Stability of distributed congestion control with heterogeneous feedback delays

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Abstract

In this note we investigate how congestion control can achieve efficient usage of network resources in the presence of heterogeneous communication delays between network users and resources. To this end, we consider a fluid flow model of network behaviour. We study the stability of the system’s behaviour under small perturbations around the target equilibrium point (local stability). We establish several criteria for stability of certain linear delay-differential equations, via a technique which essentially reduces the question to studying stability of ordinary differential equations.

These results are then used to derive sufficient conditions for local stability of the network congestion control problem. The same issue has been studied in [13], where the authors propose a conjecture according to which local stability can be ensured in a distributed way. The correctness of the conjecture is established in [13] only in degenerate cases where feedback delays coincide. Our results show that a relaxed form of the conjecture holds true for arbitrary feedback delays.

1 Introduction

Congestion control consists in regulating the rate at which traffic sources send data into a network, based on the network congestion status. The TCP protocol, and the congestion control algorithms of Jacobson (see [7]) implemented therein are an important factor of the Internet’s success, and of the performance of applications such as Web browsing. This has motivated a huge amount of work, aimed at understanding what the objectives of congestion control should be, and how to achieve them. The objective traditionally considered is to allocate bandwidth in a fair manner (see [2], [15]). A more general objective, which we adopt here, is to maximize the sums of utilities end-users (traffic sources) retrieve from their network usage (see [17] for background in economical modeling, [14] for an application of such ideas to network usage pricing, and [10] for the application of utility functions to congestion control).

Given an objective function, the first design goal for the associated control algorithms is that they drive the system state close to operating points maximising this objective function. One equally important design objective is that such algorithms should be distributed, in the sense that network resources and users exchange as little information as possible, and control decisions are taken locally at the resources inside the network and at the users at the network’s edge. More specifically, one might want to impose on any candidate congestion control architecture the following requirements: on the network side, each resource (this could be a switching fabric, buffer space or link capacity) synthesises feedback information from its own global usage pattern, unaware of either other resources or individual users identities/preferences. At the network edge, end-users choose how to send data based on feedback gathered from these resources they are using, unaware of either resources they don’t use or other users.

In the context of the Internet, a number of features make the design of both feedback synthesis and end-user control algorithms for congestion control very challenging, such as:

- the presence of different types of resources (mainly buffer space and link capacity),
- the interaction with other levels of controls (admission control and route selection),
- the dynamic character of the users population,
- delays in the feedback loops between resources and users.
In order to understand the implications of each of these features on design choices, one viable approach is to address them separately, trying to capture each in a specific model. For instance, Misra et al. [16] investigate feedback synthesis by considering a static population of TCP users all sharing a single bottleneck resource, and all having the same delay in their feedback loop, while taking into account queueing effects at the single link. Key and Massoulié [12] consider jointly admission control and congestion control in a fluid flow model with a dynamic population of users, while ignoring feedback delay, routing, and buffering, and assuming a specific form of feedback.

In a recent paper [13], R. Johari and D. Tan investigate the design of end-user congestion control algorithms, and more precisely how it should take feedback delays into account. Assuming that these delays are common to all users within classes, and differ of several orders of magnitude from class to class, they derive sufficient conditions for local stability of the system’s behaviour. It is important to ensure local stability, because it is likely to imply non-oscillatory, and hence efficient usage of the network resources.

The appeal of the sufficient conditions for local stability derived in [13] is that they are distributed in the following sense. The only information needed by end-users to make their rate adaptations, in addition to anonymous feedback (i.e., synthesised for no specific user in particular) they retrieve from the network, is their own feedback delay. In particular, there is no need for users to learn other users’ feedback delays (one could for instance imagine that all adaptation rates have to be tuned to the slowest feedback delay in the system; their result implies that for specific delay distributions, it is not so). Finally, D. Tan and R. Johari conjecture that a similar, distributed condition should be sufficient for stability when feedback delays can vary arbitrarily among users.

Our aim in the present note is to complement the results of [13] and make some progress on the investigation of this conjecture, by establishing distributed stability conditions for congestion control in the presence of heterogeneous feedback delays.

The paper is organised as follows. The network model together with relevant previous work is presented in the next section. Section 3 contains Theorem 1, giving a sufficient condition for stability of linear delay-differential equations. This then allows to prove Theorem 2, which provides distributed stability criteria for congestion control when feedback delay parameters are allowed to depend on traffic sources, but not on network links. Theorem 3, a variant of Theorem 1, is then proven in Section 4, where it is used to extend the result of Theorem 2 to the general case where feedback delays are allowed to depend on both traffic sources and network links. Whereas Theorems 2 and 4 apply to the situation where users have logarithmic utility functions, the case of general utility functions is addressed in Section 5. Conclusions are drawn in Section 6.

2 Modeling assumptions and previous work

The model considered here has been previously introduced in [9] and [10]; the results reported in this section are taken from [9], [10], [8] and [13]. A fixed population of users, indexed by \( r \in R \), competes for network bandwidth. Network resources consist in links indexed by \( l \in L \). User \( r \) requires capacity from all links along a route through the network, which is assumed to be fixed. We denote by \( l \in r \) the fact that user \( r \) sends its traffic through link \( l \). User \( r \) retrieves utility \( U_r(x_r) \) from sending data at rate \( x_r \), for some utility function \( U_r \). It is also assumed that link \( l \) incurs a cost \( C_l(y) \) when forwarding data at rate \( y \), for some cost
function $C_l$. Assuming that utilities and costs add over users and links, the total utility of a rate allocation $x := \{x_r\}_{r \in R}$ is given by

$$\mathcal{U}(x) := \sum_r U_r(x_r) - \sum_{l} C_l \left( \sum_{r \in l} x_r \right).$$

It is also assumed that link $l$ computes at all times $t$ some “spot price” of bandwidth $p_l(t)$, which we take equal to

$$p_l(t) = f_l(y_l(t)),$$

where $f_l$ is the derivative of function $G_l$, assumed to exist, and $y_l(t)$ is the rate at which data is sent through link $l$ at time $t$. Each user $r$ is then informed of the aggregate spot price $\sum_{l \in r} p_l$ along its route.

