Solutions to Putnam Exam Problems for 1 Dec.. 2001

A1) Consider a set S and a binary operation $*$ on S ; that is, $x*y$ is in S for each x and y in S . Assuming that $(x*y)*x = y$ for all x and y in S , prove that $x*(y*x) = y$ for all x and y in S .

Solution: The proof resembles the pattern-matching performed by *Macro Preprocessors* in Compilers for Computer Programming Languages like ***C*** . Matched patterns are underscored:

$$\underline{x}*(y*x) = \underline{((y*x)*y)}*(y*x) = (\underline{(y*x)}*y)*\underline{(y*x)} = y \text{ , as required.}$$

A2) $C_1, C_2, \ldots, C_n$ are n coins. For each k , coin C_k is biased so that, when tossed, it falls *Heads* with probability $1/(2k+1)$. If all n coins are tossed, what is the probability that the number of *Heads* will be odd? Express the answer explicitly as a rational function of n .

Solution: The probability is $P_n := n/(2n+1)$, as will be confirmed here by induction: Let $H_k := 1/(2k+1)$ and begin with $P_0 = 0$ and $P_1 = H_1 = 1/3$. For every $n > 0$ we find P_n by first flipping the first $n-1$ coins, getting an odd number of *Heads* among $n-1$ coins with probability P_{n-1} , even with probability $1-P_{n-1}$, and then we flip coin C_n to get an odd number of *Heads* among n coins with probability

$$P_n = H_n(1-P_{n-1}) + (1-H_n)P_{n-1} = H_n + (1-2H_n)P_{n-1} = (1 + (2n-1)P_{n-1})/(2n+1) \,.$$

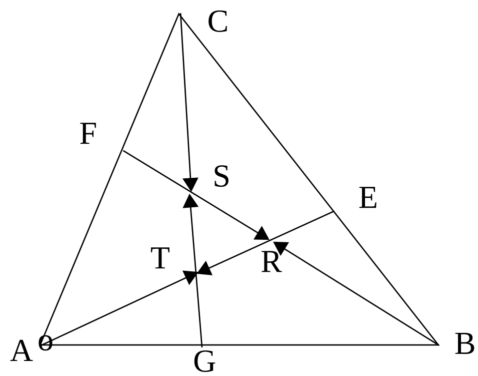
From this recurrence and the induction hypothesis $P_{n-1} = (n-1)/(2n-1)$ follows $P_n = n/(2n+1)$ as was claimed. (Where did the induction hypothesis come from? It was a guess generated by running the recurrence for several steps to see whether a pattern presented itself.)

A3) For each integer m the polynomial $P_m(x) := x^4 - (2m+4)x^2 + (m-2)^2$. For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

Solution: Either m is a perfect square, or $m/2$ is a perfect square. To see why, observe first that the factors of P_m are monic because P_m is monic; observe second that if $P_m(x)$ has a linear factor, say $(x-k)$, than $(x+k)$ must be a factor too because $P_m(-x) = P_m(x)$; and then (x^2-k^2) must be a factor. In other words, P_m factors into monic quadratic factors if it factors at all.

Now two cases must be considered according to whether each quadratic factor shares the sign-symmetry of P_m or not. In the first case, $P_m(x) = (x^2 - j)(x^2 - k)$ for some integers j and k (not necessarily positive for all we know now); matching coefficients makes $j+k = 2m+4$ and $jk = (m-2)^2$, whence follows $\{j, k\} = \{2(1+\sqrt{m/2})^2, 2(1-\sqrt{m/2})^2\}$ from which we infer that $m/2$ must be a squared integer. In the second case, when neither quadratic factor of P_m shares its sign symmetry, replacing x by $-x$ in the factorization of $P_m(x)$ must swap its factors thus: $P_m(x) = (x^2 + jx + k)(x^2 - jx + k)$ for some integers j and k . Matching coefficients again makes $j^2 - 2k = 2m+4$ and $k^2 = (m-2)^2$, whence follows $k = +(m-2)$ (but not $-(m-2)$ lest $j^2 = 8$) and $j = \pm 2\sqrt{m}$, so m must be a squared integer. In both cases P_m factors as shown.

