Probab. Th. Rel. Fields 82, 137–154 (1989)

Probability Theory and Related Fields

Isotropic Gauss-Markov Currents*

Eugene Wong and Moshe Zakai**

University of California, College of Engineering, Dept. of Electrical Engineering and Computer Sciences, Berkeley CA 94720, USA and Technion, Haifa, Israel

Summary. A natural definition of the Markov property for multi-parameter random processes (random fields) is the following. Let $\{X_t, t \in \mathbb{R}^N\}$ be a multiparameter process. For any set D in $\mathbb{R}^N$, let σ_D denote the σ-field generated by $\{X_t, t \in D\}$. The field $\{X_t, t \in \mathbb{R}^N\}$ is said to be *Markov* (or Markov of degree 1 [6], or sharp Markov) if, for any bounded open set D with smooth boundary, σ_D and σ_{D^c} are conditionally independent given $\sigma_{\partial D}$. It has been known for some time that to find interesting examples of Markov processes under this definition; it is necessary to consider generalized random functions. In this paper we show that a natural framework for the Markov property of multiparameter processes is a class of generalized random differential forms (i.e., random currents). Our principal objective is to relate the Markovian nature of an isotropic gaussian current to its spectral properties.

1. Introduction

The notion of a random r-current in $\mathbb{R}^N$, $0 \leqq r \leqq N$, was introduced by K. Ito in 1955 [5] following the deRham notion of non-random currents (cf. [9]); it was motivated by the theory of statistical turbulence. The notion of a differential r-form is a generalization of scalar ($r=0$) and vector ($r=1$) fields in $\mathbb{R}^N$ (cf. e.g. [4, 12]), and the notion of currents generalizes differential forms by considering r-forms with coefficients that are generalized functions. In particular, a random 1-current is just a random vector field with components that are generalized random fields. A random current is said to be homogeneous (isotropic) if its second-order properties are invariant under translations (rotations and reflections). In [5], Ito presented a complete characterization of the spectral measures associated with homogeneous isotropic currents in terms of two positive measures on $(0, \infty)$ and one constant, regardless of the space dimension

* Work supported by the Army Research Office, Grant No. DAAG29-85-K-0233

** Work done while at the University of California at Berkeley

N and order of the form r (cf. [16] for an exposition of Ito's results and some further results).

A natural definition of the Markov property for multi-parameter random processes (random fields) is the following. Let $\{X_t, t \in \mathbb{R}^N\}$ be a multi-parameter process. For any set D in $\mathbb{R}^N$, let σ_D denote the σ-field generated by $\{X_t, t \in D\}$. The field $\{X_t, t \in \mathbb{R}^N\}$ is said to be *Markov* (or Markov of degree 1 [7], or sharp Markov) if, for any bounded open set D with smooth boundary, σ_D and σ_{D^c} are conditionally independent, given $\sigma_{\partial D}$. As was already noted by P. Lévy, this definition leads to a restrictive class of processes and excludes some interesting processes that have a weaker Markov property. One example of such processes is the Lévy Brownian motion in $\mathbb{R}^N$ with N odd, which, as conjectured by Lévy and proved by McKean [7], has a Markov property involving $(N-1)/2$ normal derivatives of the random field, that are generalized random processes. As was shown by Wong [13], it is possible and useful to define the Markov property for generalized (distribution valued) random fields. A general definition of the Markov property, namely the germ-field Markov property, was proposed in [7]. The idea is to define the germ σ-field associated with the boundary ∂D by $\Sigma_{\partial D} = \bigcup_{\varepsilon > 0} \sigma(\partial D_\varepsilon)$ where ∂D_ε is the ε-neighborhood of ∂D and to define X_t to be *germ-field Markov* if σ_D and σ_{D^c} are conditionally independent given $\Sigma_{\partial D}$. This definition has the advantage of being easily extendible to the case where X is a generalized field, while including many processes that are not considered Markov in the classical sense. For example, for $N=1$, any collection of polynomials is germ-field Markov and so is the k^{th} integral of the Wiener process. However, neither process is Markov in the classical sense. The theory of one parameter Markov processes deals almost exclusively with processes that are Markov in the classical sense and has very few results for processes that are germ-field Markov. On the other hand, the theory of multi-parameter fields deals mainly with processes that are germ-field Markov (usually denoted just Markov; cf. [6, 11]).

The purpose of this paper is to consider the Markov property in the framework of random r-currents. This approach reflects, in a natural way, the geometric aspects of the problem and is supported by a powerful coordinate-free calculus for these objects [9]. In the particular case of zero currents, namely generalized scalar valued processes, the results of this paper also follow from known results for Markov fields (cf. e.g. [3] or Theorem 2 on p. 145 of [11]). The boundary σ-field associated with certain random currents is defined in this paper, in a direct "geometric" way, avoiding the germ-field notion. The Markov property relative to this boundary field is defined and conditions under which a Gaussian isotropic random current has this Markov property are derived. In order to be able to define such a Markov property (in contrast with the germ-field Markov property), we consider a restricted class of random currents. Roughly speaking, the random currents that we consider have the property that when integrated on bounded subsets of $(N-1)$ dimensional manifolds, they yield random variables. This gives a natural definition for the splitting sigma field associated with the boundary data. Currents for which integration on bounded subsets of $(N-1)$ manifolds yields random variables will be called *localizable* *currents*. This approach leads to a natural definition of the Markov property for such currents that seems to reflect well the geometrical nature of the problem. Scalar valued Markov (not necessarily Gaussian) processes on the plane were considered in [14], and it turns out that a natural parametrization for Markov processes on the plane is parametrization by paths rather then by points. This corresponds to localizable one forms in the plane and motivated the approach of this paper. Martingale properties of certain classes of random currents were considered in [15].

In the one-parameter case, a Gaussian stationary process $\{X_t, -\infty < t < \infty\}$ with spectral density $1/P(v^2)$, where P is a polynomial of order p, is, in general, not Markov. However, X_t, together with its $(p-1)$ derivatives, form a Markov p-dimensional vector. The notions of localizability and Markov property introduced in this paper enable us to obtain results for general Gaussian random currents that reduce to this result when specialized to the one-parameter case. In the one-parameter case, a Gaussian process with spectral density $Q(v^2)/P(v^2)$ (where the Q and P are polynomials and the order of Q is lower than that of P) is, in general, not germ-field Markov, but can be embedded in a finite-dimensional Markov process as one of its components. It seems that this result does not have a general extension to the multi-parameter case. A partial generalization to the case is discussed in the last section.

In the next section we summarize the results of [5] (cf. also [16]) and present a useful spectral representation for homogeneous and isotropic random currents [16]. The notions of random cochains, localizable currents and Markov currents are introduced in Sect. 3. Conditions for localizability on the spectral density are also given in Sect. 3. Conditions under which an isotropic random current is Markov or can be embedded in a Markov current are derived in Sects. 4 and 5. Section 4 deals with random currents that are either solenoidal or irrotational, and Sect. 5 deals with general currents.