This might for instance model the following mechanism: data packets have a dedicated field in their header, and when they reach the head of the queue in front of a link, the current value in that field is incremented by the current spot price at that link (see the Early Congestion Notification proposal [5] for such a feedback mechanism based on a single bit; multibit feedback encoding could alternatively be employed in IPv6 networks).

Remark that here, we have chosen a feedback which is a function of the instantaneous load only, whereas it is conceivable (and actually recommended in many propositions such as RED [6] and its avatars) that feedback would result from some filtering of the link state. Our choice is not only guided by economy and simplicity of modeling, but also by the following reason. As different users will receive this feedback after different delays, and such delays are known to the users but not to the link, it appears that the end-users are in a better position to perform an adequate filtering of raw link state (this argument is from F. Kelly [11] and [13]).

If transmission delays could be neglected, then each user $r$ would then have access to $\sum_{l \in r} p_l(t)$ at all times $t$, and in addition the rate $y_l(t)$ through link $l$ would coincide with the aggregate sending rate $\sum_{r \in l} x_r(t)$. In that case, the adaptation rule

$$\dot{x}_r(t) = G_r \left( U'_r(x_r) - \sum_{l \in r} p_l(t) \right), \quad (1)$$

where $G_r$ is some positive gain parameter, possibly depending on $x_r(t)$, defines dynamics such that the aggregate utility $\mathcal{U}$ always increases over time. Indeed,

$$\frac{d}{dt} \mathcal{U}(x(t)) = \sum_r G_r \left( U'_r(x_r) - \sum_{l \in r} f_l(\sum_{s \in l} x_s(t)) \right)^2.$$

If in addition the objective function $\mathcal{U}$ is strictly concave (for instance, if the individual utility functions $U_r$ are strictly concave and the cost functions are convex), then this ensures that the above algorithm converges to the unique maximum of $\mathcal{U}$.
Choosing the gain function $G_r$ proportional to $x_r$, i.e.

$$ G_r(x_r) = \kappa_r x_r $$

for some fixed, positive gain parameter $\kappa_r$ is an appealing choice, as the second term in $\dot{x}_r(t)$ is then proportional to $x_r(t) \sum_{l \in r} p_l(t)$, which is the rate at which feedback is received by user $r$, or equivalently its instantaneous charging rate. Also, a choice for the functions $U_r$ of particular interest consists in taking

$$ U_r(x_r) = w_r \log(x_r) $$

for some positive $w_r$. Indeed, this connects the optimisation problem under consideration here to the notion of proportional fairness introduced and motivated in [9]. Combining the two choices for $G_r$ and $U_r$ yields the following rate adaptation rule:

$$ \dot{x}_r(t) = \kappa_r \left[ w_r - x_r(t) \sum_{l \in r} f_l \left( \sum_{s \in l} x_s(t) \right) \right]. $$

It has been proposed by Gibbens and Kelly in [8], and is named there the “willingness to pay” strategy because of the following property: at equilibrium user $r$ would pay $w_r$ per time unit, no matter what rate it receives for that price.

One interesting feature of the algorithms (1) is that they converge to the desired optimum in a distributed manner: no explicit information exchange between users and resources is required. For instance, users need not inform anyone of their utility function.

The question raised by Johari and Tan is the following: can one preserve convergence to equilibrium in such a distributed manner in the presence of feedback delays? More precisely, they consider the case where all users implement a willingness to pay strategy (2), but the feedback rate to user $r$ at time $t$ is now given by

$$ x_r(t - D_{l,r}) \sum_{l \in r} f_l \left( \sum_{s \in l} x_s(t - D_{l,r}^+ - D_{l,s}^-) \right). $$

Here, $D_{l,r}^+$ represents the propagation delay from link $l$ to source $r$, $D_{l,s}^-$ the propagation delay from source $s$ to link $l$, and it is assumed that for all $r, l \in r$, the sum

$$ D_{l,r}^+ + D_{l,s}^- =: D_r $$

does not depend upon $l$. The motivation for this equation is that the feedback at time $t$ from link $l$ has been synthesised $D_{l,r}^-$ time units ago, and at that time the contribution of user $s$ to the load on link $l$ was the traffic rate it sent $D_{l,s}^-$ time units earlier, that is $x_s(t - D_{l,r}^+ - D_{l,s}^-)$. Then the time-delayed version of (2) is

$$ \dot{x}_r(t) = \kappa_r \left[ w_r - x_r(t - D_r) \sum_{l \in r} f_l \left( \sum_{s \in l} x_s(t - D_{l,r}^+ - D_{l,s}^-) \right) \right]. $$

