0}) \\ \xi_{\varphi_0, r_0}(\varphi, \lambda) &= \textstyle\sum_{m,l} h_{ml}(\varphi_0) h_{ml}(\varphi) \psi_m(-\lambda, r_0). \end{aligned} \tag{43}$$

Let ∂D be a smooth $n - 1$ closed surface in R^n and let $d\sigma$ be the differential surface area. For $f \in L^2(\partial D, d\sigma)$ define

$$\begin{gathered} \tilde{f}_{ml}(\lambda) = \textstyle\int_{\partial D} f(t) \psi_m(-\lambda, r(t)) h_{ml}(\varphi(t))\, d\sigma \\ t = (t_1, \cdots, t_{n-1}) \text{ coordinates for } \partial D, \end{gathered} \tag{44}$$

and let

$$\tilde{f}(\varphi, \lambda) = \textstyle\sum_{m,l} h_{ml}(\varphi) \tilde{f}_{ml}(\lambda). \tag{45}$$

Suppose X is such that $\tilde{f} \in L^2(dF\, dO)$ whenever $f \in L^2(\partial D, d\sigma)$, then we can define

$$X_{\partial D}(f) = \hat{X}(\tilde{f}) = \textstyle\sum_{m,l} \int_0^\infty \tilde{f}_{ml}(\lambda) \hat{x}_{ml}(d\lambda). \tag{46}$$

Clearly, $\{X_{\partial D}(f), f \in L^2(\partial D, d\sigma)\}$ serves to represent the surface data on ∂D. Once surface data is defined, Markovian property can again be defined.

Let X be a homogeneous Gaussian generalized random field with spectral distribution F. Suppose that whenever ∂D is a smooth closed $n - 1$ surface in R^n and $f \in L^2(\partial D, d\sigma)$ then $\tilde{f} \in L^2(dF\, d0)$. Let $\mathcal{H}(\partial D) \subset \mathcal{H}$ denotes the closed *linear manifold generated by* $\{X_{\partial D}(f), f \in L^2(\partial D, d\sigma)\}$. We say X is Markovian if for any increasing sequence of nested surfaces ∂D_1, ∂D, ∂D_2, $\mathcal{H}(\partial D_2) - P_{\partial D}\mathcal{H}(\partial D_2)$ is orthogonal to $\mathcal{H}(\partial D_1)$, where $P_{\partial D}\mathcal{H}(\partial D_2)$ denotes the image of $\mathcal{H}(\partial D_2)$ under the projection $P_{\partial D}$ on $\mathcal{H}(\partial D)$. In other words, X is Markovian, if given the surface data on ∂D inside and outside are independent since with Gaussian law orthogonality and independence are equivalent. The following result generalizes Theorem 1 for R^n:

THEOREM 2. *Let $\{X(f), f \in \mathcal{S}\}$ be a homogeneous Gaussian generalized random field on R^n with spectral distribution F. A necessary and sufficient condition for X to be Markovian is that*

$$\textstyle\int_0^\infty \psi_0(-\lambda, r) F(d\lambda) = R(r), \qquad r > 0, \tag{47}$$

defines a twice-differentiable function on $(0, \infty)$ *which satisfies*

$$\{r^{n-1}\}^{-1} \frac{d}{dr}\left[r^{n-1} \frac{dR(r)}{dr}\right] = \alpha R(r), \tag{48}$$

where α is a positive constant.

REMARK. We note that $R(r)$ need not be bounded, but when it is, the result reduces to that of Theorem 1.

PROOF. *Necessity.* Let ∂D be an $n - 1$ sphere with radius r. Since the spherical functions $h_{mn} \varepsilon L^2(\partial D, d\sigma)$, we can define

$$x_{ml}(r) = X_{\partial D}(h_{ml}), \qquad m \geqq 0, l \leqq d_m . \tag{49}$$

By an argument completely analogous to that of Lemma 1, we can show that $\{x_{ml}(r), 0 \leqq r < \infty\}$ is a family of independent one-dimensional Gauss-Markov processes. Hence, we must have

$$EX_{01}(r)X_{01}(r_0) = f_0'(\max(r, r_0))g_0'(\min(r, r_0)). \tag{50}$$

From (46) and (44), it follows that

$$X_{01}(r) = X_{\partial D}(h_{01}) = r^{n-1} \int_0^\infty \psi_0(-\lambda, r)x_{01}(d\lambda). \tag{51}$$

Therefore, from (41) and (50)

$$\begin{aligned} EX_{01}(r)X_{01}(r_0) &= (r, r_0)^{n-1} \int_0^\infty \psi_0(-\lambda, r)\psi_0(-\lambda, r_0)F(d\lambda) \\ &= f_0'(\max(r, r_0))g_0'(\min(r, r_0)) \end{aligned} \tag{52}$$

or

$$\int_0^\infty \psi_0(-\lambda, r)\psi_0(-\lambda, r_0)F(d\lambda) = f_0(\max(r, r_0))g_0(\min(r, r_0))$$

which is identical to (27). Hence, (29) holds once more and

$$\begin{aligned} \Delta_0 f_0(r) &= \{r^{n-1}\}^{-1}\frac{d}{dr}\left[r^{n-1}\frac{df_0(r)}{dr}\right] \\ &= \text{constant } f_0(r). \end{aligned} \tag{53}$$

Because $\psi_0(\lambda, 0) = 1, f_0(r) = R(r)$ and (48) follows.