2. Random Currents

In this section we summarize the results of random currents that will be needed in later sections (cf. [5, 16] for details). Let $e_1, e_2, \ldots, e_N$ denote an orthonormal basis in $\mathbb{R}^N$. We use $\mathbf{i}$ to denote a multi-index $\mathbf{i}=(i_1, \ldots, i_r)$, with $|\mathbf{i}|=r$, $|\mathbf{i}|$ to denote a rearrangement of $\mathbf{i}$ that puts the indices in increasing order and $\mathbf{i}^*$ to denote the $n-r$ multi-index complementary to $\mathbf{i}$ in increasing order. A differential r-form has the representation

$$\phi_r(t) = \sum_{|\mathbf{i}|} \phi_{\mathbf{i}}(t)\, e_{\mathbf{i}}, \qquad t \in \mathbb{R}^N \tag{2.1}$$

where $e_{\mathbf{i}} = e_{i_1} \wedge e_{i_2} \ldots \wedge e_{i_r}$ and $\phi_{\mathbf{i}}(t)$ are sufficiently smooth functions (to be clarified later). The Hodge star operator* operating on ϕ_r is the $N-r$ form

$$*\phi_r(t) = \sum_{|\mathbf{i}|} \phi_{\mathbf{i}}(t)\, e_{\mathbf{i}^*} \cdot \delta \begin{pmatrix} 1, 2, \ldots, \mathrm{N} \\ \mathbf{i}, \mathbf{i}^* \end{pmatrix}$$

where δ is 1 if $(\mathbf{i}, \mathbf{i}^*)$ is an even permutation of $1, 2, \ldots, N$ and -1 if the permutation is odd. The exterior derivative is $d\phi_r = \sum_{|\mathbf{i}|} \left(\sum_j \frac{\partial \phi_{\mathbf{i}}}{\partial t_j} e_{\mathbf{j}} \wedge e_{\mathbf{i}} \right)$. The interior product $e_{\mathbf{i}} \vee e_{\mathbf{j}}$ is defined by

$$e_{\mathbf{i}} \vee e_{\mathbf{j}} = {}^*(e_{\mathbf{i}} \wedge {}^*e_{\mathbf{j}}) \cdot (-1)^{(q-p)(N-q)} \tag{2.2}$$

$|\mathbf{i}| = p$, $|\mathbf{j}| = q$. Note that if $|\mathbf{i}| = |\mathbf{j}|$ then $e_{\mathbf{i}} \vee e_{\mathbf{j}}$ is scalar valued; in this case we shall also write $(e_{\mathbf{i}}, e_{\mathbf{j}})$ for $(e_{\mathbf{i}} \vee e_{\mathbf{j}})$ and $(e_{\mathbf{i}}, e_{\mathbf{j}})$ can take the values $0, +1, -1$ only.

Whenever convenient, we use a coordinate system to represent the forms, currents, and operations under consideration even though in nearly every case, these quantities are intrinsic and the results obtained are independent of the coordinate system used to derive them [4, 12].

Let S denote the Schwartz space of fast decreasing functions on $\mathbb{R}^N$ and S^r the space of differential r-forms as given by (2.1) with $\phi_{\mathbf{i}} \in S$. A (non-random) *r-current* [9] is a continuous linear functional on S^{N-r} or, roughly speaking, an r-current is an r-form (2.1) in which the functions $\phi_{\mathbf{i}}(t)$ are Schwartz distributions.

A *random current* [5] is a continuous linear functional from the (non-random) space S^{N-r} to the space of L^2 (possibly complex) random variables and a Gaussian random current is a random current for which these linear functionals are (possibly complex) Gaussian random variables. Let M be a random zero current, i.e., a random Schwartz distribution. M is said to be a *random measure* if for every pair $\phi, \psi \in S$.

$$EM(\phi)\, M^c(\psi) = \int_{\mathbb{R}^N} \phi(t)\, \psi^c(t)\, m(dt) \tag{2.3}$$

for some slowly increasing measure $m(dt)$ on $\mathbb{R}^N$ where $(\ )^c$ denotes the complex conjugate when applied to numbers and the complement when applied to sets. A random measure M can be extended to be a functional on indicator functions of Borel sets E in $\mathbb{R}^N$ with $EM(E') \cdot M(E'') = m(E' \cap E'')$. Therefore, in the Gaussian case the random set function M is of independent increments and $M(\phi) = \int_{\mathbb{R}^N} \phi(t)\, M(dt)$. A random r-current M_r is called a random measure of degree r if there exist slowly increasing measures on $\mathbb{R}^N$, $m_{\mathbf{i},\mathbf{j}}(dt)$, $|\mathbf{i}| = |\mathbf{j}| = r$, such that for $\phi, \psi \in S$.

$$E\{M_r(\phi \wedge ({}^*e_{\mathbf{i}})) \cdot M_r^c(\psi \wedge ({}^*e_{\mathbf{j}})) = \int_{\mathbb{R}^N} \phi(t)\, \psi^c(t)\, m_{\mathbf{i},\mathbf{j}}(dt)\}. \tag{2.4}$$

Let h denote a point in $\mathbb{R}^N$ and consider the shift $(\tau_h \phi)(t) = \phi(t+h)$. A random r current U_r is said to be homogeneous if for all h,

$$E(U_r(\phi_{N-r})\, U_r^c(\psi_{N-r})) = E(U_r(\tau_h \phi_{N-r}) \cdot U_r^c(\tau_h \psi_{N-r})). \tag{2.5}$$

Let $\rho_{\mathbf{i},\mathbf{j}}(\phi, \psi \in S)$ denote

$$\rho_{\mathbf{i},\mathbf{j}}(\phi\, \psi) = E(U_r(\phi \cdot e_{\mathbf{i}^*})\, U_r^c(\psi \cdot e_{\mathbf{j}^*})) \tag{2.6}$$

ρ is called the covariance bilinear form (or the covariance double current) associated with U. The main result regarding homogeneous random currents is as follows. Let $\hat{\phi}$ denote the Fourier transform of ϕ, $\hat{\phi}(v) = \int_{\mathbb{R}^N} \exp -i(v,t)\,\phi(t)\,dt$. The covariance bilinear form of any homogeneous r-current can be written as

$$\rho_{\mathbf{ij}}(\phi, \psi) = \int_{\mathbb{R}^N} \hat{\phi}(v) \cdot \hat{\psi}^c(v)\, m_{\mathbf{ij}}(dv). \tag{2.7}$$

Furthermore, there exists a random r-measure M_r such that

$$U_r(\phi \wedge e_{\mathbf{i}*}) = M_r(\hat{\phi} \wedge e_{\mathbf{i}*}), \tag{2.8}$$

M_r and $m_{\mathbf{i},\mathbf{j}}$ are called the random measure and spectral measure associated with U_r respectively.

Every smooth non-random vector field ($r=1$) in a domain in $\mathbb{R}^N$, $N=3$, can be represented as the sum of a constant (position independent) vector field, a gradient of a scalar potential (zero form) and the curl of a vector potential. The generalization of this representation for homogeneous random currents with general r and N will now be considered. Let the exterior derivative d and the Hodge star operator* on U be defined as follows:

$$\begin{aligned} (dU_r)(\phi) &= (-1)^{r+1}\, U_r(d\phi), \qquad \phi \in S^{N-r-1} \\ (*U_r)(\psi) &= (-1)^{r(N-r)} (U_r(*\psi))^c\, \psi \in S^r \\ \delta U_r &= (-1)^{Nr+r+1} (*d*U_r). \end{aligned} \tag{2.9}$$

The spectral measure M_r associated with the homogeneous current U via (2.8) can be decomposed into the sum of 3 spectral measures $M_r = M_r^{(0)} + M_r^{(1)} + M_r^{(s)}$ where

$$M_r^{(0)}(A) = M_r(A \cap \{0\}). \tag{2.10}$$

Let $M_r^{(u)}$ denote the difference $M_r^{(u)} = M_r - M_r^{(0)}$, and for $v \neq 0$ set

$$M_r^{(i)}(dv) = \frac{v \wedge (v \vee M_r^{(u)}(dv))}{|v|^2} = e_v \wedge (e_v \vee M_r^{(u)}(dv)), \tag{2.11}$$

$$M_r^{(s)}(dv) = \frac{v \vee (v \wedge M_r^{(u)}(dv))}{|v|^2} = e_v \vee (e_v \wedge M_r^{(u)}(dv)) \tag{2.12}$$

where $e_v = \frac{v}{|v|}$ is the unit vector in the direction of the *vector* v (which is the vector connecting the point 0 to the *point* v in $\mathbb{R}^N$). Then [5, 16]

$$M_r(dv) = M_r^{(0)}(dv) + M_r^{(i)}(dv) + M_r^{(s)}(dv).$$

Let $U^{(0)}$, $U^{(i)}$, $U^{(s)}$ denote the homogeneous random currents corresponding to the spectral measures $M^{(0)}$, $M^{(i)}$, $M^{(s)}$ respectively. Then $U = U^{(0)} + U^{(i)} + U^{(s)}$,

U^0 is called the invariant part of $U(U^{(0)}(\phi(t)\cdot e_{\mathbf{i}})=U^{(0)}(\phi(t+\tau)\cdot e_{\mathbf{i}})$ for all fixed τ in $\mathbb{R}^N$), $U^{(i)}$ is called the *irrotational* part of U, and $U^{(s)}$ is called the *solenoidal* part of U.