Denote by $\bar{x}_r$ the sending rate of user $r$ at the equilibrium point of (2), and accordingly

$$ \left\{ \begin{array}{l} p_r := \sum_{l \in r} f_l \left( \sum_{s \in l} \bar{x}_s \right) = w_r / \bar{x}_r, \\
pl := f_l \left( \sum_{s \in l} \bar{x}_s \right). \end{array} \right.$$
Introducing the variables $y_r(t) = (x_r(t) - \bar{x}_r)/\sqrt{\kappa_r \bar{x}_r}$, system (3) admits the following linearization around its equilibrium point $(\bar{x}_r)$:

$$\dot{y}_r(t) = - \left( \kappa_r p_r y_r(t - D_r) + \sum_{l \in R} \sum_{s \in I} \sqrt{\kappa_r \bar{x}_r \kappa_s \bar{x}_s} y_s(t - D_{l,r}^- - D_{l,r}^+) \right)$$  \hspace{1cm} (4)

The system studied in [13] is actually a discrete time version of system (4). We do not reproduce the conjecture made in [13] concerning the stability of the discrete time system studied there, but rather state the natural counterpart in the present continuous time setting:

**Conjecture 1** The continuous time system (4) is asymptotically stable (i.e., all solutions converge to zero) under the condition:

$$\kappa_r D_r \left( p_r + \sum_{l \in R} \sum_{s \in I} \bar{x}_s \right) < \frac{\pi}{2}, \quad r \in R.$$  \hspace{1cm} (5)

If that were true, then in order to achieve local stability, each user $r$ would only need to know its round trip delay $D_r$, its equilibrium bandwidth price $p_r$, and the associated price sensitivity $\sum_{l \in R} \sum_{s \in I} \bar{x}_s$. Such quantities could presumably be inferred by user $r$ from its feedback, as they are specific to user $r$’s route through the network. In [13], the authors show the validity of their conjecture in the situation where users can be partitioned into two groups $R_1, R_2$ say, and users $r$ in the first group $R_1$ all share the same round-trip delay parameter, i.e.

$$D_r \equiv D, \quad r \in R_1,$$

while users $r$ in the other group $R_2$ either don’t adapt or adapt “infinitely quickly”.

In the next section, we prove a weakened version of the above conjecture, namely that stability of (4) holds under the condition

$$\kappa_r D_r \left( p_r + \sum_{l \in R} \sum_{s \in I} \bar{x}_s \right) < 1,$$  \hspace{1cm} (6)

and provided the delay parameters $D_{l,r}^-, D_{l,r}^+$ depend on $r$ only. The extension to the general case where the delay parameters can depend on both $l$ and $r$ constitutes Section 4. We then show in Section 5 that, provided the round-trip delays $D_r$ do not depend on $r$, (6) is sufficient for stability not only for logarithmic utility functions but also for general utility functions $U_r$.

### 3 Source-dependent, link-independent delays

This section is aimed at establishing Theorem 2 below, namely that (6) is sufficient for stability of (4), provided the delay parameters $D_{l,r}^-, D_{l,r}^+$ depend on $r$ only. We hence assume that these do not depend on $l$, and denote them by $D_r^-, D_r^+$ in this section. Consider then the following delay-differential system

$$\dot{x}_r(t) = - \sum_s M_{rs} x_s(t - D_r^- - D_r^+), \quad r \in R,$$  \hspace{1cm} (7)

where $M := (M_{rs})$ is a given square matrix. As in the previous section, we use the notation

$$D_r := D_r^+ + D_r^-.$$
Finally, for any vector \( (y_r) \), we let \( y \) denote both the vector \( (y_r) \) and the corresponding diagonal matrix. For instance, \( D = \text{Diag}(D_r) \), \( \exp(\lambda D) = \text{Diag}(\exp(\lambda D_r)) \), etc... The result below, of independent interest, will enable us to deduce Theorem 2 as a corollary at the end of this section. It will also pave the way to the more general results of the next section.

**Theorem 1** Assume that the matrix \( M \) is Hermitian, positive definite. Assume also that the matrix \( DM := (D_r M_{rs}) \) satisfies

\[
\rho(DM) < 1
\]

where \( \rho \) denotes the spectral radius. Then the system (7) is asymptotically stable.

**Proof:** It is known (see e.g. Bellman and Cooke [1]) that such stability holds if all solutions \( \lambda \) to the characteristic equation

\[
\det(\lambda I + \exp(-\lambda D_r)M \exp(-\lambda D_r)) = 0
\]

have negative real part. By a similitude transformation, (9) is equivalent to

\[
\det(\lambda I + \exp(-\lambda D)M) = 0.
\]

In turn, for each solution \( \lambda \) to Equation (9) or Equation (10) there exists a vector \( z = (z_r) \in \mathbb{C}^R \), \( z \neq 0 \) such that the functions \( x_r(t) = z_r \exp(\lambda t) \) are solutions of the modified system

\[
\dot{x}_r(t) = -\sum_s M_{rs} x_s(t - D_r), \ r \in R.
\]

We must then show that there exist no such non-trivial exponential solutions for any \( \lambda \) with \( \Re(\lambda) \geq 0 \), where \( \Re \) denotes real part.

We first rule out the possibility of solutions with \( \Re(\lambda) = 0 \). We thus assume the existence of such a solution: there exists \( \omega \in \mathbb{R}, z = (z_r) \in \mathbb{C}^R, z \neq 0 \) such that \( x_r(t) = z_r \exp(i\omega t) \) satisfies (11). Note that the case \( \omega = 0 \) is impossible: indeed, any constant solution \( x \equiv z \) must lie in the null space of \( M \), which is reduced to zero as we have assumed \( M \) to be positive definite.