A4) Triangle ABC has area 1 . Points E, F, G lie respectively on sides BC, CA, AB in such a way that AE bisects BF at point R , and BF bisects CG at point S , and CG bisects AE at point T . Find the area of triangle RST .



Solution: The area of RST is $2/(3+\sqrt{5})^2 = (7 - 3\sqrt{5})/4 \approx 0.072949\ldots$, gotten by brute force. Move the origin to A and identify the other letters with column vectors from this origin to the corresponding points. B and C shall be our basis vectors. For some positive scalars $\gamma, \phi, \xi, \tau, \sigma$ and ρ all less than 1 , the specifications of the problem put

$$G = \gamma B\ , \quad F = (1-\phi)C\ , \quad E = \xi C + (1-\xi)B\ ,$$

$$R = (F+B)/2 = (1-\rho)E\ , \quad S = (G+C)/2 = \sigma B + (1-\sigma)F\ , \quad T = E/2 = \tau C + (1-\tau)G\ .$$

Here γ, ϕ and ξ must be chosen to satisfy the equations involving R, S and T , which are …

$2R = (1-\phi)C + B = 2(1-\rho)(\xi C + (1-\xi)B)$, whence $1-\phi = 2(1-\rho)\xi$ and $1 = 2(1-\rho)(1-\xi)$.

Eliminate ρ to obtain the equation $\xi = (1-\phi)/(2-\phi)$.

$2S = \gamma B + C = 2\sigma B + 2(1-\sigma)(1-\phi)C$, whence $\gamma = 2\sigma$ and $1 = 2(1-\sigma)(1-\phi)$.

Eliminate σ to obtain the equation $\phi = (1-\gamma)/(2-\gamma)$.

$2T = \xi C + (1-\xi)B = 2\tau C + 2(1-\tau)\gamma B$, whence $\xi = 2\tau$ and $1-\xi = 2(1-\tau)\gamma$.

Eliminate τ to obtain the equation $\gamma = (1-\xi)/(2-\xi)$.

Apparently $\gamma = \phi = \xi = 2/(3 + \sqrt{5}) = (3 - \sqrt{5})/2 \approx 0.382$, the root of $\xi^2 - 3\xi + 1 = 0$ between 0 and 1 . Now the edge-vectors of the triangle RST turn out to be …

$$[R-T, S-T] = [B, C]\begin{bmatrix} \xi & 2\xi - 1 \\ 1 - 2\xi & 1 - \xi \end{bmatrix}/2\ , \text{ whence}$$

$$\text{Area}(RST) = \text{Area}(ABC)\cdot\det(\begin{bmatrix} \xi & 2\xi - 1 \\ 1 - 2\xi & 1 - \xi \end{bmatrix})/4 = \xi^2/2 \text{ after simplification.}$$

A5) Prove that the equation $x^{n+1} - (x+1)^n = 2001$ determines its positive integer solutions x and n uniquely.

Solution: $x = 13$ and $n = 2$ satisfy the equation. To show that it has no other positive integer solutions we accumulate constraints upon x and n as follows: First, $x^{n+1} - (x+1)^n \equiv 0 \ \mathbf{mod}\ 3$; consequently $x \equiv 1 \ \mathbf{mod}\ 3$ and n must be even since trying $x \equiv 0$ or $x \equiv -1$ or n odd leads to contradictions. Of course $x > 1$. Second, x divides $x^{n+1} - (x+1)^n + 1 = 2002 = 2\cdot7\cdot11\cdot13$; consequently x must be one of 7, 13, 22, 91, 154, 286 or 2002. Moreover $x \equiv -1 \ \mathbf{mod}\ (x+1)$

