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SURNAME IN BLOCK CAPITALS Given Name(s) Student ID No.

Will you take the Putnam Exam on 1 Dec. 2007? YES:[___] NO:[___] Don't know:[___]

Solve as many problems as you can in the allotted time. A clear and complete solution is worth far more than a muddy or incomplete solution, so you may benefit from rewriting a solution. No computers, calculators nor communications devices may be used. Blank paper will be supplied; put your name and the problem's number on every sheet you wish to have graded, and hand them in with this sheet. This test's grade will not affect any other grades in this or in any other course.

Problem 1: Mathematics departments at some south-western universities received Mr. H.N.'s mischievous letters asking for the one real solution x of the two equations $(1 + x)^{17}/x = 17$ and $(1 + x)^{18}/x = 18$. Professor A.S. at one university sent Mr. H.N. the following brief solution:

$$\text{“ } 18/17 = \big((1 + x)^{18}/x \big)/\big((1 + x)^{17}/x \big) = 1 + x , \quad \text{so} \quad x = 1/17 \text{ .”}$$

Is this the only real solutions x ? Why?

Solution 1: There are NO solutions x , real or complex, because the two equations given are inconsistent. The "brief solution" merely establishes that IF x satisfied both equations it would have to be $x = 1/17$. However this x does not satisfy the first of the given equations since $(1 + 1/17)^{17}/(1/17) = 18^{17}/17^{16} > 17$; neither is the second equation satisfied. In fact, the second equation has no real solution because $(1 + x)^{18} > 18x$ for all real x ; can you see why? And the first of the given equations has just one real root $x \approx -2.2387\ldots$; can you see why?

The constants "17" and "18" are simpler than in the letters actually mailed out by Mr. H.N. And Professor A.S. teaches at a state college in a state other than ours. Really!

Problem 2: A large number of spy satellites orbit the Earth; their number is a military secret. They communicate continuously by microwaves with stations on the Earth and with each other except when the Earth's bulk interrupts the line-of-sight path microwaves need. Prove that at all times at least two satellites are each in uninterrupted communication with the same number of satellites.

Solution 2: Let N be the total number of spy satellites in orbit. Any selected one of them could conceivably be in communication with no other, or one other, or two others, or …, or $N-1$ others. If every spy satellite were in communication with a different number of others, one of them, call it S_0 , would be in communication with no other; another, call it S_1 , would be in communication with one other; another, call it S_2 , would be in communication with two others; …, and the last one, call it S_{N-1} , would be in communication with all $N-1$ others. But this can't happen because it would put S_{N-1} in communication with S_0 , which is *incommunicado*. Therefore at least two satellites must be communicating with the same number of others.

Problem 3: Let G be a multiplicative *Group* and let p be a prime. Prove that p divides the number of elements x in G that satisfy $x^p = \mathbf{1}$ unless that number is infinite or 1 .

A *Group* is a set of elements *closed* under a binary operation (here sharing the syntax of multiplication, so $\mathbf{1}$ is the group's *identity* element) that is *associative* and *invertible* but not necessarily *commutative*. For example, the 2-by-2 matrices B with det(B) = 1 form such a Group.

Solution 3: The last equation has at least one root x , namely $x = \mathbf{1}$; and if it has other roots x , but only finitely many, let g be one of those. Then partition all the roots x into two subsets:

$$C := \{ \text{roots } x \text{ that commute with } g,\ \text{satisfying } g\cdot x = x\cdot g \}.$$
$$B := \{ \text{roots } x \text{ that don't commute with } g,\ \text{so } g\cdot x \neq x\cdot g \}.$$

Evidently the function $f(x) := g\cdot x$ defines a permutation of the set C , and every one of that permutation's cycles $x \to g\cdot x \to g^2\cdot x \to \ldots \to g^{p-1}\cdot x \to g^p\cdot x = x$ has length p ; the length cannot be smaller because it would have to divide the prime p . (Can you see why?) Therefore p divides the number of roots x in C . Similarly $h(x) := g\cdot x\cdot g^{-1}$ defines a permutation of the set B , and each of that permutation's cycles has length p too (can you see why?), so p divides the number of roots x in B too. Therefore the total number of roots x , if neither infinite nor just 1 , must be a multiple of p as claimed.

