Problem 1: Suppose the real function $f(x)$ is continuous on the real x-axis and satisfies thereon $2f(x) = \int_{x-1}^{x+1} f(t)\,dt$ and $f(x) \to 0$ as $|x| \to \infty$. Prove that $f \equiv 0$.

Proof 1: If f could take positive values, it would have to take its maximum value $M = f(\bar{x})$ at some real $\bar{x}$ since $f \to 0$ at the ends of the real axis. But then $M = f(\bar{x}) = \int_{\bar{x}-1}^{\bar{x}+1} f(t)\,dt/2 \le M$ with equality in the last inequality only if the continuous $f(x) = M$ throughout the interval $\bar{x}-1 \le x \le \bar{x}+1$. By induction, $f \equiv M$ everywhere on the real axis instead of $f \to 0$ at the ends. Therefore f cannot take positive values; negative values are ruled out similarly. End of proof.

This is R. Chapman's solution of M. Chamberland's problem; see p. 678 of SIAM Review 38 #4 (Dec. 1996). A weaker hypothesis that $f(x) \to 0$ as $x \to +\infty$ also implies $f \equiv 0$, but the proof is far harder.

Problem 2: The equation of a *Central Quadric Surface* in *Euclidean* n-space is $\mathbf{x}^T H \mathbf{x} = 1$ wherein H is a real symmetric n-by-n matrix. Suppose no eigenvalue of H is repeated. Prove that any *Rectangular Parallelepiped* all of whose vertices lie on that surface must have edges parallel to all *Principal Axes* of that surface. (The principal axes run along eigenvectors of H .)

Proof 2: A rectangular parallelepiped centered at $\mathbf{c}$ has 2^n vertices $\mathbf{c} \pm \mathbf{r}_1 \pm \mathbf{r}_2 \pm \mathbf{r}_3 \pm \dots \pm \mathbf{r}_n$ where $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots$ and $\mathbf{r}_n$ are mutually orthogonal vectors of nonzero perhaps diverse lengths, so $\mathbf{r}_i^T\mathbf{r}_j = 0$ if $i \ne j$. No generality is lost by assuming that the vectors $\mathbf{r}_j$ are all parallel to coordinate axes since this can be accomplished by an orthonormal change of coordinates. That all vertices lie on the surface implies 2^n equations of the form $\mathbf{v}^T H \mathbf{v} = 1$ as $\mathbf{v}$ runs over all vertices $\mathbf{c} \pm \mathbf{r}_1 \pm \mathbf{r}_2 \pm \mathbf{r}_3 \pm \dots \pm \mathbf{r}_n$. Let's look at equations from four vertices of the form $\mathbf{u} \pm \mathbf{r}_i \pm \mathbf{r}_j$:
$(\mathbf{u}-\mathbf{r}_i-\mathbf{r}_j)^T H(\mathbf{u}-\mathbf{r}_i-\mathbf{r}_j) = (\mathbf{u}-\mathbf{r}_i+\mathbf{r}_j)^T H(\mathbf{u}-\mathbf{r}_i+\mathbf{r}_j) = (\mathbf{u}+\mathbf{r}_i-\mathbf{r}_j)^T H(\mathbf{u}+\mathbf{r}_i-\mathbf{r}_j) = (\mathbf{u}+\mathbf{r}_i+\mathbf{r}_j)^T H(\mathbf{u}+\mathbf{r}_i+\mathbf{r}_j) = 1$.
These imply first that $(\mathbf{u}\pm\mathbf{r}_i)^T H \mathbf{r}_j = 0$ and then that $\mathbf{r}_i^T H \mathbf{r}_j = 0$, which tells us that all the off-diagonal elements of H vanish in an orthonormal coordinate system whose axes are parallel to the edges of the rectangular parallelepiped. Then the diagonal elements of H are its eigenvalues and, since they are assumed distinct, the coordinate axes are its eigenvectors each determined uniquely to within a nonzero scalar multiplier. End of proof.

Problem 3: The equation of an *Ellipsoid* in *Euclidean* n-space is $\mathbf{x}^T H \mathbf{x} = 1$ wherein H is a real symmetric n-by-n matrix with all n eigenvalues positive and $n \ge 3$. Suppose no eigenvalue of H is repeated, and suppose the (hyper)plane whose equation is $\mathbf{p}^T\mathbf{x} = 0$ for some $\mathbf{p}^T \ne \mathbf{o}^T$ intersects that ellipsoid in a circle (if $n = 3$), sphere (if $n = 4$), or hypersphere (if $n \ge 5$). Prove that $n = 3$.

