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SOME RESULTS CONCERNING THE ZERO-CROSSINGS OF GAUSSIAN NOISE*

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1. Introduction. Let $x(t)$ be a zero-mean stationary Gaussian process, with covariance function of the form

$$Ex(t)x(t+\tau) = \rho(\tau) = 1 - \frac{\tau^2}{2} + \frac{a}{6}|\tau|^3 + O(\tau^4). \tag{1}$$

Let ξ be a random variable denoting the interval between two successive zeros of $x(t)$. The problem of finding the probability distribution of ξ is of considerable interest and remains largely unsolved. (For further references and a more detailed discussion, see [1] and [2].) In this paper we present some explicit results concerning a zero-mean Gaussian process with covariance function that is a special case of (1), ($a = 4/\sqrt{3}$).

Let $F(t) = \text{Prob}\,(\xi \leqq t)$ and $q(t) = dF(t)/dt$. The principal results of this paper are that for a zero-mean Gaussian process with covariance function given by

$$\rho(\tau) = \tfrac{3}{2}e^{-|\tau|/\sqrt{3}}(1 - \tfrac{1}{3}e^{-(2/\sqrt{3})|\tau|}), \tag{2}$$

$F(t)$ and $q(t)$ can be expressed explicitly in terms of complete elliptic integrals. These results appear as (24) and (25) below.

It has been known for some time that for zero-mean Gaussian processes with covariance functions of the form given by (1), $q(0^+) = Ca$. Longuet-Higgins has given various bounds for C, the best ones being [3]

$$\frac{1.1556}{6} < C < \frac{1.158}{6}.$$

The results of this paper suffice to show that in fact

$$C = \left(\frac{37}{32}\right)\frac{1}{6} = \frac{1.15625}{6}. \tag{3}$$

2. Some preliminary relationships. Let $x(t)$ be a zero-mean Gaussian process with covariance function given by (2). It is assumed that a separable version is being considered. Then $x(t)$ is almost surely differentiable,

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and we shall denote its derivative by $\dot{x}(t)$ $(\dot{x}(t) = dx(t)/dt)$. Now, let $\tau(y_0)$ be defined by

$$\tau(y_0) = \min \{t; t > 0, x(t) = 0 \mid x(0) = 0, \dot{x}(0) = y_0\}, \tag{4}$$

where the condition $x(0) = 0$ is understood to be in the horizontal window sense [4]. Now, let

$$\phi(y_0, t) = \text{Prob}\,\{\tau(y_0) > t\} \tag{5}$$

and

$$p_h(y_0)\, dy_0 = \text{Prob}\,\{\dot{x}(0) \in (y_0, y_0 + dy_0) \mid x(0) = 0\}. \tag{6}$$

In (6) the conditioning is again in the horizontal window sense. Then, $F(t)$ can be expressed as

$$F(t) = 1 - \text{Prob}\,(\xi > t) = 1 - \int_{-\infty}^{\infty} p_h(y_0)\phi(y_0, t)\, dy_0. \tag{7}$$

Now, $p_h(y_0)$ can be derived as in [4]. For the process being considered, we have

$$p_h(y_0) = \frac{|y_0|}{2} e^{-(1/2)y_0^2}. \tag{8}$$

Therefore,

$$F(t) = 1 - \int_0^{\infty} y_0 e^{-(1/2)y_0^2} \phi(y_0, t)\, dy_0, \tag{9}$$

where we have used the symmetry $\phi(y_0, t) = \phi(-y_0, t)$.

3. A representation of $x(t)$. Let $\eta(t)$ be a standard Brownian motion $(E\eta^2(t) = t)$. Define $z(t)$ by

$$z(t) = \int_0^t \eta(s)\, ds, \qquad t \geqq 0. \tag{10}$$

The covariance function of $z(t)$ is given by

$$R_z(s, t) = Ez(s)z(t) = \tfrac{1}{2}s^2 t - \tfrac{1}{6}s^3, \qquad t \geqq s. \tag{11}$$

Therefore, the normalized covariance function is given by

$$\rho_z(s, t) = \frac{R_z(s, t)}{\sqrt{R_z(s, s)R_z(t, t)}} = \frac{3}{2}\sqrt{\frac{s}{t}} - \frac{1}{2}\left(\frac{s}{t}\right)^{3/2}, \qquad t \geqq s. \tag{12}$$