Let $e_1, \ldots, e_N$ be an orthonormal basis at the point v in $\mathbb{R}^N$ such that $e_1 = e_v$, where e_v denotes the unit vector in the direction from the origin to v. Then, every r-form $\phi = \Sigma\, \phi_{\mathbf{i}} \cdot e_{\mathbf{i}}$ can be decomposed into two components, one component s is the sum over all multi-indices $\mathbf{i}=[\mathbf{i}]$ such that $i_1 = 1$ and the other component is the sum over all $\mathbf{i}=[\mathbf{i}]$ such that $i_1 \neq 1$. Then $M_r^{(i)}(dv)$ as defined by (2.11) is the sum of all components of $M_r^{(u)}$ in which $i_1 = 1$ and $M_r^{(s)}(dv)$ is the sum over all multi-indices $\mathbf{i}$ which do not include $i_1 = 1$. This decomposition of a current into a) the sum of components, including (locally at point v) e_v in $e_{\mathbf{i}}$, and b) the sum of components which locally at v do not include e_v in $e_{\mathbf{i}}$ plays a key role in the representation of isotropic currents which we consider next.

Let G denote the full group of orthogonal transformations (rotations and reflections) in $\mathbb{R}^N$. For $g \in G$, let $\sigma_g U_r$ denote the transformation on U_r induced by g, i.e.

$$\sigma_g[U_r(\phi_{N-r})] = U_r(\sigma_g[\phi_{N-r}])$$

and $\sigma_g[\phi_{N-r}]$ is the transformation on the differential form ϕ_{N-r} induced by g $(\sigma_g(\phi_{\mathbf{i}}(t)\cdot e_{i_1} \wedge \ldots \wedge e_{i_r}) = \phi_{\mathbf{i}}(g\cdot t)\cdot g^{-1} e_{i_1} \wedge \ldots \wedge g^{-1} e_{i_r})$. A random current U_r is said to be isotropic if, for all $g \in G$, ϕ, $\psi \in S^{N-r}$

$$E(U_r(\phi)\, U_r^c(\psi)) = E\, \sigma_g\, U_r(\phi)\, (\sigma_g\, U_r \cdot (\psi))^c. \tag{2.13}$$

Consider the spectral measure $m_{\mathbf{i},\mathbf{j}}(dv)$, $|\mathbf{i}| = |\mathbf{j}| = r$ associated with a homogeneous and isotropic current, and assume for the remainder of the paper, that $M^{(0)} \equiv 0$. Assume that at point v in $\mathbb{R}^N$ the coordinate system is such that $e_v = e_1$, then it follows from isotropy that $m_{\mathbf{i},\mathbf{j}}(dv) = 0$ for $[\mathbf{i}] \neq [\mathbf{j}]$. It also follows from isotropy that there are at most two different values for $m_{i,i}$: one for $1 \in \mathbf{i}$ and another for $1 \notin \mathbf{i}$. This yields the following:

The Ito representation ([5], cf. [16].) Let $e_1, e_2, \ldots, e_N$ be any orthonormal coordinate system. Then the spectral measure of any homogeneous and isotropic current is expressible as:

$$m_{\mathbf{i},\mathbf{j}}(dv) = (e_v \vee e_{\mathbf{i}}, e_v \vee e_{\mathbf{j}})\, m^{(i)}(dv) + (e_v \wedge e_{\mathbf{i}}, e_v \wedge e_{\mathbf{j}})\, m^{(s)}(dv). \tag{2.14}$$

(Recall that for $|\mathbf{i}| = |\mathbf{j}|$, $(e_{\mathbf{i}}, e_{\mathbf{j}}) = (e_{\mathbf{i}} \vee e_{\mathbf{j}})$), where $m^{(i)}(dv)$ and $m^{(s)}(dv)$ are two spherically invariant measures on $\mathbb{R}^N$. That is, if g is any rotation or reflection in $\mathbb{R}^N$, then $m^{(i)}(gA) = m^{(i)}(A)$ and $m^{(s)}(gA) = m^{(s)}(A)$ for every Borel set in $\mathbb{R}^N$. For every spherical invariant measure $m(dv)$ we can write

$$m(dv) = \lambda^{N-1} \cdot \eta(d\theta)\, F(d\lambda) \tag{2.15}$$

where $\lambda=|v|$, $\theta=\frac{v}{\lambda}$ is a point on the unit sphere in $\mathbb{R}^N$ and $\eta(d\theta)$ is the uniform measure on S^{N-1} (the unit sphere in $\mathbb{R}^N$) with total measure $\eta(S^{N-1}) =2\cdot\pi^{N/2}\left(\Gamma\left(\frac{N}{2}\right)\right)^{-1}$. Hence

$$m^{(i)}(dv)=\lambda^{N-1}\eta(d\theta)\,F^{(i)}(d\lambda), \tag{2.16}$$

$$m^{(s)}(dv)=\lambda^{N-1}\eta(d\theta)\,F^{(s)}(d\lambda) \tag{2.17}$$

and $m_{\mathbf{i},\mathbf{j}}(dv)$ is determined by the two measures on $(0,\infty)$, $F^{(i)}$ and $F^{(s)}$ on $(0,\infty)$. If $r=0$, then $m^{(i)}\equiv 0$ and if $r=N$ then $m^{(s)}\equiv 0$.

A Sample Function Representation for U_r [16]. If U is a homogeneous and isotropic r-current, then there exist a random $(r-1)$ measure $\hat{Y}(dv)$ and a random $(N-r-1)$ measure $\hat{Z}(dv)$ such that $\hat{Y}$ and $\hat{Z}$ are uncorrelated and for every $\phi_{N-r}\in S^{N-r}$ and $\mathbf{i}$, with $|\mathbf{i}|=r$

$$U_r(\phi_{N-r})=\int_{\mathbb{R}^N}\hat{\phi}_{N-r}\wedge\frac{iv}{|v|}\wedge\hat{Y}(dv)+\int_I R^N*\hat{\phi}_{N-r}\wedge\frac{iv}{|v|}\wedge\hat{Z}(dv) \tag{2.18}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ: if $\phi=\Sigma\phi_{\mathbf{i}}\cdot e_{\mathbf{i}}$ then $\hat{\phi}=\Sigma\hat{\phi}_{\mathbf{i}}\,e_{\mathbf{i}}$ where $\hat{\phi}_{\mathbf{i}}$ is the Fourier transform of function $\phi_{\mathbf{i}}(t)$). $\hat{Y}(dv)=\sum_{|\mathbf{i}|}\hat{Y}_{\mathbf{i}}(dv)$, $|\mathbf{i}|=r-1$ and

$$E\hat{Y}_{\mathbf{i}}(dv)\,(\hat{Y}_{\mathbf{j}}(dv'))^c=\begin{cases}0, & \text{if } [\mathbf{i}]\neq[\mathbf{j}]\\ m^{(i)}(dv\cap dv'), & \mathbf{i}=\mathbf{j}\end{cases} \tag{2.19}$$

where $m^{(i)}(dv)$ is as in (2.16). For $|i|=N-r-1$,

$$E\hat{Z}_{\mathbf{i}}(dv)\,(\hat{Z}_{\mathbf{j}}(dv'))^c=\begin{cases}0, & \text{if } [\mathbf{i}]\neq[\mathbf{j}]\\ m^{(s)}(dv\cap dv'), & \mathbf{i}=\mathbf{j}\end{cases} \tag{2.20}$$