Problem 4: Column vector **x** is distributed *Randomly* and *Uniformly* over the surface of the unit sphere in N-dimensional *Euclidean* space. What is the *Expected (Mean) Value* of the squared length of **x**'s orthogonal (⊥) projection onto a K-dimensional plane through **o** at the sphere's center?

Solution 4: The expected value of the projection's squared length is K/N . Here is why: Choose any fixed orthonormal coordinate system for which the selected K-dimensional plane consists of vectors whose last K – N components vanish. The orthogonal projection upon that plane is accomplished by setting a vector's last K – N components to zeros. The components $\xi_1, \xi_2, \ldots, \xi_N$ of **x** in that coordinate system are random variables; by symmetry, each has the same probability distribution as any other because exchanging coordinates is a *Reflection* of the sphere onto itself which, since the distribution is *Uniform*, leaves it unchanged. Their squares sum to 1 since **x** lies on the surface of the unit sphere; therefore the expected value of each square ξ_j^2 must be 1/N , and the expected value of the sum of any K squares must be K/N .

The foregoing solution takes for granted that the expected value of a sum of random variables is the sum of their expected values regardless of whether these random variables are *correlated* (not *independent*). See texts on Probability and Statistics, or the text for Math. 55 (*Discrete Mathematics and its Applications*, 4th ed. (1999) by K. Rosen, pp. 275-8), for an easy proof.

Problem 5: Here the notation $f^n(x)$ denotes the n-fold Iteration of $f(x)$; in other words, $f^0(x) := x$ and $f^{n+1}(x) := f(f^n(x))$ for $n = 0, 1, 2, 3, \dots$ in turn. Suppose that an interval $\mathbf{X}$ on the real axis includes one of its end-points but not the other, and that $f(x)$ maps $\mathbf{X}$ to itself continuously, and for each x in $\mathbf{X}$ there is a least positive integer N(x) such that $f^{N(x)}(x) = x$. Why can no function f but the identity function f^0 satisfy all these suppositions?

Solution 5: The only possibility is the identity function $f^0(x) = x$. To prove this, let us first call any function f that satisfies the problem's requirements "Eligible". Identity function f^0 is obviously Eligible, so we must rule out all other candidates f . To do so we shall show why any other Eligible f must be a continuous decreasing bijection of $\mathbf{X}$ onto itself satisfying $f^2 = f^0$.

Suppose f is Eligible; then it maps $\mathbf{X}$ onto itself since f takes every value x in $\mathbf{X}$. To see why f maps one-to-one, suppose $f(x) = f(y)$; then $x = f^{N(y)\cdot N(x)}(x) = f^{N(x)\cdot N(y)}(y) = y$. As a continuous *bijection*, f must be monotonic, either increasing or decreasing.

If f is increasing then $f = f^0$ because otherwise, were $x < f(x)$ at some x in $\mathbf{X}$, we could infer that $x < f(x) < f^2(x) < \dots < f^{n-1}(x) < f^n(x)$ for every integer $n \ge 0$, and *mutatis mutandis* were $x > f(x)$, so that f could not be Eligible. Therefore an Eligible increasing $f = f^0$.

Were f decreasing then f^2 would be increasing and Eligible since $(f^2)^{N(x)}(x) = f^{2N(x)}(x) = x$, whence would follow that $f^2 = f^0$. This is impossible because a decreasing Eligible f would have to swap the end-points of $\mathbf{X}$, which includes only one of them.

If the problem is modified by including either both or neither end-point of $\mathbf{X}$ in $\mathbf{X}$, the Eligible functions f will include those many decreasing continuous functions that swap $\mathbf{X}$'s endpoints and whose graphs are their own reflections in the diagonal line $y = x$ because $f^2 = f^0$. The foregoing solution is Dan Velleman's solution on pp. 85-6 of the *Amer. Math. Monthly* **100** #1 (Jan. 1993) to a problem posed by A.B, Boghossian.