As before, let $x(t)$ be a zero-mean Gaussian process with covariance function given by (2). Comparing (2) and (12), we see that $x(t)$ must have the same probability laws as $\sqrt{3}\, e^{-\sqrt{3}t} z(e^{(2/\sqrt{3})t})$. From (10) this means that $x(t)$ has the representation

$$x(t) = \sqrt{3} e^{-\sqrt{3}t} \int_0^{\exp(2/\sqrt{3})t} \eta(s)\, ds, \tag{13}$$

where $\eta(s)$ is again a standard Brownian motion. Furthermore, we can rewrite (13) as

$$\begin{aligned} x(t) &= \sqrt{3} e^{-\sqrt{3}t} \int_0^1 \eta(s)\, ds + \sqrt{3} e^{-\sqrt{3}t} \int_1^{\exp(2/\sqrt{3})t} \eta(s)\, ds \\ &= \sqrt{3} e^{-\sqrt{3}t} \int_0^1 \eta(s)\, ds + \sqrt{3} e^{-\sqrt{3}t} (e^{(2/\sqrt{3})t} - 1)\eta(1) \\ &\quad + \sqrt{3} e^{-\sqrt{3}t} \int_1^{(2/\sqrt{3})t} [\eta(s) - \eta(1)]\, ds. \end{aligned} \tag{14}$$

Now, we note that

$$x(0) = \sqrt{3} \int_0^1 \eta(s)\, ds \tag{15}$$

and

$$\dot{x}(0) = 2\eta(1) - 3 \int_0^1 \eta(s)\, ds. \tag{16}$$

We further note that $\eta(s)$ being a Brownian motion, $\eta(s) - \eta(1)$ and $\eta(s-1)$ are identical in law. Thus, $x(t)$ can be written as

$$\begin{aligned} x(t) &= e^{-\sqrt{3}t} x(0) + \frac{3}{2} e^{-\sqrt{3}t} (e^{(2/\sqrt{3})t} - 1) x(0) \\ &+ \frac{\sqrt{3}}{2} e^{-\sqrt{3}t} (e^{(2/\sqrt{3})t} - 1)\dot{x}(0) + \sqrt{3} e^{-\sqrt{3}t} \int_0^{\exp(2/\sqrt{3})t-1} \eta(s)\, ds, \end{aligned} \tag{17}$$

where $\eta(s)$ is again a standard Brownian motion. (Note that the $\eta(s)$ in (17) and the $\eta(s)$ in (13) through (16) are not the same except in law.)

4. The distribution of intervals between zeros. For a standard Brownian motion $\eta(s)$, define σ by

$$\sigma = \min\left\{t;\ t > 0,\ t + \int_0^t \eta(s)\, ds = 0\right\}. \tag{18}$$

In a very interesting paper McKean [5] has obtained explicit expressions concerning the distribution of σ. Specifically, he has shown [5, §3, (6)] that

$$f(y,t)\,dy\,dt = \text{Prob}\{\sigma\in(t,t+dt),[\eta(\sigma)+1]\in(-y,-y+dy)\}$$

(19)
$$= dy\,dt\,\frac{3}{\sqrt{2\pi}}\frac{y}{t^2}e^{-(2/t)(1-y+y^2)}\int_0^{4y/t}\frac{e^{-3\theta/2}}{\sqrt{\pi\theta}}\,d\theta, \qquad y>0.$$

Now, $\tau(y_0)$ as defined by (4) can be related to σ as given by (18), through (17). In what follows, we make free use of the fact that $\eta(t)$ and $c\eta(t/c^2)$ have the same law when $\eta(t)$ is a standard Brownian motion and $c > 0$. While $\eta(t)$ always denotes a standard Brownian motion in the following derivation, $\eta(t)$ from one line to the next need not be the same except in law. Let $g(t) = e^{(2/\sqrt{3})t} - 1$, and $g^{-1}(t) = \sqrt{3}/2\log(1+t)$. Then, from (4) and (17) we have

$$\tau(y_0) = \min\left\{t;\, t>0,\, \frac{y_0}{2}g(t) + \int_0^{g(t)}\eta(s)\,ds = 0\right\}$$

$$= \min\left\{g^{-1}(t);\, t>0,\, \frac{y_0}{2}t + \int_0^{t}\eta(s)\,ds = 0\right\}$$

$$= \min\left\{g^{-1}(t);\, t>0,\, \frac{y_0}{2}t + \int_0^{t}c\eta\left(\frac{s}{c^2}\right)ds = 0\right\}$$

$$= \min\left\{g^{-1}(t);\, t>0,\, \frac{y_0}{2}t + c^3\int_0^{t/c^2}\eta(s)\,ds = 0\right\}$$