and $m^{(s)}(dv)$ is as in (2.17). The first term in the right hand side of (2.18) is the irrotational component of U and the second term is the solenoidal component. $\hat{Y}=0$ if $r=0$ and $\hat{Z}=0$ if $r=N$. For intermediate values of r, $\hat{Y},\hat{Z}$ are defined as follows:

$$\begin{aligned}\hat{Y}(dv)&=e_v\vee M_r(dv)\,a+e_v\wedge W^a(dv)\\ \hat{Z}(dv)&=*(e_v\wedge M_r(dv))+*(e_v\vee W^b(dv))\end{aligned} \tag{2.21}$$

where M_r is the random measure associated with U_r, $W^a(dv)$ is a random $(r-2)$ measure independent of M_r and satisfying

$$EW_i^a(dv)\,W_j^a(dv')=\begin{cases}0, & [i]\neq[j]\\ m^{(i)}(dv\cap dv'), & i=j, \quad |i|=(r-2)\end{cases} \tag{2.22}$$

and $W^b(dv)$ is a random $(r+2)$ measure independent of M_r and W^a, satisfying

$$EW_i^b(dv)\,W_j^b(dv')=\begin{cases}0, & [i]\neq[j]\\ m^{(s)}(dv\cap dv'), & i=j, \quad |i|=(r+2).\end{cases} \tag{2.23}$$

3. Random Cochains, Localization and the Markov Property for Currents

Let D be a bounded open set with smooth (C^∞) boundary ∂D; the σ-field generated by a random r-current U_r on D is defined to be the σ-field generated by $\{U_r(\phi), \text{supp. } \phi \subset D\}$ and similarly (with D replaced by D^c) for the σ-field generated by U_r on D^c. In order to consider the Markov property of random r-currents relative to D, ∂D, D^c we have to define what is meant by the boundary data. The notion of the boundary data will now be defined for a class of currents for which this data has a concrete meaning (cf. also [14] and [15]) and conditions will be derived under which homogeneous currents belong to this class.

Let V be an r-dimensional C^∞ manifold in $\mathbb{R}^N$ and let c be a chain in V, (i.e., c is a finite linear combination of simplexes on V) then ([9] Sects. 9, 10) c defines a (non-random) $N-r$ current in the sense of deRham and there exists a sequence of $(N-r)$ differential forms ψ^k, $k=1, 2, \ldots, \psi^k \in S^{N-r}$ such that $\psi^k \to c$ in the sense that

$$\int_{\mathbb{R}^N} \psi^k \wedge \phi \to \int_c \phi \tag{3.1}$$

for every $\phi \in S^r$ and the convergence is uniform for any collection of forms ϕ that are bounded on S^r. A collection of forms ϕ on S^r is said to be bounded if the support of all the ϕ's is contained in the same compact set K where K itself is contained in the domain of some coordinate system and all the partial derivatives of every coefficient of each ϕ are bounded in absolute value on K.

Definition. A random r-current U_r is said to be a *random r-cochain* if, whenever c is a bounded r-chain and $\psi^k \to c$, $\psi^k \in S^{N-r}$ then the sequence of random variables $U_r(\psi^k)$ is a Cauchy sequence in L^2.

If X is an r-cochain, $r=N-1$, and ∂D is a C^∞ compact manifold, then $X(c)$ is a well-defined random variable for a rich class of r-dimensional subsets c of ∂D and this will be used later to define the σ-field of the boundary data on ∂D. For the case where $r \leqq N-2$, the fact that X is an r-cochain will enable us, as will be shown later, to define the sigma-field of the boundary data on ∂D. We shall also consider cases where U_r is not extendible to an r-cochain, but can be extended to become an $(N-1)$-cochain, which will then suffice to construct the σ-field of the boundary data. For example, consider an l-current in $N=3$ dimensional space such that integration along lines does not yield random variables, but integration first along lines and then along a perpendicular path yields concrete random variables parametrized by $N-1$ dimensional subsets of ∂D. These r.v. can then be used to define the sub-σ-field of the boundary data.

Let U_r be an r-current and let a_{N-r-1} be a fixed $N-r-1$ covector, then for any fixed 1-vector b_1 and any $\phi \in S$, $\phi \cdot b_1 \wedge a_{N-r-1}$ is in S^{N-r}, hence $(a_{N-r+1} \wedge U_r)$ $(\phi \wedge b_1) = U_r(\phi \wedge b_1 \wedge a_{N-r+1})$ and $(a_{N-r+1} \wedge U_r)$ is a well-defined $(N-1)$-current.

Definition. An r-current U_r, $1 \leqq r \leqq N-1$, is said to be *localizable* if for every fixed $(N-r-1)$-covector a_{N-r-1}, the $(N-1)$-current $a_{N-r-1} \wedge U_r$ is a cochain and for every $(r-1)$ covector a_{r-1} the $(N-1)$-current $a_{r-1}\wedge(*U_r)$ is a cochain. A zero current U is localizable if $a_{N-1}\wedge U$ is a cochain and an N-current is localizable if $*U$ is localizable. Note that for $1\leqq r\leqq N-1$, every r-cochain is localizable.

Let U be a localizable random r-current and V a $C^\infty(N-1)$-manifold in $\mathbb{R}^N$. Let V_c denote the collection of bounded $(N-1)$ chains in V.

Definition. The *boundary data* of U on V is the sub-σ-field generated by

$$\int_c a_{N-r-1}\wedge U \quad \text{and} \quad \int_c a_{r-1}\wedge(*U)$$

for all $c\in V_c$ and all fixed covectors a_{N-r-1} and a_{r-1}, i.e.,

$$\sigma_V(U)=\sigma\{\int_c a_{N-r-1}\wedge U, \int_c a_{r-1}\wedge(*U), c\in V_c\}. \tag{3.2}$$

Remark. It is indeed natural to define $\sigma_{\partial D}(U)$ via $U(c)$, $c\in V_c$. The reasons for including $(*U)(c)$ in the definition are as follows: (a) Let $\phi=\sum_{|\mathbf{i}|}\phi_{\mathbf{i}}(t)\,e_{\mathbf{i}}$ be an r-form with $\phi_{\mathbf{i}}(t)$ random and well-behaved (say c^∞) then the σ-field generated by ϕ on ∂D will be

$$\sigma\{\phi_{\mathbf{i}}(t), t\in\partial D, |\mathbf{i}|=r\}$$

and the extension of this to currents calls for the inclusion of $*U$. (b) For scalar valued Markovian fields X, i.e., zero currents, an important role is played by (cf. e.g. [7, 8], Chapt. 3 of [11, 3]). More specifically, the minimal splitting σ-field generated by the integral of $\partial X/\partial n$ (and higher derivatives) on subsets of the boundary which is an $N-1$ dimensional manifold is the same as the integral of $*dX$ on this subset. Consequently for the particular case of zero currents, the results of this paper also follow from the previously known results cited above. It is when the Markov property of higher order currents is considered that the advantage of the calculus of currents becomes clear.

We define now, the Markov property for localizable currents, as follows:

Definition. Let $U=(U_{r_1}^{(1)}, U_{r_2}^{(2)}, \ldots, U_{r_p}^{(p)})$ be a collection of localizable currents of order $r_1, \ldots, r_p$ respectively (which may be equal or not). U is said to be Markov if for every connected and bounded domain D with smooth boundary ∂D, the σ-fields generated by U on D and D^c are independent given the boundary σ-field generated by $\{U_{r_i}, i=1,2,\ldots,p\}$.

Remark. This definition generalizes the notion of "Markov of order p" given in [7] (cf. also [8, 14]).

We conclude this section with some conditions under which homogeneous currents are cochains or localizable. Consider the case where U is an homogeneous current. Assume that the spectral measure $m_{i,\mathbf{j}}(d\nu)$ associated with U is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$.

$$m_{\mathbf{i},\mathbf{j}}(d\nu)=\mu_{\mathbf{i},\mathbf{j}}(\nu)\,d\nu \tag{3.3}$$

Proposition 3.1. *Let*

$$|\mu_{\mathbf{i},\mathbf{j}}(v)| \leqq \frac{K}{|v|^{2m_0} P(|v|^2)} \tag{3.4}$$

for all **i**, **j** *with* $|\mathbf{j}| = |\mathbf{i}| = r$ *where* $P(\cdot)$ *is a polynomial of degree* $p \geqq 0$ *with non-negative coefficients and* $P(0) \neq 0$.