After rearrangement, one obtains that such a solution also satisfies the system of ordinary differential equations

\[
\dot{x}_r(t) = -\sum_s M_{rs} x_s(t) + D_r \theta_r \sum_s M_{rs} x_s(t),
\]

where

\[
\theta_r = \begin{cases} 
(1 - \exp(-i\omega D_r))/i\omega D_r & \text{if } D_r > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

This system can be written in matrix form as

\[
(I - D\theta M)\dot{x} = -Mx.
\]

We now make use of the following lemma:
Lemma 1 Under the assumptions of Theorem 1, and for $\theta$, given by (13), the system (14) is asymptotically stable.

Proof: This will follow if the eigenvalues of the matrix $M^{-1}(I - D\theta M)$ all have positive real parts. Indeed, this is equivalent to the fact that the eigenvalues of the matrix $(I - D\theta M)^{-1}M$ all have positive real parts, which is necessary and sufficient for asymptotic stability of (14). Assume first that all delays $D_r$ are strictly positive. By a similitude transformation, and because the two diagonal matrices $D$ and $\theta$ commute, it is in turn equivalent to establish positivity of the real parts of the eigenvalues of the matrix

$$N := D^{1/2}M^{-1}D^{-1/2} - \theta D.$$

Now, by Lyapunov’s second method (see e.g. [3]), the desired property holds iff one can find a Hermitian, positive definite matrix $H$ such that the Hermitian matrix $HN + N^\dagger H$ is positive definite, where $N^\dagger = (N_{rs}) = (\overline{N}_{sr})$. Taking $H = D^{-1}$ yields

$$HN + N^\dagger H = 2D^{-1/2}M^{-1}D^{-1/2} - \Re(\theta).$$

The eigenvalues of the Hermitian matrix $D^{-1/2}M^{-1}D^{-1/2}$ are the reciprocals of the eigenvalues of the matrix $DM$. By assumption, the spectral radius of this matrix is less than 1. Also, it follows from the fact that $M$ is Hermitian, positive definite, that $DM$ admits real and positive eigenvalues. Thus the eigenvalues of $D^{-1/2}M^{-1}D^{-1/2}$ are all real, positive, and strictly larger than one; let $1 + \varepsilon$ be the smallest of them, with $\varepsilon > 0$.

On the other hand, it holds that

$$|\Re(\theta_r)| \leq |\theta_r| = \left|\frac{\sin(\omega D_{r}/2)}{\omega D_{r}/2}\right|,$$

and the latter is smaller than 1 for all real values of the argument $\omega D_{r}/2$. Combining the two evaluations, one obtains that for all $x \in \mathbb{C}^R, x \neq 0$,

$$x^\dagger (HN + N^\dagger H)x \geq 2x^\dagger x(1 + \varepsilon - \sup_r \Re(\theta_r)) \geq 2\varepsilon x^\dagger x.$$

Lyapunov’s second method criterion is thus verified, i.e. $HN + N^\dagger H$ is positive definite, and hence the result of the Lemma holds when the delay parameters $D_r$ are strictly positive. To handle the general case where $D_r$ can be zero for some $r$, choose $\delta > 0$ sufficiently small that $\rho((D + \delta)M) < 1$ (such $\delta$ exists since the eigenvalues of a matrix depend continuously on the entries of the matrix). The conclusion of the Lemma holds provided all the eigenvalues of

$$N := (D + \delta)^{1/2}M^{-1}(D + \delta)^{-1/2} - \theta D$$

have positive real parts. Applying Lyapunov’s second method with $H = (D + \delta)^{-1}$ amounts then to checking that

$$2 \left( (D + \delta)^{-1/2}M^{-1}(D + \delta)^{-1/2} - \Re(\theta)D/(D + \delta) \right)$$

is positive definite, which follows from the same argument as previously. □
A consequence of the Lemma is that our candidate solution \( x(t) = z_r \exp(i \omega t) \) to the original system (14) must converge to zero, being a solution to the modified system (11) which is now known to be stable. However this is possible only if \( z_r \equiv 0 \) for all \( r \), and we have thus ruled out the possibility of solutions \( \lambda \) to (9) with \( \Re(\lambda) = 0 \). In order to proceed, we need the following

**Lemma 2** Solutions \( \lambda \) to Equation (9) must be such that either \( \Re(\lambda) < 0 \), or

\[
|\lambda| \leq \sup_r \sum_s |M_{rs}|. \tag{15}
\]

**Proof:** Given a solution \( \lambda \), there exists \( z \in \mathbb{C}^n, z \neq 0 \) such that for all \( r \),

\[
\lambda z_r = \exp(-\lambda D_r) \sum_s M_{rs} z_s.
\]

Choose \( r \) such that \( |z_r| = \sup_s |z_s| \). Dividing the previous equation by \( z_r \), it then follows that

\[
|\lambda| \leq \exp(-\lambda D_r) \sum_s |M_{rs}|.
\]

If \( \Re(\lambda) \geq 0 \), \( \exp(-\lambda D_r) \leq 1 \), and the claim of the Lemma follows. \( \square \)

Let \( n(D) \) denote the number of solutions to (9) such that \( \Re(\lambda) \geq 0 \). As follows from the previous Lemma and from the first part of the proof, all such solutions must lie in the domain \( C = \{ z : \Re(z) > 0 \text{ and } |z| < 2 \sup_r \sum_s |M_{rs}| \} \). Let \( F(\lambda; D) \) denote the left-hand side of (9). The function \( F \) is analytic, entire in \( \lambda \), so that the number \( n(D) \) is finite and, by Cauchy’s theorem, is given by the variation of the argument of \( F(z; D) \) as \( z \) runs along the boundary of the domain \( C \). Letting \( \gamma \) denote the contour of domain \( C \), this is formally written as

\[
n(D) = [\arg F(z; D)]_{|\gamma|}.\]