and $n+1$ is odd, so $2002 \equiv 0 \ \mathbf{mod}\ (x+1)$, which means $x+1$ divides 2002 too. Of the seven possibilities listed above for x, only $x = 13$ satisfies this last constraint. Then $n = 2$ works, as a few minutes of arithmetic confirm. Can any other even n satisfy that given equation? Not if n is very big, since $13^{n+1} - 14^n < 0$ for all sufficiently large n. A computer or programmable calculator could save time otherwise dissipated in thought by testing values of n until they got sufficiently large; but that is disallowed, so let us persist in our analysis. Suppose $n = 2m+2$ worked for some integer $m > 0$; this would mean $2001 = 13^3 - 14^2 = 13^{2m+3} - 14^{2m+2}$, whence $(14^{2m} - 1)14^2 = (13^{2m} - 1)13^3$. This last equation would require 13^3 to divide

$$14^{2m} - 1 \ = \ (1+13)^{2m} - 1 \ = \ 2m\cdot 13 + m(2m-1)\cdot 13^2 + (\dots)\cdot 13^3 \ ,$$

whence 13^2 would have to divide $m\cdot(2 + (2m-1)\cdot 13)$, and therefore m would have to be a multiple of 169. But when $m \ge 169$ we must have $n = 2m+2 \ge 340$; and then (with $x = 13$)

$$2001 = \ x^{n+1} - (x+1)^n \ = \ x^n\cdot\big(x - (1 + 1/x)^n\big) \ \le \ x^n\cdot\big(13 - (1 + 1/13)^{340}\big) \ < 0 \ .$$

It can't happen.

A6) Can an arc of a parabola inside a circle of radius 1 have length greater than 4 ?

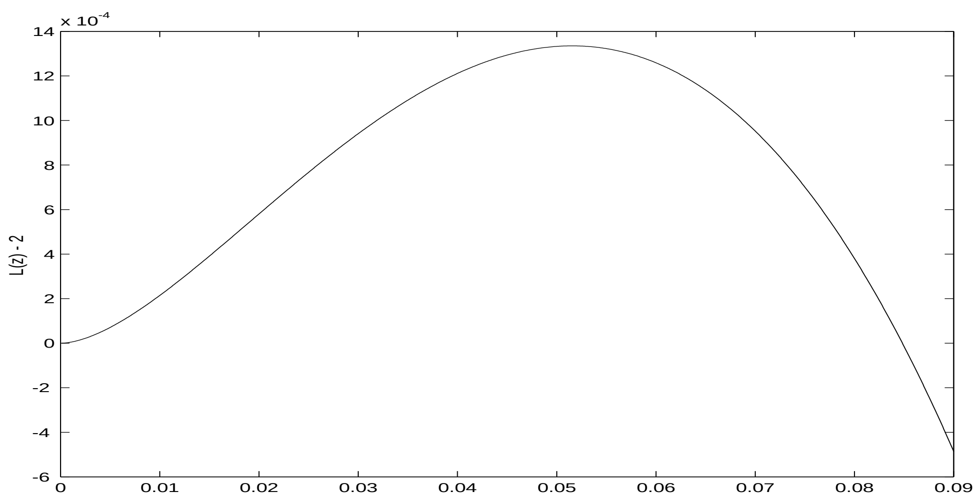
Solution: YES is the short answer. The long answer is tedious and will be only outlined. In the (x,y)-plane where the circle's equation is $y^2 = 1-x^2$, the parabola in question has the equation $y^2 = 4z^2(1+x)$ for a very tiny constant $z > 0$. The arc in question runs from $x = -1$ up to $x = T := 1 - 4z^2$. Let us consider only the upper half of that arc, the half above the x-axis, since the lower half is just the upper half's reflection in the x-axis. We shall show that this half-arc's length L exceeds 2 for all z tiny enough. In fact, along this half-arc whereon $y = 2z\sqrt{1+x}$,