Problem 6: Define a *Plane Parabolic Convex Body* (PPCB) to be a convex body drawn in the plane and bounded only by a parabola from whose every point a ray (half-line) extends infinitely into the PPCB's interior parallel to the parabola's axis. No constraint is imposed upon any axis' orientation nor upon the distance from any parabola's focus to its directrix nor upon the location of the focus in the plane. A PPCB includes its boundary parabola. Explain why no finite number of PPCBs, some of them overlapping, can possibly cover all of the plane.

Solution 6: Finitely many PPCBs' parabolas can have only finitely many axes; choose a line £ parallel to none of them. This £ intersects every PPCB in a finite segment, a point, or not at all, so the PPCBs can cover only a finite number of finite intervals in £ . Because the rest of £ must be uncovered, the plane cannot be covered entirely by PPCBs.

Problem 7: Can uncountably many non-intersecting copies of the digit " 8 " with whatever orientations (including " *8* " and " ∞ " *etc*.) and sizes (including " 8 " and " 8 " and " 8 " *etc*. but not just a dot "·") you like be packed into the plane? Justify your answer; don't just guess.

Solution 7: No, at most countably many non-intersecting copies will pack into the plane; here is why: Choose two points with rational coordinates, one strictly inside one loop and another point strictly inside the other loop, of each " 8 ". Thus each copied " 8 " is associated with four rational numbers, and hence with an ordered octuple of eight integers. No two " 8's " can share the same octuple because they cannot intersect. Thus the set of " 8's " is placed in one-to-one correspondence with a subset of the set of all ordered octuples of integers. This set is countable since the set of all integers is countable. Therefore the set of copied " 8's " is countable too.

Problem 8: Which of $\int_0^\pi \exp(\sin^2(x))\,dx$ and $3\pi/2$ is the bigger, and why?

Solution 8: $\int_0^\pi \exp(\sin^2(x))\,dx \approx 5.5084\ldots > 4.7123\ldots \approx 3\pi/2$. Hereunder is a proof that establishes the inequality with almost no numerical computation:

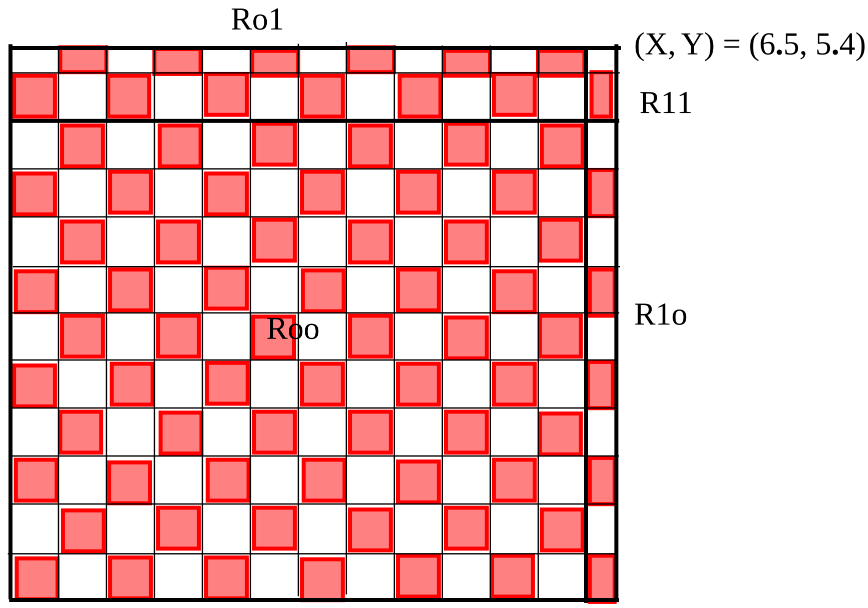
$$\begin{aligned}
\int_0^\pi \exp(\sin^2(x))\,dx &= \int_0^{\pi/2} \exp(\sin^2(x))\,dx + \int_{\pi/2}^\pi \exp(\sin^2(x))\,dx = \\
&= \int_0^{\pi/2} \exp(\sin^2(x))\,dx + \int_0^{\pi/2} \exp(\cos^2(x))\,dx = \\
&= \int_0^{\pi/2} \exp(\sin^2(x))\,dx + \int_0^{\pi/2} \exp(1-\sin^2(x))\,dx = \\
&= \sqrt{e}\cdot\left(\int_0^{\pi/2} \big(\exp(\sin^2(x)-1/2) + \exp(1/2-\sin^2(x))\big)\,dx\right) = \\
&= 2\sqrt{e}\cdot\left(\int_0^{\pi/2} \cosh(1/2-\sin^2(x))\,dx\right) \\
&> 2\sqrt{e}\cdot\left(\int_0^{\pi/2} 1\,dx\right) = \pi\cdot e^{1/2} = \pi\cdot(1 + 1/2 + 1/8 + \ldots) > 3\pi/2 .
\end{aligned}$$