For any \( \varepsilon \in [0, 1] \), the assumptions of Theorem 1 hold with \( D \) replaced by \( \varepsilon D_r \). Indeed, one only has to check that the spectral radius of \( \varepsilon DM \) is less than 1, which follows directly from \( \rho(\varepsilon DM) = \varepsilon \rho(DM) \) and \( \rho(DM) < 1 \). Thus, for all \( \varepsilon \in [0, 1] \), there is no zero of \( F(\cdot; \varepsilon D) \) on the contour \( \gamma \). We thus extend the former relation to

\[
n(\varepsilon D) = [\arg F(z; \varepsilon D)]_{|\gamma|}.
\]

Finally, it is easily seen that \( D \to F(\cdot; D) \) is continuous in \( D \), for the topology of uniform convergence of functions on compact sets. In particular, the contour \( \gamma \) is compact, and thus the number \( n(\varepsilon D) \) does not depend on \( \varepsilon \), provided \( \varepsilon \in [0, 1] \): as \( \varepsilon \) varies, the winding number of the contour \( \{ F(z; \varepsilon D), z \in \gamma \} \) around zero cannot change unless for some \( \varepsilon \), this contour contains zero, which we know does not happen. In order to show that \( n(D) = 0 \), by setting \( \varepsilon \) to zero, it is thus enough to show that the number \( n \) corresponding to null delays equals zero. For null delays, Equation (9) reads

\[
\det(\lambda I + M) = 0.
\]

Thus, each solution \( \lambda \) to (9) is such that \(-\lambda \) is an eigenvalue of \( M \). Under Theorem 1’s assumptions, the eigenvalues of \( M \) are real positive, so that all zeros of \( F(\cdot; 0) \) have negative real parts. Hence \( n(0) = 0 \), and thus \( n(D) = n(0) = 0 \). \( \square \)
Remark 1 Consider the following time-linearisation of Equation (11):
\[ \dot{x}_r(t) = -\sum_s M_{rs} [x_s(t) - D_s \dot{x}_s(t)]. \]

When matrix $M$ is Hermitian positive definite, stability of this linearized system is actually equivalent to the further condition $\rho(DM) < 1$: indeed, stability holds iff the matrix $(I - DM)M^{-1} = M^{-1} - D$ has eigenvalues with positive real parts, or equivalently if the matrix $M^1D^{-1}$ has eigenvalues with real part larger than 1. As such a matrix has real positive eigenvalues for real $D$ and Hermitian positive definite $M$, this is equivalent to $\rho(M^{-1}D^{-1}) > 1$, or again to $\rho(DM) < 1$. Theorem 1 thus gives conditions ($M$ Hermitian, positive definite) under which stability of the system (11) of delay-differential equations is implied by stability of the time-linearization of that system.

A direct application of Theorem 1 yields

**Theorem 2** Under the assumption (6), provided that for all $r, l,$
\[ p_r > 0, \quad \kappa_r > 0, \quad p'_l \geq 0, \]
and the delay parameters $D^r_{l', r}, D^{-}_l r$ do not depend on $l$, then the system (4) is stable.

**Proof:** Indeed, the system (4) is of the form (7), with
\[ M_{rs} = I_{r=s} \kappa_r p_r + \sum_l I_{r \in E_l} I_{s \in E_l} p'_l \sqrt{\kappa_r \kappa_s} \bar{x}_r \bar{x}_s. \]

The matrix $M = (M_{rs})$ is Hermitian, being real and symmetric; also, it is easily seen to be positive definite, given that all the coefficients $p'_l$ are non-negative, and all the products $\kappa p_r$ are positive. It only remains to verify that the spectral radius of $DM$ is less than one. By a similitude transformation, this coincides with the spectral radius of $M' := \sqrt{\kappa/\bar{x}} DM \sqrt{\bar{x}/\kappa}$. Note that the $(r, s)$ entry of matrix $M'$ is given by
\[ M'_{rs} = \kappa_r D_r \left( I_{r=s} p_r + \sum_l I_{r \in E_l} I_{s \in E_l} p'_l \bar{x}_s \right). \]

Finally, the spectral radius of a matrix $M'$ is bounded above by the maximum of the absolute row sum, i.e.
\[ \rho(M') \leq \sup_r \sum_s |M'_{rs}| \]
and that the latter is strictly less than one follows directly from (6). \hfill \Box

### 4 General case: source- and link-dependent delays

The previous technique can in fact be extended to prove the weak version of the conjecture. Indeed, Theorem 1 admits the following variant:

**Theorem 3** Consider the delay-differential system
\[ \dot{x}_r(t) = -\sum_{l \in L} \sum_s M^{(l)}_{rs} x_s(t - D^r_{l, r} - D^{-}_{l, s}), \]
(16)
where the non-negative delay parameters \( D_{l,r}^+, D_{l,r}^- \) are such that

\[
D_{l,r}^+ + D_{l,r}^- = D_r, \quad r \in R, \quad l \in r.
\]

Assume in addition that each matrix \( M^{(l)} \) is Hermitian, with non-negative eigenvalues, and that for some \( l \), \( M^{(l)} \) is positive definite. Then the system (16) is asymptotically stable if the matrix

\[
N := D \sum_l \left| M^{(l)} \right|
\]

admits a spectral radius strictly less than 1.