$$\begin{aligned} L(z) &= \int_{-1}^{T} \sqrt{\big(1 + (dy/dx)^2\big)}\, dx = \int_{-1}^{T} \sqrt{\big(1 + z^2/(1+x)\big)}\, dx \quad \text{at } T = 1 - 4z^2 \\ &= \sqrt{(T+1)}\cdot\sqrt{(T+1 + z^2)} + z^2\cdot\ln\big(\big(\sqrt{(T+1)} + \sqrt{(T+1 + z^2)}\big)/z\big) \\ &= \sqrt{(2 - 4z^2)}\cdot\sqrt{(2 - 3z^2)} + z^2\cdot\ln\big(\big(\sqrt{(2 - 4z^2)} + \sqrt{(2 - 3z^2)}\big)/z\big)\ . \end{aligned}$$

$L(z) > 2$ for all z tiny enough because $L(z) \to 2$ as $z \to 0+$ and, differentiating,

$$L'(z)/z = 2\cdot\ln\big(\big(\sqrt{(2 - 4z^2)} + \sqrt{(2 - 3z^2)}\big)/z\big) - 8\sqrt{\big((2 - 3z^2)/(2 - 4z^2)\big)} \ \to +\infty \ \text{ as } z \to 0+ \ .$$

<u>L(z) – 2</u> :



$L(z)$ need not be obtained explicitly to establish that $L(0) = 2$ and determine that $L'(z)/z > 0$ for all z tiny enough, but the foregoing computations of L and L' require no cleverness and help to satisfy curiosity about how much the half-arc length $L(z)$ exceeds 2 and for which z. Numerical computation indicates that $L(z) > 2$ only while $0 < z < 0.08483287\ldots$, and $\max\{L\}$ is $L(0.051546\ldots) = 2.001335\ldots$. The short answer YES is barely true.

Alternate Solution: Due to Austin Shapiro, it does not require explicit evaluation of an integral unfamiliar to many calculus students. Instead his solution reduces problem A6 to a familiar divergent series. Instead of drawing a sufficiently narrow parabolic arc inside the unit circle, his solution fixes the parabola and draws a circle sufficiently big to enclose an arc of that parabola longer than twice the circle's diameter. The equation of the fixed parabola P is $y = x^2$. The equation of the circle C_n is $x^2 + (y-n)^2 = n^2$ for a big positive integer n to be determined later.

Because C_n has diameter $2n$ and center at $(x, y) = (0, n)$, circle C_n and parabola P intersect tangentially at $(x, y) = (0, 0)$, where their common tangent is the x-axis. Their two other intersections are at $(x, y) = (\pm\sqrt{2n-1}, 2n-1)$. As x runs from $-\sqrt{2n-1}$ to $+\sqrt{2n-1}$, the point $(x, y) = (x, x^2)$ on the arc of P within C_n runs down from $(-\sqrt{2n-1}, 2n-1)$ to $(0, 0)$ and back up to $(+\sqrt{2n-1}, 2n-1)$, staying within C_n because, just on that arc,

$$x^2 + (y-n)^2 - n^2 = x^2 + (x^2-n)^2 - n^2 = -x^2(2n-1-x^2) \le 0 .$$

Since P and C_n are each its own reflection in the y-axis, we can solve problem A6 by showing that the length L_n of the half-arc of P within the right-hand D-shaped semicircle of C_n exceeds its diameter $2n$ when n is big enough. L_n exceeds the sum of the lengths of secants joining consecutive intersections of P with C_k for $k = 0, 1, 2, \ldots, n$ in turn; consequently

$$L_n - 2n \ge \sqrt{2} + \sum_{1\le k\le n-1} \sqrt{\left((\sqrt{2k+1} - \sqrt{2k-1})^2 + ((2k+1) - (2k-1))^2 \right)} - 2n .$$