Here is a much shorter proof sent in by Prof. Kent Merryfield of Cal. State Univ. @ Long Beach:

$$\int_0^\pi \exp(\sin^2(x))\,dx > \int_0^\pi (1+\sin^2(x))\,dx = \pi + \pi/2 .$$

Problem 9: Let a TILE be any rectangle one inch wide of arbitrary length, not necessarily an integral number of inches long. Suppose a set of TILEs of possibly diverse lengths barely cover some bigger rectangle R without overlapping; the one-inch sides of the TILEs need not all be parallel. Why must at least one side of R be some integral number of inches long?

Solution 9: To explain this, position R with its bottom along the positive x-axis and its left side along the positive y-axis of a Cartesian plane painted like a checkerboard with 1/2-by-1/2 inch Squares alternately red and white. The Square with one corner at the origin (0, 0) and another at (1/2, 1/2) has some color; suppose it is red. Every TILE in the covering of R covers equal areas of red and white since the tile is exactly two Squares wide. Therefore R covers equal areas of red and white too. Now suppose, for the sake of an argument by contradiction, that the upper right corner of R fell at a point (X, Y) neither of whose coordinates was an integer. Then R could be cut into four subrectangles thus: (See the picture.)



Roo has diagonally opposite corners at $(0, 0)$ and $(\lfloor X\rfloor, \lfloor Y\rfloor)$.
Ro1 has diagonally opposite corners at $(0, \lfloor Y\rfloor)$ and $(\lfloor X\rfloor, Y)$.
R1o has diagonally opposite corners at $(\lfloor X\rfloor, 0)$ and $(X, \lfloor Y\rfloor)$.
R11 has diagonally opposite corners at $(\lfloor X\rfloor, \lfloor Y\rfloor)$ and (X, Y) .
($\lfloor X\rfloor$ = floor(X) = the biggest integer no bigger than X .)

The first three subrectangles would cover equal areas of red and white, but the fourth would turn out to cover more red than white. (Do you see why?) This would contradict the inference that R covers as much red as white, so in fact at least one of X and Y must be an integer.

Problem 10: Suppose B and C are matrices, not necessarily square, of the same dimensions. Explain why $\det(I - B^T\cdot C) = \det(I - C\cdot B^T)$ wherein " I " is an appropriate identity matrix, not necessarily of the same dimensions in each instance, and superscript " ^{T} " means "transpose".

Solution 10: $\det(I - B^T\cdot C) = \dots$

$$= \det\left(\begin{bmatrix} I - B^TC & B^T \\ O & I \end{bmatrix} \cdot \begin{bmatrix} I & O \\ C & I \end{bmatrix}\right) = \det\left(\begin{bmatrix} I & B^T \\ C & I \end{bmatrix}\right) = \det\left(\begin{bmatrix} I & O \\ C & I \end{bmatrix} \cdot \begin{bmatrix} I & B^T \\ O & I - CB^T \end{bmatrix}\right) = \det(I - C\cdot B^T) .$$

Problem 11: $P(x)$ is a polynomial in x with coefficients all integers no bigger than K in magnitude; and $P(z) = 0$ for some $|z| \ge K+1$. What polynomial must $P(x)$ be, and why?