(a) *If* $m_0 = 0$ *and* $(N-r) \leqq p$ *then* U *is an r-cochain.*

(b) *If* $N - r \leqq p + m_0$ *and* $N > 2m_0$ *then* U *is an r-cochain.*

(c) *If* $N > 2m_0$ *and* $p + m_0 \geqq 1$ *then* U *is localizable.*

Proof. Let c be a chain as discussed earlier, since c defines an $(N-r)$ current it possesses a Fourier transform, say $C(v) = \sum_{\mathbf{i}} C_{\mathbf{i}}(v)\, e_{\mathbf{i}}$. Assuming that $P(0) > 0$, then (3.4) implies that

$$|\mu_{\mathbf{i},\mathbf{j}}(v)| \leqq K_1 (1 + |v|^2)^{-p}$$

for some K_1. In order to show that $U(\psi^k)$ converges in L^2, it suffices to show that for all **i**, $|\mathbf{i}| = N - r$,

$$\int_{\mathbb{R}^N} |C_{\mathbf{i}}(v)|^2 \frac{K_1}{(1+|v|^2)^p}\, dv < \infty \tag{3.5}$$

(since $e^{-|v|^2/k} C(v)$, $k \to \infty$, can serve as $\hat{\psi}^k$). For $p \geqq N - r$, (3.5) follows from part b of Theorem IX.39 of [10], which proves part a. Regarding part b, we have to show that for all **i**, $|\mathbf{i}| = N - r$,

$$\int_{\mathbb{R}^N} |C_{\mathbf{i}}(v)|^2 \frac{K_1}{(1+|v|^2)^{p+m_0}}\, dv < \infty \tag{3.6}$$

and

$$\int_{|v| \leqq 1} |v|^{-2m_0}\, dv < \infty. \tag{3.7}$$

Now, (3.6) follows by the same argument as (3.5). As for (3.7), by (2.15)

$$\int_{|v| \leqq 1} |v|^{-2m_0}\, dv = N \cdot \pi^{N/2} (\Gamma(N + \tfrac{1}{2}))^{-1} \int_0^1 \lambda^{-2m_0} \lambda^{N-1}\, d\lambda$$

and (3.7) is satisfied for $N - 1 - 2m_0 \geqq 0$. Finally, (c) follows from (a) and (b) by specializing to $r = N - 1$.

4. Isotropic Gauss Markov Currents I

We shall use $(*d)^m X$ to denote $(*d)^0 X = X$, $(*d)^1 X = *dX$, $(*d)^2 X = *d(*dX)$, etc. Note that if X is an r-current, then $(*d)^m X$ is an r-current for m even and an $N - r - 1$ current for m odd.

Theorem 4.1. *Let X be a Gaussian homogeneous and isotropic r-current in $\mathbb{R}^N$, $r \geqq 1$, with $F_0 \equiv 0, F^{(s)} \equiv 0$, and $F^{(i)}(d\lambda) = (\lambda^{2m_0} P(\lambda^2))\, d\lambda$ where m_0 is an integer satisfying*

$$-1 \leqq m_0 \leqq (N-3)/2$$

and P is a polynomial of order p with $P(0) \neq 0$, i.e.

$$m_{\mathbf{i},\mathbf{j}}(dv) = ((e_v \vee e_{\mathbf{i}}), (e_v \vee e_{\mathbf{j}})) \cdot (|v|^{2m_0} P(|v|^2))^{-1} \cdot dv_1\, dv_2 \dots dv_N.$$

Then there exists an $(r-1)$ current Y such that $X = dY$ and

$$\{(^*d)^m Y, m = 0, 1, \dots, (p+m_0)\}$$

is Markov.

Remark. Theorem 4.3 extends the result of this theorem to $m_0 = [(N-1)/2]$.

Proof. (a) Note that by Proposition 3.1, $(^*d)^m X$, $0 \leqq m \leqq p + m_0 - 1$, is localizable. Set

$$Y(\phi) = \int_{\mathbb{R}^N} \hat{\phi}(v) \wedge \frac{1}{|v|} \hat{Y}(dv) \tag{4.1}$$

where $\hat{Y}$ is as defined in equations (2.17) and (2.18) and $\phi \in S^{N-r+1}$, then $X = dY$ and by Proposition 3.1 Y is also localizable.

(b) For further reference we note the following two well known lemmas:

Lemma 4.1. *Let W be the Fourier transform of a random measure M;*

$$E\{W(\phi)\, W^c(\psi)\} = \int_{\mathbb{R}^N} \hat{\phi}(v)\, \hat{\psi}^c(v)\, m(dv) \tag{4.2}$$

$\phi, \psi \in S$. Assume that m is spherically invariant ($m(A) = m(gA)$) and consequently $m(d\lambda) = \eta(d\theta)\, \rho(d\lambda)$. Then

$$E\{W(\phi)\, W^c(\psi)\} = \frac{(N-1)\,\pi^{\frac{1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)} \lim_{\substack{a \to 0 \\ b \to \infty}} \int_a^b \int_{\mathbb{R}^N \times \mathbb{R}^N} \phi(x)\, \psi^c(y) \cdot \frac{J_{(N-2)/2}(2\pi\lambda|x-y|)}{(2\pi\lambda|x-y|)^{(N-2)/2}} \lambda^{N-1}\, dx\, dy\, \rho(d\lambda) \tag{4.3}$$

where J_n is the Bessel function of order n. Furthermore, for any multi-indices α, β

$$\left(\frac{\partial^\alpha}{\partial X^\alpha} = \frac{\partial^{\alpha_N}}{\partial X_N^{\alpha_N}} \cdots \frac{\partial^{\alpha_1}}{\partial (X_1)^{\alpha_1}}, |\alpha| = \Sigma \alpha_i\right)$$

$$E\left\{W\left(\frac{\partial^\alpha}{\partial X^\alpha}\phi\right) W^c\left(\frac{\partial^\beta}{\partial X^\beta}\psi\right)\right\} = (-1)^{|\alpha|}(-1)^{|\beta|} \frac{(N-1)\,\pi^{\frac{1}{2}}}{\Gamma\frac{(N+1)}{2}} \cdot \lim_{\substack{a \to 0 \\ b \to \infty}} \int_a^b \int_{\mathbb{R}^N \times \mathbb{R}^N} \phi(x)\, \psi^c(y) \cdot \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial x^\beta} \frac{J_{(N-2)/2}(2\pi\lambda|x-y|)}{(2\pi\lambda|x-y|)^{(N-2)/2}} \lambda^{N-1} \rho(d\lambda). \tag{4.4}$$

The proof of this result is straightforward and therefore omitted.