To further reduce the right-hand side the inequality $(\sqrt{u} - \sqrt{v})/(u-v) = 1/(\sqrt{u} + \sqrt{v}) > 1/\sqrt{2u + 2v}$, easily proved valid for all distinct positive u and v, is applied twice in succession to obtain

$$\begin{aligned} L_n - 2n &> \sqrt{2} + \sum_{1\le k\le n-1} \sqrt{\left((2/\sqrt{8k})^2 + (2)^2 \right)} - 2n \\ &= \sqrt{2} - 2 + \sum_{1\le k\le n-1} \left(\sqrt{\left(1/(2k) + 4 \right)} - 2 \right) \\ &> \sqrt{2} - 2 + \sum_{1\le k\le n-1} (1/(2k))/\sqrt{16 + 1/k} > \sqrt{2} - 2 + \left(\sum_{1\le k\le n-1} 1/k\right)/\sqrt{68} . \end{aligned}$$

The last $(\sum\ldots)$ is the harmonic series and diverges as $n \to +\infty$; therefore $L_n - 2n \to +\infty$ so that $L_n - 2n > 0$ for all sufficiently big n. (Actually any $n \ge 60$ is sufficiently big.) End of proof.

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B1) Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ into the squares of an n-by-n grid so that the k-th row, from left to right, is

$$(k-1)n + 1, \quad (k-1)n + 2, \quad \dots, \quad (k-1)n + n .$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black. (A checkerboard coloring is one possibility.) Prove that, for any such coloring, the sum of the numbers on the red squares equals the sum of the numbers on the black squares.

Solution: Let $N(k, j) := (k-1)n + j$ be the number written into column j of row k for $1 \le j \le n$ and $1 \le k \le n$. Set $S(k,j) := +1$ if red is the color in column j of row k ; set $S(k, j) := -1$ if the color is black. $\sum_j S(k, j) = 0$ for every k since each row has as many $+1$s as -1s ; similarly $\sum_k S(k, j) = 0$ for every j . Problem B1 is solved by proving that $\sum_j \sum_k S(k, j)\cdot N(k, j) = 0$;

$$\sum_j \sum_k S(k, j)\cdot N(k, j) = \sum_j \sum_k S(k, j)\cdot((k-1)n + j) = \sum_j \sum_k S(k, j)\cdot(k-1)n + \sum_j \sum_k S(k, j)\cdot j$$
$$= \sum_k \left(\sum_j S(k, j)\right)\cdot(k-1)n + \sum_j \left(\sum_k S(k, j)\right)\cdot j = 0 + 0 \quad \text{as claimed.}$$

B2) Find all pairs of real numbers (x, y) satisfying the system of equations

$$1/x + 1/(2y) = (x^2 + 3y^2)(3x^2 + y^2) \quad \text{and} \quad 1/x - 1/(2y) = 2(y^4 - x^4) .$$

Solution: Adding and subtracting the given equations and multiplying up transforms them into $2 = x\cdot Q(x^2, y^2)$ and $1 = y\cdot Q(y^2, x^2)$ wherein $Q(x, y) := x^2 + 10xy + 5y^2$. Then we find $2\pm1 = x\cdot Q(x^2, y^2) \pm y\cdot Q(y^2, x^2) = (x\pm y)^5$, whence follows $x+y = \sqrt[5]{3}$ and $x-y = 1$. Therefore the sole solution-pair (x, y) has $\underline{x = (\sqrt[5]{3} + 1)/2}$ and $\underline{y = (\sqrt[5]{3} - 1)/2}$.