**Proof:** The proof follows closely that of Theorem 1, and thus we shall only detail the parts which significantly differ. Stability of system (16) holds provided all solutions \( x(t) \) to (16) which have the exponential form \( x_r(t) = z_r \exp(\lambda t) \) are such that either \( z = (z_r) = 0 \), or \( \Re(\lambda) < 0 \). Equivalently, all zeros \( \lambda \) of the function \( F \) given by

\[
F(\lambda) = \det \left( \lambda I + \sum_{l \in L} \exp(-\lambda D_{l,r}^+) M^{(l)} \exp(-\lambda D_{l,r}^-) \right)
\]

(17)

have negative real part.

We first rule out the existence of such zeros with null real part. In order to do so, we assume the existence of \( \omega \in \mathbb{R} \), and \( z \in \mathbb{C}^R \) such that \( x_r(t) = z_r \exp(i\omega t) \) satisfies (16). By the same type of manipulation as in the proof of Theorem 1, \( x_r \) must then satisfy the system of ordinary differential equations

\[
\dot{x}_r(t) = -\sum_{l \in L} \sum_s M_{ls}^{(l)} \theta_{rs}^{(l)} x_s(t) + \sum_{l \in L} \sum_s M_{ls}^{(l)} \theta_{rs}^{(l)} \frac{1}{i\omega} \left[ 1 - \frac{\exp(-i\omega(D_{l,r}^+ + D_{l,s}^-))}{\theta_{rs}^{(l)}} \right] \dot{x}_s(t),
\]

where \( \theta_{rs}^{(l)} \) could be any nonzero complex number. Choosing

\[
\theta_{rs}^{(l)} = \exp(-i\omega(D_{l,r}^+ - D_{l,s}^-)),
\]

after simplification of the ratio

\[
\frac{\exp(-i\omega(D_{l,r}^+ + D_{l,s}^-))}{\theta_{rs}^{(l)}},
\]

the former equation can be rewritten in matrix form as

\[
\dot{x}(t) = -\sum_l e^{i\omega D_{l,r}^+} M^{(l)} e^{-i\omega D_{l,r}^-} x(t) + \left( \frac{1 - \exp(-i\omega D)}{i\omega} \right) \sum_l e^{i\omega D_{l,r}^+} M^{(l)} e^{-i\omega D_{l,r}^-} \dot{x}(t).
\]

Introducing the notations

\[
\begin{align*}
N^{(l)} & := e^{i\omega D_{l,r}^+} M^{(l)} e^{-i\omega D_{l,r}^-}, \\
N & := \sum_l N^{(l)}, \\
\theta_r & := \frac{1 - \exp(-i\omega D_r)}{i\omega D_r},
\end{align*}
\]

this can be expressed more compactly as

\[
(I - D\theta N)\dot{x} = -Nx.
\]
Thus, the system (18) is asymptotically stable provided the real parts of the eigenvalues of the matrix 
\(N^{-1}(I - DBN)\) are all positive. Remark that this system is exactly of the same form as system (14). Thus, in view of Lemma 1, it is asymptotically stable if matrix \(N\) is Hermitian, positive definite, and the spectral radius of \(DN\) is strictly less than 1. Since we have assumed that each matrix \(M^{(l)}\) is Hermitian, with non-negative eigenvalues, it follows that each matrix \(N^{(l)}\) as defined above is also Hermitian, with non-negative eigenvalues (in particular, \(N^{(l)}\) admits the same eigenvalues as \(M^{(l)}\), as it is obtained from \(M^{(l)}\) via a similitude transformation). Moreover, since we have assumed that for some \(l\), \(M^{(l)}\) is positive definite, the corresponding matrix \(N^{(l)}\) is also positive definite. Thus the matrix \(N = \sum_l N^{(l)}\) is Hermitian, positive definite. The following Lemma will be used to check that the spectral radius of \(DN\) is indeed smaller than 1.

**Lemma 3** Let \(A\) and \(B\) be two matrices the entries of which all satisfy
\[
|A_{rs}| \leq B_{rs}
\]
(and hence the \(B_{rs}\) are real, non-negative). Then the spectral radius of \(A\) is smaller than or equal to that of \(B\).

The proof follows easily from the Perron-Frobenius theory of non-negative matrices and is thus omitted here. As a consequence, the spectral radius of \(DN\) is less than that of \(|DN|\), in turn less than that of \(D\sum_j |M^{(j)}|\). As we have assumed that the latter is less than 1, all the assumptions of Lemma 1 are satisfied, and thus system (18) is asymptotically stable. Hence, there do not exist solutions \(\lambda\) to the characteristic equation (17) with \(\Re(\lambda) = 0\). The proof is then concluded via a continuity argument, exactly as that of Theorem 1.

We are now ready to establish the main result of this note, namely that the weak form of Conjecture 1 holds true.

**Theorem 4** The system (4) is asymptotically stable under Assumption (6), provided that all parameters \(\kappa, p_r\) are positive and \(p_l\) are non-negative.

**Proof:** For notational convenience we identify here the set of links \(l\) with \(\{1, \ldots, L\}\), for some \(L > 0\). The system (4) can be written as
\[
\dot{x}_r(t) = -\sum_s \sum_{0 \leq l \leq L} M^{(l)}_{rs} x_s(t - D_{l,r}^+ - D_{l,r}^-),
\]
where
\[
M^{(l)}_{rs} = \kappa_r p_r I_{rs}, \quad M^{(l)}_{rs} = p_l \sqrt{\kappa_r \bar{x}_r \kappa_s \bar{x}_s} I_{l,r} I_{l,s}, \quad 1 \leq l \leq L,
\]
and \(D_{0,r}^+ = D_{1,r}^-, \, D_{0,r}^- = D_{1,r}^+\) for all \(r\). This is indeed of the form (16). Also, for \(l \geq 1\), the matrices \(M^{(l)}\) are Hermitian, non-negative definite provided \(p_l \geq 0\), while \(M^{(0)}\) is positive definite when the parameters \(\kappa, p\) are positive. Finally, the spectral radius of matrix \(D\sum_j |M^{(j)}|\) is bounded from above by the maximum of the absolute row sum of matrix \(\sqrt{\kappa} \sqrt{\bar{x}} D\sum_j |M^{(j)}| \sqrt{\bar{x}} / \kappa\), and this is indeed less than 1 under condition (6).