(20)
$$= \min\left\{g^{-1}(t);\, t>0,\, \frac{y_0}{2c}\left(\frac{t}{c^2}\right) + \int_0^{t/c^2}\eta(s)\,ds = 0\right\}$$

$$= \min\left\{g^{-1}(t);\, t>0,\, \frac{4t}{y_0^2} + \int_0^{4t/y_0^2}\eta(s)\,ds = 0\right\}$$

$$= \min\left\{g^{-1}\left(\frac{y_0^2t}{4}\right);\, t>0,\, t + \int_0^{t}\eta(s)\,ds = 0\right\}$$

$$= g^{-1}\left(\frac{y_0^2}{4}\sigma\right) = \frac{\sqrt{3}}{2}\log\left(1+\frac{y_0^2\sigma}{4}\right).$$

Therefore, from (5), (19) and (20) we have

(21)
$$\varphi(y_0,t) = \text{Prob}\left\{g^{-1}\left(\frac{y_0^2}{4}\sigma\right) > t\right\} = \text{Prob}\left\{\sigma > \frac{4}{y_0^2}g(t)\right\}$$
$$= \int_{(4/y_0^2)g(t)}^{\infty}ds\int_0^{\infty}dy f(y,s),$$

where $f(y,s)$ is given by (19). It follows from (9) that

$$F(t) = 1 - \int_0^\infty y_0 e^{-(1/2)y_0^2} \varphi(y_0, t)\, dy_0$$

(22)

$$= 1 - \int_{g(t)}^\infty ds \int_0^\infty dy_0 \int_0^\infty dy\, \frac{4}{y_0}\, e^{-(1/2)y_0^2} f\left(y, \frac{4s}{y_0^2}\right)$$

and

$$q(t) = \dot{g}(t) \int_0^\infty dy_0 \int_0^\infty dy\, \frac{4}{y_0}\, e^{-(1/2)y_0^2} f\left(y, \frac{4g(t)}{y_0^2}\right). \tag{23}$$

With the substitution of (19), the integrals in (22) and (23) can be evaluated. The results are (see the Appendix)

$$F(t) = 1 - \frac{3}{2\pi}\left\{\frac{[1 - 2r^2(t)]^{3/2}}{3 - 2r^2(t)}\, \pi_1\left(-\frac{3}{4} + \frac{1}{2}r^2(t), r(t)\right) + \frac{2\sqrt{1 - 2r^2(t)}}{3 - 2r^2(t)}\, K(r(t))\right\} \tag{24}$$

and

$$q(t) = \frac{\sqrt{3}}{4\pi}\left\{\frac{[1 - 2r^2(t)]^{1/2}}{[1 - r^2(t)][1 + 2r^2(t)]}\, E(r(t)) + \frac{[1 - 2r^2(t)]^{1/2}}{[3 - 2r^2(t)]}\left[\frac{K(r(t)) - E(r(t))}{r^2(t)}\right] + \frac{8[1 - 2r^2(t)]^{3/2}}{[3 - 2r^2(t)]^2[1 + 2r^2(t)]}\left[\pi_1\left(-\frac{3}{4} + \frac{1}{2}r^2(t), r(t)\right) - K(r(t))\right]\right\}, \tag{25}$$

where

$$r(t) = [\tfrac{1}{2}(1 - e^{-(1/\sqrt{3})t})]^{1/2}, \tag{26}$$

and

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \tag{27}$$

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi, \tag{28}$$

$$\pi_1(\nu, k) = \int_0^{\pi/2} \frac{1}{[1 + \nu \sin^2 \varphi]\sqrt{1 - k^2 \sin^2 \varphi}}\, d\varphi \tag{29}$$

are complete elliptic integrals.

It is easy to see from (25) that

$$q(0^+) = \left(\frac{37}{32}\right)\frac{1}{6}\left(\frac{4}{\sqrt{3}}\right), \tag{30}$$

which verifies (3), since (2) corresponds to $a = 4/\sqrt{3}$. Further, $q(t) \to 0$ exponentially as $t \to \infty$. In fact,

$$\lim_{t\to\infty} e^{(1/2\sqrt{3})t} q(t) = \frac{\sqrt{3}}{4\pi} K\left(\frac{1}{\sqrt{2}}\right). \tag{31}$$

Appendix. Let $\psi(t)$ be defined by

$$\psi(t) = \int_0^\infty \int_0^\infty \frac{4}{y_0} e^{-(1/2)y_0^2} f\left(y, \frac{4t}{y_0^2}\right) dy_0 \, dy, \tag{A-1}$$

where $f(y, 4t/y_0^2)$ can be found from (19) to be

$$f\left(y, \frac{4t}{y_0^2}\right) = \frac{3}{\sqrt{2\pi}} \frac{y_0^4 y}{16t^2} e^{-(y_0^2/2t)(1-y+y^2)} \int_0^{y_0^2 y/t} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}} d\theta. \tag{A-2}$$