B3) For any positive integer n let $s(n)$ denote the integer closest to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} (2^{s(n)} + 2^{-s(n)})/2^n .$$

Solution: The sum is 3 . Why? Observe first that $s(n) = m$ just when $m(m-1) < n \le m(m+1)$ because $(m \pm 1/2)^2 = m(m \pm 1) + 1/4$. Therefore the range $1 \le n < \infty$ of summation can be broken into disjoint subintervals of the form $T(m)+1 := m(m-1) + 1 \le n \le m(m+1) = T(m+1)$ for $m = 1, 2, 3, \dots$. Then the sum in question takes the form

$$\begin{aligned}
\sum_1^{\infty} (2^{s(n)} + 2^{-s(n)})/2^n &= \sum_{m=1}^{\infty} \sum_{n=T(m)+1}^{T(m+1)} (2^{s(n)} + 2^{-s(n)})/2^n \\
&= \sum_{m=1}^{\infty} \sum_{n=T(m)+1}^{T(m+1)} (2^m + 2^{-m})/2^n \\
&= \sum_{m=1}^{\infty} (2^m + 2^{-m}) \sum_{n=T(m)+1}^{T(m+1)} 2^{-n} \\
&= \sum_{m=1}^{\infty} (2^m + 2^{-m}) (2^{-T(m)} - 2^{-T(m+1)}) \quad \dots \text{ geom. series' sum} \\
&= \sum_{m=1}^{\infty} (2^{-m(m-2)} - 2^{-m(m+2)}) \quad \text{after some algebra} \\
&= \sum_{m=1}^{\infty} 2^{-m(m-2)} - \sum_{M=3}^{\infty} 2^{-M(M-2)} \\
&= 2 + 1 , \quad \text{as claimed.}
\end{aligned}$$

B4) Let S denote the set of rational numbers other than –1, 0 and 1 . Define $f : S \to S$ by $f(x) := x - 1/x$. Prove or disprove that $\cap_1^\infty f^{(n)}(S) = \varnothing$ wherein the n-fold composition $f^{(n)} := \underbrace{f \circ f \circ \ldots \circ f}_{\leftarrow\ n\ \rightarrow}$, and $f^{(n)}(S)$ is the set of all values taken by $f^{(n)}(s)$ as s ranges over S .

Solution: We shall prove that the intersection $\cap_1^\infty f^{(n)}(S)$ is empty. The set S consists of rational numbers m/n with integers $n \neq 0$, $m \neq 0$, $|m| \neq |n|$ and $GCD(|m|, |n|) = 1$. For all such rational numbers define $K(m/n)$ to be the sum of the exponents in the prime factorization of $|m \cdot n|$. For example, $K(-8/45) = 6$. Observe that $f(m/n) = (m-n)(m+n)/(m \cdot n)$ lies in S whenever m/n does, and that $K(f(m/n)) > K(m/n)$ because $GCD(|(m-n)(m+n)|, |m \cdot n|) = 1$. Therefore every s in $f^{(n)}(S)$ has $K(s) > n$; and each s in $f^{(n)}(S)$ is absent from $f^{(K(s))}(S)$.

B5) Let µ and ß be given real numbers strictly between 0 and 1/2 , and let g be a continuous real-valued function such that $g(g(x)) = \mu \cdot g(x) + ß \cdot x$ for all real x . Prove that $g(x) = c \cdot x$ for some constant c .