5 General utility functions and constant round-trip delays

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If we no longer assume that the user utility functions are logarithmic, a natural rate adaptation rule for user $r$, in the presence of delays, might then be the following:

$$\dot{x}_r(t) = \kappa_r \left[ x_r(t)U'_r(x_r(t)) - x_r(t - D) \sum_{l \in R} f_l \left( \sum \lambda_l(t - D^{l^-}_{i,r} - D^{l^-}_{i,s}) \right) \right].$$

(19)

Indeed, when delays are set to zero, and assuming strict convexity of the $U'_r$, such dynamics will converge to the system optimum, while in the presence of delays, the term $x_r(t)U'_r(x_r(t))$ is computed from the current sending rate $x_r(t)$, the sum of the other terms being the rate of feedback received at time $t$. Let as before $\bar{x}_r$, $p_r$, $p'_l$ denote the quantities of interest at equilibrium. The linearization of (19) around the equilibrium point can be described in terms of the variables $y(t) = (x_r(t) - \bar{x}_r)/\sqrt{\kappa_r \bar{x}_r}$ as

$$\dot{y}_r(t) = \kappa_r \left( U'_r(\bar{x}_r) + \bar{x}_r U''_r(\bar{x}_r) \right) y_r(t) - \kappa_r p_r y_r(t - D) - \sum_{l \in R} p'_l \sum_{s \in l} \sqrt{\kappa_r \bar{x}_r \kappa_s \bar{x}_s} (t - D^{l^-}_{i,r} - D^{l^-}_{i,s}),$$

(20)

In order to study the stability of this system, we shall need the following result.

**Theorem 5** Consider the system (16), where the delay parameters $D^{l^-}_{i,r}$, $D^{l^-}_{i,s}$ are non-negative, and are no longer assumed to be such that the sum $D^{l^-}_{i,r} + D^{l^-}_{i,s}$ does not depend on $l$. Assume that the matrices $|M^{[l]}| := (|M^{[l]}|)$ are symmetric. Assume also that the matrix $M := \sum_{l \in L} M^{[l]}$ is Hermitian, positive definite. Then the system (16) is asymptotically stable provided the matrix

$$N := \frac{1}{2} \sum_{l \in L} (D^{l^-}_{i,r} + D^{l^-}_{i,s}) |M^{[l]}| + |M^{[l]}|(D^{l^-}_{i,r} + D^{l^-}_{i,s})$$

(21)

admits a spectral radius strictly less than 1.

**Proof:** As in the proof of Theorem 3, we only detail the parts that differ from the proof of Theorem 1. Stability of system (16) holds provided all solutions $x(t)$ to (16) which have the exponential form $x(t) = z \exp(\lambda t)$ are such that either $z = (z_r) = 0$, or $\Re(\lambda) < 0$. Equivalently, all zeros $\lambda$ of the function $F$ given by

$$F(\lambda) = \det \left( \lambda I + \sum_{l \in L} \exp(-\lambda D^{l^-}_{i,r}) M^{[l]} \exp(-\lambda D^{l^-}_{i,s}) \right)$$

have negative real part.

We first rule out the existence of such zeros with null real part. In order to do so, we assume the existence of $\omega \in \mathbb{R}$, and $z \in \mathbb{C}^d$ such that $x_r = z_r \exp(\omega t)$ satisfies (16). By the same type of manipulation as in the proof of Theorem 1, $x_r$ must then satisfy the system of ordinary differential equations

$$\dot{x}_r(t) = \sum_{l \in L} \sum_{s \in l} M^{[l]}_{rs} x_s(t) + \sum_{l \in L} \sum_{s \in l} M^{[l]}_{rs} \frac{1 - \exp(-\omega(D^{l^-}_{i,r} + D^{l^-}_{i,s}))}{\omega} \bar{x}_s(t).$$

Let $N^{[l]}$ denote the matrix with entries

$$N^{[l]}_{rs} = M^{[l]}_{rs} \frac{1 - \exp(-\omega(D^{l^-}_{i,r} + D^{l^-}_{i,s}))}{\omega(D^{l^-}_{i,r} + D^{l^-}_{i,s})}.$$
The former system can then be written in matrix form as
\[
    \dot{x} = -Mx + \left( \sum_{l \in L} (D_{l}^{-} N^{(l)} + N^{(l)} D_{l}^{-\ast}) \right) x.
\]  

(22)

It is stable provided the eigenvalues of \( P := M^{-1}(I - \sum_{l \in L} D_{l}^{-} N^{(l)} + N^{(l)} D_{l}^{-\ast}) \) all have positive real parts. By Lyapunov’s second method, this holds iff there exists a positive definite, Hermitian matrix \( H \) such that \( HP + P^\ast H \) is positive definite. We have assumed that \( M \) is Hermitian positive definite, hence it is enough to show that \( MP + P^\ast M \) is positive definite. Cancelling products \( MM^{-1}, \) the latter can be expressed as
\[
    I - \sum_{l \in L} D_{l}^{-} N^{(l)} + N^{(l)} D_{l}^{-\ast} + I - \sum_{l \in L} D_{l}^{-\ast} N^{(l)} + N^{(l)} D_{l}^{-\ast}.
\]