Solution 11: P must be the zero polynomial; *i.e.*, $P(x) = 0$ for all x. To see why, suppose $P(x) := c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ with coefficients c_j all integers no bigger than $K > 0$ in magnitude, and with $c_n \ne 0$ for the sake of argument. If $P(z) = 0$ then $n \ge 1$ and

$$|z|^n \le |c_n|\cdot|z|^n = |\sum_{j<n} c_j z^j| \le \sum_{j<n} |c_j|\cdot|z|^j \le K\cdot\sum_{j<n} |z|^j = K\cdot(|z|^n - 1)/(|z| - 1) .$$

And if $|z| \ge K+1$ too then $|z|^n \le K\cdot(|z|^n - 1)/(|z| - 1) \le K\cdot(|z|^n - 1)/K = |z|^n - 1$, which is not possible. Therefore every $c_j = 0$, as claimed.

The foregoing proof establishes that, if the coefficients c_j of $P(x)$ satisfy $|c_j/c_n| \le K$ for $0 \le j \le n-1$, then every zero z of P satisfies $|z| < K+1$. This is one of many theorems that restrict the zeros of a polynomial to a region in the complex plane determined relatively easily from the polynomial's coefficients. For more see M. Marden's book *The Geometry of the Zeros of a Polynomial in a Complex Variable* (1949) Amer. Math. Soc.

This problem arises from the following task: Suppose you are given a well-formed but horrendously complicated expression entailing finitely many additions, subtractions and multiplications involving integers and one variable x; and suppose all you wish to know is whether this expression simplifies to zero. The simplification process that reduces the given expression to a polynomial of the form $\sum_j c_j x^j$ with explicit integer coefficients c_j can take a long time, especially if the expression includes many subexpressions like $(813x + 709)^{587}$. However, a crude overestimate K of the magnitudes of all coefficients c_j can be calculated comparatively quickly; for instance, no coefficient of $(813x + 709)^{587}$ exceeds $(813+709)^{587}$. Although K may turn out to be a gargantuan integer, substituting $K+1$ for x and then calculating the given complicated expression may well take far less time than simplifying it, and the calculation's result will be zero just when the expression would simplify to zero. To speed up the calculation further, perform it in *modular arithmetic* several times, each time *modulo* a different large prime no wider than the computer's natural wordsize; the *Chinese Remainder Theorem* ensures that ordinary arithmetic would produce zero if and only if modular arithmetic produces zero for sufficiently many primes. How many is "sufficiently many" depends upon K and the (overestimated) degree of the given expression. The different primes can be employed concurrently by interleaving their modular arithmetics in one deeply pipelined processor and/or by exercising several processors in parallel, thus reducing the time until a decision is reached.

Problem 12: Let $y = Y(x)$ be any nontrivial solution of a second-order differential equation $(p\cdot y')' + r\cdot y/p = 0$ in which $(\ldots)' = d(\ldots)/dx$, and $p(x) > 0$, $r(x) > 0$ and $r'(x) > 0$ for all $x \ge 0$. Prove that $|Y(x)| < M$ for some finite constant M and all $x > 0$.

Proof 12: Set $£(x, y, z) := y^2 + (p(x)\cdot z)^2/r(x)$ and note that $£(x, Y, Y')' = -(p\cdot Y')^2\cdot r'/r^2 \le 0$, so $£(x, Y, Y')$ must decrease as x increases, and therefore $Y(x)^2 < M^2 := £(0, Y(0), Y'(0))$.

$£$ is called a *Lyapunov Function* because $£(x, Y, Y')$ decreases along every solution Y of the given differential equation. Such functions were introduced by A. Lyapunov early in the 20th century to characterize the stability of solutions of differential equations. If $£(x, y, z)$ decreases to 0 so must $|y|$, but this need not happen unless the behaviors of p and r are further constrained. Lyapunov functions can be too numerous to find easily.