Solution: Let ı denote the identity function $ı(x) = x$ and let g_n denote the n-fold composition $g_n(x) = g(g_{n-1}(x))$ starting from $g_0 = ı$ and $g_1 = g$. Now the given equation takes the form $g_2 = \mu \cdot g_1 + ß \cdot g_0$, whence repeated substitution yields $g_{n+1} = \mu \cdot g_n + ß \cdot g_{n-1}$, and therefore $[g_{n+1}, g_n] = [g_n, g_{n-1}]G = [g, ı]G^n$ where $G := \begin{bmatrix} \mu & 1 \\ ß & 0 \end{bmatrix}$. Its eigenvalues are $ç := (\sqrt{(\mu^2+4ß)} + \mu)/2$ and $¢ := -(\sqrt{(\mu^2+4ß)} - \mu)/2$ with $0 < -¢ < ç < 1$, $0 < ç+¢ = \mu < 1/2$ and $0 < ß = -¢ \cdot ç < 1/2$. Its eigenvalue/vector decomposition leads to $G^n = \begin{bmatrix} 1 & -1 \\ -¢ & ç \end{bmatrix} \cdot \begin{bmatrix} ç^n & 0 \\ 0 & ¢^n \end{bmatrix} \cdot \begin{bmatrix} ç & 1 \\ ¢ & 1 \end{bmatrix} /(ç-¢)$ for all integers n , positive and negative. Evidently $G^n \to O$ as $n \to +\infty$, and consequently so does $[g_{n+1}(x), g_n(x)] = [g(x), x]G^n \to [0, 0]$ for each real x . Because g is continuous, we may take the limit in the equation $g(g_n(x)) = g_{n+1}(x)$ to infer that $g(0) = 0$. However, if $x \neq 0$ then $0 \neq ß \cdot x = g(g(x)) - \mu \cdot g(x)$ and therefore $g(x) \neq 0$; and then $g_n(x) \neq 0$ for every $n \geq 0$.

More generally, g takes every value in its range just once. To see why, suppose $g(x) = g(x')$; then $x = (g(g(x)) - \mu \cdot g(x))/ß = (g(g(x')) - \mu \cdot g(x'))/ß = x'$. Consequently the function inverse to g must be $g_{-1}(y) := (g(y) - \mu \cdot y)/ß$; for any y in the range of g , the equation $y = g(x)$ has just the one solution $x = g_{-1}(y)$, and this g_{-1} is continuous too. Therefore g must be a strictly monotonic function, either strictly increasing or strictly decreasing; and since $g(0) = 0$ we infer that the sign of $g(x)/x$ must be the same for all $x \neq 0$. Now two cases must be considered:

In the first case, $g(x)/x < 0$ for all $x \ne 0$. Then we can infer that $g(x) = ¢\cdot x$ as follows: For any x for which $g(x) - ¢\cdot x \ne 0$ we would find from the eigen-decomposition of G^n that

$$[g_{n+1}(x), g_n(x)]/ç^n \to [g(x), x]\begin{bmatrix}1 & -1\\ -¢ & ç\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}ç & 1\\ ¢ & 1\end{bmatrix}/(ç-¢) = [ç, 1](g(x) - ¢\cdot x)/(ç-¢) \ne [0, 0] ,$$

whence $g_{n+1}(x)/g_n(x) \to ç > 0$, when actually every $g_{n+1}(x)/g_n(x) = g(g_n(x))/g_n(x) < 0$. This contradiction implies that $g(x) = ¢\cdot x$ for all x if ever $g(x)/x < 0$.

In the second case $g(x)/x > 0$ for all $x \ne 0$. In this case we wish to let $n \to -\infty$ in the formula $[g_{n+1}, g_n] = [g, ı]G^n$, interpreting $g_n(x) = g_{-1}(g_{n+1}(x))$ as the $|n|$-fold composition of the inverse function g_{-1} when $n < 0$. Before doing so, let's find out whether the range of g , the domain of g_{-1} , is the whole real axis. Because g is strictly increasing, it could be bounded above by a finite least upper bound L only if $g(x) \to L$ as $x \to +\infty$; but then taking limits in the equation $g(g(x)) = \mu\cdot g(x) + ß\cdot x$ would lead to the contradiction $L \ge g(L) = \mu\cdot L + \infty$. Similarly we infer that g is unbounded below. Therefore the domain of g_{-n} is the whole real axis for $-n = -1$ and then for all $-n \le -1$. Moreover, the formula $g(y) = \mu\cdot y + ß\cdot g_{-1}(y)$ that defined g_{-1} can be composed to yield $g_{-n+1} = \mu\cdot g_{-n} + ß\cdot g_{-n-1}$ which, when rewritten $[g_{-n+1}, g_{-n}] = [g_{-n}, g_{-n-1}]G$, vindicates the formula $[g_{1-n}, g_{-n}] = [g, ı]G^{-n}$ we shall use to deduce that $g(x) = ç\cdot x$: For any x for which $ç\cdot x - g(x) \ne 0$ we would find from the eigen-decomposition of G^{-n} that as $n \to +\infty$