Substituting (A-2) in (A-1) and letting $r = y_0/2\sqrt{t}$, we have

$$\begin{aligned}\psi(t) &= \frac{12}{\pi\sqrt{2}} \int_0^\infty \int_0^\infty yr^3 e^{-2r^2(1-y+y^2)} e^{-2tr^2} \left[\int_0^{4yr^2} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}} d\theta\right] dy \, dr \\ &= -\frac{6}{\pi\sqrt{2}} \frac{d}{dt}\left\{\int_0^\infty \int_0^\infty yre^{-2r^2(t+1-y+y^2)} \left[\int_0^{4yr^2} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}} d\theta\right] dy \, dr\right\} \\ &= -\frac{3}{2\pi} \frac{d}{dt}\left\{\int_0^\infty \frac{y^{3/2} \, dy}{[t + 1 - y + y^2] \sqrt{t + (1 + y)^2}}\right\}.\end{aligned} \tag{A-3}$$

Now, let $H(t)$ be defined by

$$H(t) = \frac{3}{2\pi} \int_0^\infty \frac{y^{3/2} \, dy}{[t + 1 - y + y^2] \sqrt{t + (1 + y)^2}}. \tag{A-4}$$

Then, we have

$$\begin{aligned}\psi(t) &= -\frac{d}{dt} H(t) \\ &= \frac{3}{2\pi} \int_0^\infty \frac{y^{3/2} \, dy}{(t + 1 - y + y^2) \sqrt{t + (1 + y)^2}} \\ &\qquad \cdot \left\{\frac{1}{(t + 1 - y + y^2)} + \frac{1}{2} \frac{1}{[t + (1 + y)^2]}\right\}.\end{aligned} \tag{A-5}$$

From (22) and (23) it is easily seen that

$$F(t) = -\int_{g(t)}^\infty \psi(s) \, ds = 1 - H(g(t)) \tag{A-6}$$

and

$$q(t) = \dot{g}(t)\psi(g(t)) \qquad (g(t) = e^{(2/\sqrt{3})t} - 1). \tag{A-7}$$

Proceeding to evaluate $H(t)$, we make a change in the variable of integration in (A-4):

$$y = \sqrt{1+t}\left(\frac{1-\cos\varphi}{1+\cos\phi}\right). \tag{A-8}$$

The result is

$$H(t) = \frac{3}{8\pi}\frac{1}{(1+t)^{1/4}}\int_0^\pi \frac{(1-\cos\varphi)^2\,d\varphi}{[1-\nu(t)\sin^2\varphi]\sqrt{1-k^2(t)\sin^2\varphi}}, \tag{A-9}$$

with

$$\nu(t) = \frac{1}{2} + \frac{1}{4\sqrt{1+t}}, \tag{A-10}$$

$$k^2(t) = \frac{1}{2} - \frac{1}{2\sqrt{1+t}}. \tag{A-11}$$

Therefore,

$$\begin{aligned} H(t) &= \frac{3}{8\pi}\frac{1}{(1+t)^{1/4}}\int_0^\pi \frac{(1+\cos^2\phi-2\cos\phi)\,d\phi}{[1-\nu(t)\sin^2\phi]\sqrt{1-k^2(t)\sin^2\phi}} \\ &= \frac{3}{4\pi}\frac{1}{(1+t)^{1/4}}\int_0^{\pi/2} \frac{(1+\cos^2\phi)\,d\phi}{[1-\nu(t)\sin^2\phi]\sqrt{1-k^2(t)\sin^2\phi}} \\ &= \frac{3}{4\pi}\frac{1}{(1+t)^{1/4}}\left\{\left[2-\frac{1}{\nu(t)}\right]\right. \\ &\quad \cdot\int_0^{\pi/2}\frac{d\phi}{[1-\nu(t)\sin^2\phi]\sqrt{1-k^2(t)\sin^2\phi}} \\ &\quad \left. + \frac{1}{\nu(t)}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2(t)\sin^2\phi}}\right\} \\ &= \frac{3}{4\pi(1+t)^{1/4}}\left\{\left[2-\frac{1}{\nu(t)}\right]\pi_1(-\nu(t),k(t)) + \frac{1}{\nu(t)}K(k(t))\right\}. \end{aligned} \tag{A-12}$$

Using (A-12) in (A-6) yields (24).