Lemma 4.2. *If, the assumptions of the previous lemma $\rho(d\lambda)=d\lambda/P(\lambda^2)$ where P is a polynomial of order q, then*

$$E\{W(P(-\Delta)\,\phi)\cdot W^c(\psi)\} = \int_{\mathbb{R}^N} \hat{\phi}(t)\,\hat{\psi}^c(t)\,dt = \int_{\mathbb{R}^N} \phi(t)\,\psi^c(t)\,dt \tag{4.5}$$

where Δ is the Laplacian operator $\Sigma_i \frac{\partial^2}{\partial t_i^2}$. Furthermore, let D be a bounded domain in $\mathbb{R}^N$, let $\psi \in S$ be fixed and assume that the support of ψ is in the interior of D. Then there exists a function $R(x,\psi)$, $x \in D^c$ such that

(i) *$R(\cdot,\phi)$ is C^∞ on D^c.*

(ii) *$D(-\Delta)\,R(t,\psi)=0 \quad t\in D^c$.*

(iii) *For any $\phi\in S$ with support in D^c,*

$$E(W(\phi)\,W^c(\psi)) = \int_{\mathbb{R}^N} R(t,\psi)\,\phi(t)\,dt$$

(iv) $$E\left(W\left(\frac{\partial^\alpha}{\partial t^\alpha}\phi\right)W^c(\psi)\right) = \int_{\mathbb{R}^N} \phi(t)\frac{\partial^\alpha}{\partial t^\alpha}R(t,\psi)\,dt.$$

Proof of Lemma 4.2. Equation (4.5) follows from Eq. (4.2). Turning to $R(\cdot,\psi)$ for a fixed x, we let $R(t,\psi)$ denote the generalized function satisfying

$$E\{W(\phi)\,W^c(\psi)\} = R(\phi,\psi)$$

in the distribution sense. Then, since the supports of ϕ and ψ are disjoint, it follows by (4.6) that

$$E(W(P(-\Delta)\,\phi)\,W^c(\psi))=0$$

and over D^c

$$P(-\Delta)\,R(\cdot,\phi)=0$$

in the distribution sense. The existence and properties of $R(\cdot,\phi)$ follow now from the fact that $P(-\Delta)$ is hypoelliptic (cf. e.g. [1], p. 66) and weak solutions are in fact C^∞ solutions.

(c) We now show the Markov property for $r=1$, by reducing the problem to an interior Dirichlet problem and applying the results of part (b). Set $P_1(x) = x^{1+m_0}P(x)$. Since $m_0 \geqq -1$, P_1 is also a polynomial.

Let D be a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂D. Consider the problem of solving $P_1(-\Delta)\,f(t)=0$, $t\in D$ subject to the boundary conditions $\{(\partial^m f(t)/\partial t^m)=h^m(t),\ t\in\partial D,\ m=0,\ldots,m_0+p\}$ ([1] p. 91–93, [7]). It is more convenient for our purposes to rewrite the boundary conditions as follows: let $h^m(t)$, $t\in\partial D$, $0\leqq m\leqq p+m_0$ denote a zero form on ∂D for m even and an $N-1$ differential form on ∂D for m odd. It is required that $((*d)^m f(t))_{\partial D}$, i.e., the restriction of $(*d)^m f(t)$ to the boundary, satisfy

$$(*d)^m f(t) = h^m(t), \qquad t\in\partial D, \qquad 0\leqq m\leqq p+m_0. \tag{4.6}$$

Note that $(*d)^{2m}=(\Delta)^m$ that $*df$ is an $(N-1)$ form, and the integral of $*df$ on subsets of the surface $t_1=0$, $(t=t_1,\ldots,t_N)$ yields the integral of the normal derivative of $f(t)$ on this surface.

For smooth boundary data the Green function solution to $P_1(-\Delta)f=0$ subject to (4.8) is as follows [1]: given any t in the interior of D, there exist $g_t^m(\cdot)$, $0\leqq m\leqq p+m_0$ that are smooth $(N-1)$ differential forms on ∂D for m even and smooth zero forms for m odd, parametrized by t, such that

$$f(t)=\sum_{m=0}^{p+m_0}\int_{\partial D} h^m\wedge g_t^m \tag{4.7}$$

solves $P_1(-\Delta)f=0$ in D and satisfies (4.6), c.f. [3] for an extension of the solution of $P_1(-\Delta)f=0$ subject to boundary data which are generalized functions. In our case, we have the function $R(t,\psi)$, $t\in D$, the support of ψ is in D^c, and for $P_1(-\Delta)R(t,\psi)=0$, $t\in D$. Equation (4.7) is, in this case

$$R(t,\psi)=\sum_{m=0}^{p+m_0}\int_{\partial D}(*d)^m R(\cdot,\psi)\wedge g_t^m(\cdot). \tag{4.8}$$

If Y_t were a smooth random process, we could write

$$E\left(Y_t-\sum_{m=1}^{p+m_0}\int_{\partial D}(*d)^m Y\wedge g_t^m)\,Y^c(\psi)\right)=R(t,\psi)-\sum_{m=0}^{p+m_0}\int_{\partial D}(*d)^m R(\theta,\psi)\wedge g_t^m(\theta) \tag{4.9}$$

The right hand side of the last equation is zero by (4.8) and therefore, the Gaussian random variables $Y(\phi)$ and

$$Y_t-\sum_{m=1}^{p+m_0}\int_{\partial D}(*d)^m Y\wedge g_t^m \tag{4.10}$$

are independent. Consequently, $Y(\psi)$ and Y_t are independent, given the boundary data $\{(*d)^m Y, m=0,\ldots,p+m_0$ on $\partial D\}$. Hence $\{(*d)^m Y, m=0,\ldots,p+m_o\}$ is Markov. In the general case, (still $r=1$), Y_t is a generalized random variable. Let $\phi\in S$ and the support of ϕ is in the interior of D. Then, by (4.8) and the smoothness of g

$$E(Y(\phi)\,Y^c(\psi))=R(\phi,\psi)=\sum_{m=0}^{p+m_0}\int_{\partial D}(*d)^m R(\cdot,\psi)\wedge g_\phi^m(\cdot) \tag{4.11}$$

where $g_\psi=\int_{\mathbb{R}^N} g(t)\,\psi(t)\,dt$. Note that by (2.18) the stochastic integrals

$$\int_{\partial D}(*d)^m Y(\cdot)\wedge g_\phi^m(\cdot)$$

are well defined and satisfy

$$EY^c(\psi) \int_{\partial D} (*d)^m Y(\cdot) \wedge g_\phi^m(\cdot) = \int_{\partial D} (*d)^m R(\cdot, \phi) \wedge g_\phi^m(\cdot).$$

Consequently, by (4.11)

$$E\left\{\left(Y(\phi) - \sum_{m=0}^{p+m_0} \int_{\partial D} (*d)^m Y \wedge g_\phi^m(\cdot)\right) \cdot Y^c(\psi)\right\} = 0 \tag{4.12}$$

and therefore the Gaussian random variables $Y(\phi)$ and $Y(\psi)$ are conditionally independent, given the boundary data, proving Theorem 4.1 for $r=1$.

(d) Until now we have considered the case $r=1$, for which Y is a zero form. For general r, Y as defined by (4.1) is an $(r-1)$ current with *independent components*. That is, $Y=\Sigma Y_{\mathbf{i}} \cdot e_{\mathbf{i}}$, $|\mathbf{i}|=r-1$, where $Y_{\mathbf{i}}$ is defined as follows: For $\phi=\Sigma \phi_{\mathbf{i}*}$ where ϕ is an $N-(r-1)$ form, set $Y_{\mathbf{i}}(\phi_{\mathbf{i}*})=Y(\phi_{\mathbf{i}*})$ and for $[\mathbf{j}] \neq [\mathbf{i}]$, $Y_{\mathbf{i}}(\phi_{\mathbf{j}*})=0$, then $Y_{\mathbf{i}}$ and $Y_{\mathbf{j}}$ are independent ($[\mathbf{i}] \neq [\mathbf{j}]$). Because of the independence of $Y_{\mathbf{i}}$, $|\mathbf{i}|=r-1$, it suffices to prove the theorem for some fixed multi-index $\mathbf{i}_0$, $|\mathbf{i}_0|=r-1$. Now, $Y_{\mathbf{i}_0}$ can be considered as a zero current and the results of the previous parts of the proof apply. The only question that remains to be settled is whether the sub-σ-fields generated by $\{(*d)^m Y_{\mathbf{i}_0}, 0 \leqq m \leqq p+m_0\}$ with $Y_{\mathbf{i}_0}$ considered as an r-form and as a zero form are the same. Let $Y_{\mathbf{i}_0}^0$ denote Y_{i_0} considered as a zero current, namely for all $\phi \in S$. Then we get

$$Y_{\mathbf{i}_0}^0(\phi) = Y_{\mathbf{i}_0}(\phi \cdot e_{\mathbf{i}_0^*}). \tag{4.13}$$

Therefore the sub-σ-fields $\sigma\{Y_{\mathbf{i}_0}^0(\phi), \phi \in S, \operatorname{supp} \phi \subset D\}$ and $\sigma\{Y_{\mathbf{i}_0}(\phi \cdot e_{\mathbf{i}*}), \phi \in S, \operatorname{supp} \phi \subset D\}$ are the same. Turning to the boundary data, let Y be a 1-form and let $Y_{\mathbf{i}}=Y_1$. Then the integration of the one form Y_1 along a path in a hyperplane perpendicular to the t_1-direction is zero while the integration of Y_1^0 will yield a nonzero random variable. However, by (3.2) the $N-1$ form $*Y_1$ can be integrated on a subset of the hyperplane perpendicular to t_1. Consequently the boundary data of Y_1 and Y_1^0 are the same on subsets of an $(N-1)$ hyperplane. Similarly, by (3.2), the same arguments extend to subsets of an $N-1$ manifold, and consequently the boundary data of Y_1 coincide with that of Y_1^0. The same arguments apply to the case where Y is an r-form, $r>1$, which completes the proof.