$$[g_{1-n}(x), g_{-n}(x)]\cdot¢^n \to [g(x), x]\begin{bmatrix}1 & -1\\ -¢ & ç\end{bmatrix}\cdot\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}ç & 1\\ ¢ & 1\end{bmatrix}/(ç-¢) = [¢, 1](ç\cdot x - g(x))/(ç-¢) \ne [0, 0] ,$$

whence $g_{1-n}(x)/g_{-n}(x) \to ¢ < 0$, when actually every $g_{1-n}(x)/g_{-n}(x) = g(g_{-n}(x))/g_{-n}(x) > 0$. This contradiction implies that $g(x) = ç\cdot x$ for all x if ever $g(x)/x > 0$. End of proof for B5.

Of all the problems on this exam, I think B5 comes closest to what Mathematicians like me do for a living.

B6) Assume that $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an increasing sequence of positive real numbers such that $a_n/n \to 0$ as $n \to +\infty$. Must there exist infinitely many positive integers n such that $a_{n-j} + a_{n+j} < 2a_n$ for $j = 1, 2, \ldots,$ and $n-1$?

Solution: Yes. To see why, plot the points (n, a_n) in the (x, y)-plane and let C be the upper boundary of the convex hull of all those points. C consists of segments of lines lying above all those points except for the points that lie on C . Could only finitely many of them lie on C ? If so, there would be a last point, say (N, a_N) , beyond which C would be a semi-infinite line segment lying barely above all points (n, a_n) with $n > N$. Therefore this segment's slope would be $s := \sup_{n>N} (a_n - a_N)/(n-N) > 0$. But $(a_n - a_N)/(n-N) \to 0$ as $n \to +\infty$ because $a_n/n \to 0$, so actually $s = \max_{n>N} (a_n - a_N)/(n-N) = (a_M - a_N)/(M-N)$ for some $M > N$, implying that (N, a_N) could not be the last point on C after all. Therefore C passes through infinitely many boundary vertices (n, a_n) . Each such boundary vertex lies above every line segment joining two points $(n-j, a_{n-j})$ and $(n+j, a_{n+j})$ on or under C for $1 \le j < n$; this means $a_n > (a_{n-j} + a_{n+j})/2$ for every $j = 1, 2, \ldots, n-1$, as the problem requires.

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I am no exception to the complaint "Everybody wants to be an editor". I have taken the liberty of redrafting some of the problems on this year's Putnam Exam. There are several reasons to do so. One is that I eschew jargon as much as possible because several of my students are not yet comfortable with it; for instance, I prefer the sequence " a_1, a_2, a_3, …, a_n, … " to " $(a_n)_{n\ge1}$ " and "n-by-n" to "nxn". I avoid using an unadorned " *a* " as a variable lest it be confused with the indefinite article, and decline to omit "and" from " for j = 1, 2, …, and n–1 " lest someone read "or" in its place. I use the future tense instead of the present in "If all n coins are tossed, what is the probability that the number of *Heads* will be odd?" because after the toss the probability is either 0 or 1 . The last two commas in "Points E, F, G lie, respectively, on sides …" are undeserved. "Prove that there are unique positive integers …" is too easy because every integer is unique. In B3 I typed "s(n)" instead of using symbols absent from some computers' fonts. I use " := " for assignment or definition to distinguish it from the predicate " = ". And so on. I am a nit-picker; I would have to be one to solve B5 fully correctly. W. K.