The function $\psi(t)$ can be found by differentiating $H(t)$. However, it is somewhat simpler to proceed directly from (A-5). Making the change in variable of integration (A-8) in (A-5), we find

$$\begin{aligned} \psi(t) &= \frac{3}{32\pi}\frac{1}{(1+t)^{5/4}}\int_0^\pi \frac{\sin^4\phi}{[1-\nu(t)\sin^2\phi]\sqrt{1-k^2(t)\sin^2\phi}} \\ &\quad \cdot\left\{\frac{1}{1-\nu(t)\sin^2\phi} + \frac{1}{2}\frac{1}{[1-k^2(t)\sin^2\phi]}\right\}d\phi. \end{aligned} \tag{A-13}$$

Changing variables a second time ($z = \sin^2 \phi$), we obtain

$$\psi(t) = \frac{3}{32\pi} \frac{1}{(1+t)^{5/4}} \int_0^1 \frac{z^2}{(1 - \nu z)\sqrt{z(1-z)(1-k^2 z)}} \cdot \left\{ \frac{1}{1 - \nu z} + \frac{1}{2} \frac{1}{(1 - k^2 z)} \right\} dz, \tag{A-14}$$

where $\nu = \nu(t)$, $k^2 = k^2(t)$ are given by (A-10) and (A-11). Equation (A-15) can be rewritten by partial fraction expansion as

$$\begin{aligned} \psi(t) = \frac{3}{32\pi} \frac{1}{(1+t)^{5/4}} \int_0^1 & \frac{1}{\sqrt{z(1-z)(1-k^2 z)}} \cdot \left\{ \frac{1}{\nu^2} \left[1 - \frac{2}{(1 - \nu z)} + \frac{1}{(1 - \nu z)^2} \right] \right. \\ & \left. + \frac{1}{2k^2 \nu} \left[1 + \frac{k^2}{\nu - k^2} \frac{1}{(1 - \nu z)} - \left(\frac{\nu}{\nu - k^2} \right) \frac{1}{1 - k^2 z} \right] dz \right\}. \end{aligned} \tag{A-16}$$

To proceed further, we note that

$$\begin{aligned} & \frac{1}{(1 - \nu z)^2 \sqrt{z(1-z)(1-k^2 z)}} \\ & \quad = \frac{\nu^2}{(\nu - 1)(\nu - k^2)} \frac{d}{dz} \left[\frac{\sqrt{z(1-z)(1-k^2 z)}}{(1 - \nu z)} \right] \\ & \quad + \left[1 - \frac{1}{2} \frac{(\nu^2 - k^2)}{(\nu - k^2)(\nu - 1)} \right] \frac{1}{(1 - \nu z)\sqrt{z(1-z)(1-k^2 z)}} \\ & \quad + \frac{1}{2(\nu - 1)\sqrt{z(1-z)(1-k^2 z)}} - \frac{\nu}{2(\nu - 1)(\nu - k^2)} \sqrt{\frac{1 - k^2 z}{z(1-z)}} \end{aligned} \tag{A-17}$$

and

$$\begin{aligned} & \frac{1}{(1 - k^2 z)\sqrt{z(1-z)(1-k^2 z)}} \\ & \quad = \frac{2k^2}{k^2 - 1} \frac{d}{dz} \sqrt{\frac{z(1-z)}{1 - k^2 z}} - \frac{1}{(k^2 - 1)} \sqrt{\frac{1 - k^2 z}{z(1-z)}}. \end{aligned} \tag{A-18}$$

Using (A-17) and (A-18) in (A-16) and simplifying the results (including the transformation $z = \sin^2 \phi$), we obtain

$$
\text{(A-19)}\quad
\begin{aligned}
\psi(t) = \frac{3}{16\pi(1+t)^{5/4}}\Bigg\{&\frac{2\nu(t)-1}{2\nu^2(t)[1-\nu(t)]} \\
&\cdot\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2(t)\sin^2\phi}}\left[\frac{1}{1-\nu(t)\sin^2\phi}-1\right] \\
&+\frac{1}{2\nu(t)k^2(t)}\int_0^{\pi/2}\left[\frac{1}{\sqrt{1-k^2(t)\sin^2\phi}}-\sqrt{1-k^2(t)\sin^2\phi}\right]d\phi \\
&+\frac{1}{2(1-\nu(t))(1-k^2(t))}\int_0^{\pi/2}\sqrt{1-k^2(t)\sin^2\phi}\,d\phi\Bigg\}.
\end{aligned}
$$

Combining (A-7) and (A-19) yields (25).

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