Theorem (4.1) dealt with irrotational currents. For solenoidal currents we have:

Corollary 4.2. *Let U be an isotropic and homogeneous r-current in $\mathbb{R}^N$, $r \leqq N-1$ with $F_0=0$, $F^{(i)}=0$ and $F^{(s)}(d\lambda)=(\lambda^{2m_0} P(\lambda^2)\, d\lambda)$, with m_0 an integer satisfying $-1 \leqq m_0 \leqq (N-3)/2$ and $P(\cdot)$ a polynomial of order p with $P(0) \neq 0$, i.e.*

$$m_{\mathbf{i},\mathbf{j}} = ((e_v \wedge e_{\mathbf{i}}), e_v \wedge e_{\mathbf{j}})\,(|v|^{2m_0} P(|v|^2))^{-1}\, dv_1, \ldots, dv_N$$

Then there exists an $(r+1)$ current Z such that $U={*d*Z}$ and $\{(d^*)^m Z, m=0, 1, \ldots, (p+m_0)\}$ is Markov.

Proof. Note first that it follows from (2.14) or (2.17) that $*U$ is also homogeneous and isotropic. Set $*U=X$, then X satisfies all the conditions of Theorem 4.1. Set $Z=*Y$ where Y is as defined in Theorem 4.1, hence $\{(*d)^m *Z,\ m=0, 1, \ldots, (p+m_0)\}$ is Markov. Now $(*d)^m *Z=*(d*)^m Z$ and $\{(d*Z)^m,\ m=0, \ldots, (p+m_0)\}$ is Markov since the sub-σ-field generated by a localizable current, say W, on ∂D (or D or D^c) and the one generated by $*W$ are the same.

Theorem 4.3. *Let X be as in Theorem 4.1 (Cor. 4.2) with $m_0=[(N-1)/2]$, then the results of Theorem 4.1 (Cor. 4.2) hold.*

Remark. For the case $r=1$, $P(\lambda^2)=1$, and N odd, the one current X is the gradient of Lévy Brownian motion (cf. Eq. 7.2 of [5] or p. 129 of [11]), the zero form Y introduced in the proof is, in fact, the Lévy Brownian motion and Theorem 4.3 reduces to the result of McKean regarding the Markov property for Y for odd N.

Proof. By Proposition (3.1), $(*d)X$ is localizable for $0 \leqq m \leqq (p+m_0-1)$ as in the case of Theorem 4.1. Assume first, $r=1$; in order to assure that Y is localizable, set

$$Y_t=\int_{\mathbb{R}^N} \frac{e^{ivt}-e^{ivt_0}}{|v|} \hat{Y}(dv)$$

where t_0 is some fixed point in $\mathbb{R}^N$ and $\hat{Y}(dv)$ is as defined in (2.19). Note that in this case we still have $dY=X$. Furthermore, $E|Y_t|^2<\infty$. Therefore Y_t is a concrete random variable. Set $EY_t Y_s^c=R(t,s)$, then

$$R(t,s)=\int_{\mathbb{R}^N} \frac{(e^{ivt}-e^{ivt_0})(e^{-ivt}-e^{ivt_0})}{|v|^2} \frac{dv}{|v|^{2m_0} P(|v|^2)}.$$

Operating on the t variable with the differential operator $(-\Delta)^{m_0+1}(P(-\Delta)$ yields, in the distribution sense,

$$((-\Delta)^{m_0+1} P(-\Delta))_t R(t,s)=\int_{\mathbb{R}^N} e^{ivt}(e^{-ivs}-e^{-ivt_0} dv=\delta(t-s)-\delta(t-t_0)$$

Therefore, by the hypoellipticity of $(-\Delta)^{m_0+1}P(-\Delta)$ (p. 66 of [1]), it follows $R(t,s)$ is C^∞ outside the two points $t=s$ and $t=t_0$. For the case where D is a bounded domain on t and D does not include the point t_0, we continue as in the proof of Theorem 4.1. If $t_0 \in D$ then we have to consider the exterior Dirichlet problem rather than the interior one (as in [7]). The rest of the proof is the same as that of Theorem 4.1.

5. Isotropic Gauss Markov Currents II

In this section we consider the case where both components, $F^{(i)}$ and $F^{(s)}$ are present.

Theorem 5.1. *Let X be a Gaussian homogeneous isotropic r-current with $F^{(i)}(d\lambda) = F^{(s)}(d\lambda) = P(\lambda^2))\, d\lambda$, where $P(\cdot)$ is a polynomial of order p. Let m_0 denote the lowest power of the polynomial $(P(x) = \sum_{m_0}^{p} \alpha_q x^q)$. Then for $N-1 \geqq 2m_0$,*

$$\{(*d)^j X, 0 \leqq j \leqq p-1\}$$

is Markov.

Remark. The special case $r=0$, $p=1$ is known as the Free Euclidean Field.

Proof. By Proposition 3.1, $\{(*d)^j X, 0 \leqq j \leqq p-1\}$ is localizable. Let e be any unit vector; then it can be verified that $(e \vee e_{\mathbf{i}}, e \vee e_{\mathbf{j}}) + (e \wedge e_{\mathbf{i}}, e \wedge e_{\mathbf{j}}) = (e_{\mathbf{i}}, e_{\mathbf{j}})$. Therefore, since $F^{(i)}(d\lambda) = F^{(s)}(d\lambda)$, it follows from (2.4) that the components of X are independent, i.e., $X_{\mathbf{i}}(\phi)$ is independent of $X_{\mathbf{j}}(\psi)$ whenever $[\mathbf{i}] \neq [\mathbf{j}]$. Because of the independence of $X_{\mathbf{i}}$, $|\mathbf{i}|=r$, it suffices to prove the result for some fixed multi-index $\mathbf{i}_0$. From here on, the proof is the same as the proof of Theorem 4.1 and therefore omitted.

Remark. While Theorems 4.1 and 4.3 are similar in the sense that to the given r-form X we adjoin an $(r-1)$ form plus other forms which are $(N-r-1)$ and r-forms, in Theorem 5.1, no $(r-1)$ form is needed. The proofs of the three theorems are similar and is based on concentrating on the process with independent components which generate the r-current.

A partial converse to Theorem 5.1 is the following:

Proposition 5.2. *Let X be a Gaussian homogeneous and isotropic* 1*-current with*

$$F^{(i)}(d\lambda) = c_1 \lambda^{N-1} (\alpha^2 + \lambda^2)^{-1} d\lambda \tag{5.1}$$

$$F^{(s)}(d\lambda) = c_2 \lambda^{N-1} (\alpha^2 + \lambda^2)\, d\lambda. \tag{5.2}$$

Assume that $c_1 \neq 0$, $c_2 \neq 0$ then X is Markov if and only if $c_1 = c_2$.