Sufficiency. Assume $R(r) = \int_0^\infty \psi_0(-\lambda, r)F(d\lambda)$ satisfies (48). Then,

$$\Delta_z R(|z - z_0|) = \alpha R(|z - z_0|), \qquad z \neq z_0 . \tag{54}$$

For any smooth closed $n - 1$ surface ∂D separating z and z_0, (54) can be treated as an exterior Dirichlet problem with boundary data on ∂D. Let $G(z, z')$ be the Green's function for this Dirichlet problem, then

$$R(|z - z_0|) = \int_{\partial D} H(z, z')R(|z' - z|), \qquad z \,\varepsilon\, D^+, z_0 \,\varepsilon\, D^- \cup \partial D \tag{55}$$

where $H(z, z') = \partial n' G(z, z')$ is the outward normal derivative of $G(z, z')$ with respect to z' on ∂D. Let $\{\partial D_1, \partial D, \partial D_2\}$ be an increasing family of nested surfaces. Then

$$f_D(z') = \int_{\partial D_2} H(z, z')f(z)\, d\sigma \tag{56}$$

maps $L^2(\partial D_2, d\sigma)$ into $L^2(\partial D, d\sigma)$, so that $X_{\partial D}(f_D)$ is well-defined whenever $f \varepsilon L^2(\partial D_2, d\sigma)$. Now, $X_{\partial D}(f_D)$ is the projection of $X_{\partial D_2}(f)$ on $\mathcal{H}(\partial D)$ because

$X_{\partial D}(f_D) \in \mathcal{H}(\partial D)$ and

$$
\begin{aligned}
&E[X_{\partial D_2}(f) - X_{\partial D}(f_D)]X_{\partial D}(g) \\
&\quad = \int_{\partial D_2} \int_{\partial D} R(|z - z'|)f(z)g(z')\,d\sigma\,d\sigma' \\
&\qquad - \int_{\partial D} \int_{\partial D} R(|z - z'|)f_D(z)g(z')\,d\sigma\,d\sigma' \\
&\quad = \int_{\partial D_2} d\sigma \int_{\partial D} d\sigma' f(z)g(z')[R(|z - z'|) - \int_{\partial D} H(z, z'')R(|z'' - z'|)\,d\sigma''] \\
&\quad = 0.
\end{aligned} \tag{57}
$$

Similarly, we can show

$$E[X_{\partial D_2}(f) - X_{\partial D}(f_D)]X_{\partial D_2}(g) = 0, \qquad g \in L^2(\partial D_1, d\sigma). \tag{58}$$

Therefore, $[X_{\partial D_2}(f) - P_{\partial D}X_{\partial D_2}(f)]$ is orthogonal to $X_{\partial D_2}(g)$ for every $g \in L^2(\partial D_1, d\sigma)$. This proves that X is Markovian. This proof for sufficiency parallels closely the arguments of McKean [6].

Equation (48) can be readily solved. Corresponding to a non-negative F measure in (47), there are only two possible forms for R. These are

$$\text{(a)} \qquad R(r) = A\,\frac{J_{n/2-1}(\nu_0 r)}{(\nu_0 r)^{n/2-1}}$$

$$\text{(b)} \qquad R(r) = A\,\frac{K_{n/2-1}(\nu_0 r)}{(\nu_0 r)^{n/2-1}}.$$

Case (a) corresponds to an $F(\lambda)$ which has a single jump at $\lambda = \nu_0^2$, and was already covered by Theorem 1. Case (b) corresponds to an unbounded F

$$F(d\nu^2) = \frac{A}{\nu_0^n}\,\frac{\nu^{n-1}\,d\nu}{1 + (\nu/\nu_0)^2}. \tag{59}$$

It is interesting to note that $n = 1$, which has been excluded from our discussion so far, corresponds to a bounded spectral distribution. One readily recognizes that in that case

$$R(\tau) = A\nu_0^{-1} \int_0^\infty \cos \nu\tau \{1 + (\nu/\nu_0)^2\}^{-1}\,d\nu = \tfrac{1}{2}\pi A e^{-\nu_0|\tau|}$$

which is the well-known covariance function for the Ornstein-Uhlenbeck process.

Theorem 3 can be readily generalized to include S^n. It is probably also true for H^n, although we have no proof of that.

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