The desired property will hold provided the spectral radius of matrix
\[
    \sum_{l \in L} D_{l}^{-} N^{(l)} + N^{(l)} D_{l}^{-\ast} + D_{l}^{-\ast} N^{(l)} + N^{(l)} D_{l}^{-\ast}
\]
is strictly less than 2. In view of Lemma 3, it is enough to check that the spectral radius of
\[
    \sum_{l \in L} D_{l}^{-} |N^{(l)}| + |N^{(l)}| D_{l}^{-\ast} + D_{l}^{-\ast} |N^{(l)}| + |N^{(l)}| D_{l}^{-\ast}
\]
is less than 2. Using the fact that \( |N^{(l)}| \leq |M_{rs}^{(l)}| \), and the assumption that the matrices \( |M^{(l)}| \) are symmetric, stability of system (22) follows then from the assumption that the spectral radius of matrix \( N \) as defined by (21) is less than 1. A continuity argument allows to conclude as in the proof of Theorem 1.

We now apply the previous result to derive sufficient stability conditions for the system (20).

**Theorem 6** Assume that the delay parameters \( D_{l,r}^{+}, D_{l,r}^{-} \) are all such that
\[
    D_{l,r}^{+} + D_{l,r}^{-} = D, \quad r \in R, \ l \in r
\]
for some scalar \( D \geq 0 \). Then the system (20) is asymptotically stable under Assumption (6), provided that for all \( r \) and \( l, \)
\[
    \bar{x}_r > 0, \quad U_r^{\mu} (\bar{x}_r) < 0, \quad p_{l,r} \geq 0.
\]

Note that conditions \( U_r^{\mu} (\bar{x}_r) < 0 \) and \( p_{l,r} \geq 0 \) will typically be met when the functions \( U_r \) are strictly concave, and the functions \( C_l \) are convex.

**Proof:** As in the proof of Theorem 4, the set of links \( l \) is identified with \( \{1, \ldots, L\} \), for some \( L > 0 \). The system (4) can be written as
\[
    \dot{x}_r (t) = -\sum_{s} \sum_{-1 \leq l \leq L} M_{rs}^{(l)} x_s (t - D_{l,r}^{-} - D_{l,s}^{+}),
\]

where
\[
    M^{(-1)} = -\kappa(U_r^{\mu} (\bar{x}) + \bar{x} U_r^{\mu'} (\bar{x})), \quad M^{(0)} = \kappa p, \quad M_{rs}^{(l)} = p_{l,r} \sqrt{\kappa x_{r} \kappa x_{s,1\leq l \leq 1}} I_{l \in r} I_{l \in s}, \quad 1 \leq l \leq L.
\]
and $D_{0,r}^+ = D_{1,r}^+$, $D_{0,r}^- = D_{1,r}^-$, $D_{-1,r}^+ = D_{-1,r}^-$ for all $r$. This is indeed of the form (16). Also, for $l \geq 1$, the matrices $M^{(l)}$ are symmetric, non-negative provided $p_l^0 \geq 0$, while $M^{(0)}$ is positive definite when the parameters $\kappa_r$, $p_r$ are positive. The corresponding matrix $M$ is given by

$$M = \kappa(-U'(\bar{x}) - \bar{x}U''(\bar{x}) + p) + \sum_{1 \leq i \leq L} M^{(l)}.$$  

It follows that $M$ is positive definite if for all $r$,

$$\kappa_r(-U'_r(\bar{x}_r) - \bar{x}_rU''_r(\bar{x}_r) + p_r) > 0.$$  

Note that the parameters of the system at equilibrium must satisfy $p = U'_1(\bar{x})$, hence $M$ is positive definite under the assumption that $\kappa > 0$ and $\bar{x}_rU''_r(\bar{x}_r) < 0$. Finally, the corresponding matrix $N$ as defined by (21) can be expressed as

$$N = D \sum_{0 \leq i \leq L} M^{(l)},$$  

because we have assumed that all round trip delays $D_t$ coincide with a single scalar $D$, and the additional delay terms $D_{-1,r}^+$, $D_{-1,r}^-$ are all equal to zero. Finally, the spectral radius of matrix $N$ is bounded from above by the maximum of the absolute row sum of matrix $\sqrt{\kappa/\bar{x}} N \sqrt{\bar{x}/\kappa}$, and this is indeed less than one under condition (6).

6 Conclusions

In this note, we have established sufficient conditions for stability of linear delay-differential equations (Theorems 1, 3 and 5). The technique developed in the corresponding proofs essentially amounts to proving stability of delay-differential systems by studying ordinary differential equations (cf. also Lemma 1 and Remark 1).

These results have in turn been used to derive sufficient conditions for local stability of a fluid flow model of congestion control (Theorems 2, 4 and 6). The appeal of these sufficient conditions is that they require each network user to tune its own gain parameter based on characteristics of its own path through the network. Our work thus complements previous results of [13] by applying to more general choices of delay parameters. In particular, Theorem 4 applies to the case of a network with users having arbitrary propagation delays to and from any links in the network, hence establishing a relaxed form of Conjecture 1, adapted from a conjecture made in [13]. In view of Theorem 6, it seems plausible that the same conditions also imply stability of the system (20), which describes the dynamics of congestion control algorithms related to general user utility functions, when delay parameters can take arbitrary values as in Theorem 4. This question is the subject of ongoing work.

References