Proof. By Theorem 5.1 for $c_1 = c_2$, X is Markov. Conversely, it is well known (cf. [11]:, p. 179) that an homogeneous vector valued generalized random field $\{X(t),\ t \in \mathbb{R}^N,\ X \in \mathbb{R}^d\}$ with spectral density matrix $f(v)$ satisfying $\|f^{-1}(v)\| (1+|v|^2)^{-k} \in L_1$ for some k is germ-field Markov if and only if for any x, $y \in \mathbb{R}^d\, x^T f^{-1}(v) y$ is a polynomial in v. We show now that X is not germ-field Markov. Let I denote the unit $N \times N$ matrix and $B(v)$ the $N \times N$ matrix with entries

$$(B(v))_{i,j} = v_i v_j / |v|^2.$$

By Ito's representation (2.14) and (5.1), (5.2) $m(dv) = C(v)\, dv$ where

$$C(v) = (c_2 I + (c_1 - c_2) B(v)) \cdot (\alpha^2 + |v|^2)^{-1}.$$

We show now that $C^{-1}(v)$ is given by $D(v)$ where:

$$D(v) = \left(\frac{1}{c_2} I + \left(\frac{1}{c_1} - \frac{1}{c_2}\right) B(v)\right)(\alpha^2 + |v|^2). \tag{5.3}$$

The proof is as follows, by direct multiplication:

$$D(v)\,C(v)=I+\left(\frac{c_1-c_2}{c_2}I+\frac{c_2-c_1}{c_1}\right)B(v)-\frac{(c_2-c_1)^2}{c_1 c_2}B^2(v). \tag{5.4}$$

Now

$$B^2(v)=B(v) \text{ since}$$

$$B^2(v)=\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}(v_1,\dots,v_N)\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}(v_1,\dots,v_N)\cdot\frac{1}{|v|^4}=\frac{1}{|v|^2}\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}(v_1,\dots,v_N)=B(v).$$

Substituting into (5.4) yields $D\cdot C=I$.

The $(i, i)^{\text{th}}$ entry of $D(v)$ is

$$(\alpha^2+|v|^2)\left(\frac{1}{c_2}+\left(\frac{1}{c_1}-\frac{1}{c_2}\right)\frac{v_i^2}{|v|^2}\right)$$

which is a polynomial in v only if $c_1=c_2$. Hence for $c_1\neq c_2$, $c_1\neq 0$, $c_2\neq 0$, X is not germ-field Markov and consequently not Markov in the definition of the present paper. Note that for $c_1=0$ or $c_2=0$, $c(v)$ is singular and the result quoted from p. 179 of [11] is not applicable.

Proposition 5.3. *Let U be a Gaussian homogeneous and isotropic r-current with $F^{(i)}(d\lambda)=F^{(s)}(d\lambda)=Q(\lambda^4)/P(\lambda^2)$ where P is a polynomial or order p and Q is a polynomial of order $q<(p-1)/2$. Then U can be embedded in a Gauss-Markov, homogeneous and isotropic r-current,* i.e., *there exists an isotropic and homogeneous Gauss Markov r-current X such that U is a linear combination of $\{(*d)^j X, (d^*)_j X, 0\leqq j\leqq p-1\}$.*

Proof. Note that if A is an r-form, then $(*d)^k A$ is an r-form for even values of k and an $(N-r-1)$ form for odd values of k and consequently we can take linear combinations of $(*d)^{k_1}(U_r$, $(*d)^{k_2}U_r$ etc. if all k_j are odd or all k_j are even.

Let X be as defined by Theorem 5.1 with the same P as in the present proposition. Then by Theorem 5.1,

$$(*d)^j X, \quad 0\leqq j\leqq p-1$$

is Markov. Applying Theorem 5.1 to $*X$ yields that $(*d)^j*X$, $0\leqq j\leqq p-1$ and consequently $(d^*)^j X$, $0\leqq j\leqq p-1$ are also Markov. We claim that

$$\{(*d)^j X, (d^*)^j X, j=0, 1, \dots, p-1\} \tag{5.5}$$

is also Markov. To see this, note first that the σ-fields generated by $(*D)^j X$ and $*(*d)^j X=(*d)^j*X$, $j=0, 1, \dots, p-1$ on D (and on D^c) are the same. Now the σ-field generated by $(d^*)^j X$ on ∂D is in D (and D^c) by continuity, hence (cf. Lemma 1.3a of [6]) (5.5) is Markov. Let Δ be the generalized Laplacian

$\Delta = d\delta + \delta d$, then $\Delta = \pm((^*d)2 \pm (d^*)^2)$ where the + or − signs are determined by N and r. Therefore, since $\delta^2 = d^2 = 0$,

$$\Delta^j = \pm((^*d)^{2j} \pm (d^*)^{2j}).$$

Note that if X is an r-current, so is $\Delta^j X$ and since $X(\phi) = M(\hat{\phi})$ then ([5] or Eq. 4.13 of [16]), $(\Delta^j X)(\phi) = \pm M(|v|^{2j}\hat{\phi})$. Let $Q(|v|^4) = |Q_1(|v|^2)|^2$, set $\tilde{U}(\phi) = M(Q_1(|v|^2)\hat{\phi})$, then $\tilde{U}$ is a linear combination of $\Delta^j X$, hence a linear combination of $((^*d)^j X,\ (d^*)^j X,\ 0 \leqq j \leqq p-1)$. Now U as defined in the statement of the present proposition has the same second-order properties, as $\tilde{U}$ and since U and $\tilde{U}$ are both Gaussian and they have the same probability law. Therefore U can be written as a linear combination of components of a Markov process, which completes the proof.

References

1. Agmon, S.: Lectures on elliptic boundary value problems. Princeton, N.J.: D. Van Nostrand Co. 1965
2. Albeverio, S., Høegh-Krohn, R., Holden, H.: Markov processes on infinite dimensional spaces, Markov fields, and Markov consurfaces. Proceedings, Bremen Conf., Arnold, L. and Kotelenez, P. (eds.). Dordrecht: Reidel 1985
3. Dobrushin, R.L., Surgailis, D.: On the innovation problem for Gaussian Markov Random Fields. Z. Wahrscheinlichkeitstheor. Verw. Geb., **49**, 275–291 (1979)
4. Flanders, H.: Differential Forms. New York: Academic Press 1963
5. Ito, K.: Isotropic random current. Proceedings, Third Berkeley Symp. on Math. Stat. and Probab. pp. 125–32, 1956
6. Mandrekar, V.: Markov properties for random fields. Probab. Anal. Rel. Topics. Bharucha-Reid, A.T. (ed.), vol. 3, pp. 161–93. New York London: Academic Press 1983
7. McKean, H., Jr.: Brownian motion with a several-dimensional time. Theory Probab. Appl. **8**, 335–54 (1963)
8. Pitt, L.D.: A Markov property for Gaussian processes with a multidimensional parameter. Arch. Rational Mech. Anal. **43**, 367–91 (1971)
9. deRham, G.: Differentiable Manifolds. Berlin Heidelberg New York: Springer 1982
10. Reed, M., Simon, B.: Methods of modern mathematical physics II: Fourier analysis, self-adjointness. New York London: Academic Press 1975
11. Rozanov, Y.A.: Markov Random Fields. New York London: Academic Press 1982
12. Westenholtz, C. von: Differential forms in mathematical physics. Amsterdam: North Holland 1981
13. Wong, E.: Homogeneous Gauss-Markov random fields. Ann. Math. Stat. **40**, 1625–34 (1969)
14. Wong, E., Zakai, M.: Markov processes on the plane. Stochastics **15**, 311–333 (1985)
15. Wong, E., Zakai, M.: Multiparameter martingale differential forms. Probab. Th. Rel. Fields **74**, 429–453 (1987)
16. Wong, E., Zakai, M.: Spectral representation of isotropic random currents. To appear
17. Yaglom, A.M.: Some classes of random fields in n-dimensional space related to stationary random processes. Theory Prob. Appl. **2**, 273–320 (1957)

Received February 15, 1987