Proof 3: Without loss of generality (do you see why?) suppose the orthonormal coordinate system has $\mathbf{p}$ as its last coordinate vector. Then the (hyper)plane of vectors satisfying $\mathbf{p}^T\mathbf{x} = 0$

consists of vectors $\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ 0 \end{bmatrix}$ each of which has zero for its last component. In that (hyper)plane the equation of the circle or (hyper)sphere of intersection with the ellpsoid is $\mathbf{x}^T H \mathbf{x} = \mu \cdot \mathbf{y}^T \mathbf{y} = 1$ for some scalar $\mu > 0$. Therefore $H = \begin{bmatrix} \mu I & \mathbf{c} \\ \mathbf{c}^T & \beta \end{bmatrix}$ for some column $\mathbf{c} \neq \mathbf{o}$ (else H would have μ as a repeated eigenvalue) and scalar ß. The characteristic polynomial of H turns out to be $\det(\theta I - H) = (\theta-\mu)^{n-2} \cdot \big((\theta-\mu)(\theta-ß) - \mathbf{c}^T\mathbf{c} \big)$. (Can you see why?) Now we find that $n-2 \le 1$ lest μ be a repeated eigenvalue of H. Therefore $n = 3$ as claimed.

But the proof is not yet complete. How do we know for $n = 3$ that some nonzero $\mathbf{p}$ exists whose plane intersects the ellipsoid in a circle? Here is how such a $\mathbf{p}$ can be found. Since H has three distinct positive eigenvalues, we consider vectors $\mathbf{p}$ orthogonal to the middle eigenvalue's eigenvector. As $\mathbf{p}$ rotates in the plane **E** of the two extreme eigenvalues' eigenvectors, the plane **P** whose equation is $\mathbf{p}^T\mathbf{x} = 0$ rotates about the middle eigenvector, which always lies in **P**. It intersects the ellipsoid in an ellipse one of whose principal axes runs along that middle eigenvector. The other principal axis varies continuously, while lying in the plane **E**, from the shortest to the longest principal axis of the ellipsoid. For some two pairs of directions of $\pm\mathbf{p}$ this varying principal axis must coincide in length with the middle principal axis; either choice for $\pm\mathbf{p}$ makes the ellipse a circle. End of proof.

Problem 4: A particle moves on the real x-axis with varying velocity $v(x)$ in such a way that the uniformly weighted average velocity over any interval $y \le x \le z$ is determinable from the velocities at the interval's ends thus: Average $= (2/3)\cdot\big(v(y)^2 + v(y)\cdot v(z) + v(z)^2 \big)/\big(v(y) + v(z)\big)$. Must the particle's acceleration be constant? Explain why or why not.

Solution 4: The short answer is "Yes, the acceleration must be constant." Here is why: Let $x(t)$ be the position of the particle at time t, so that $v(x(t)) = x'(t)$. If $x(\theta) = y$ and $x(\tau) = z$, the Average in question is

$$\text{Average} = \int_y^z v(x)\,dx/(z-y) = \int_\theta^\tau v(x(t))^2 dt\,/\big(x(\tau) - x(\theta)\big) = \int_\theta^\tau v(x(t))^2 dt\,/\!\int_\theta^\tau v(x(t))\,dt\ .$$

Abbreviate $V(t) := v(x(t))$ to write the relation between the Average and the end-velocities thus:

$$3\cdot(V(\theta) + V(\tau))\cdot\int_\theta^\tau V(t)^2 dt = 2\cdot\big(V(\theta)^2 + V(\theta)\cdot V(\tau) + V(\tau)^2 \big)\cdot\int_\theta^\tau V(t)\,dt\ .$$

Differentiate with respect to τ three times and then set $\theta := \tau$ to get simply $3\cdot V(\tau)^2\cdot V''(\tau) = 0$. Since V varies it cannot be zero everywhere, so $V'' = 0$, which implies that the acceleration V' must be constant, except possibly for a jump where $V = 0$. But can $V = 0$ somewhere? …

This, M. Renardy's solution of M.S. Klamkin's problem, appears on pp. 525-6 of *SIAM Review* **38** #3 (1996) but with no consideration of whether v can vanish nor where on the x-axis the given relation with Average can hold.

Certainly $v(x)$ cannot reverse sign lest, wherever the particle reversed course, $v(x)$ would not be a single-valued function of position x. Therefore the velocity could vanish only at a point where the particle had slowed to a stop and then accelerated away without reversing direction, giving the graph of $V(t)$ a V-shape. But this, as it turns out, would violate the given relationship with the Average. If $V \neq 0$ everywhere, V' would be everywhere constant, forcing $V' \equiv 0$; do you see why? Therefore the given Average relationship can hold on only part of the real x-axis.

Problem 5: Suppose the letters "O" and "T" are drawn with infinitesimally thin lines. Show that uncountably infinitely many copies (in diverse sizes) of the letter "O" can be placed in the plane without overlapping, but only countably infinitely many copies of "T" can be so placed. (This problem was supplied by Prof. C.C. Pugh.)

A *Countable* (or *Denumerable*) infinite set is one whose members can each be labelled with an integer without using any integer more than once. The existence of uncountably infinite sets was first exposed by Cantor at the end of the nineteenth century; his first example was the set of real numbers between 0 and 1 .

Solution 5: Concentric circles with all real radii strictly between any two positive real numbers provide uncountably infinitely many non-overlapping copies of the letter "O" . To prove that uncountably infinitely many non-overlapping copies of the letter "T" cannot be placed in the plane, let us assume otherwise for the sake of argument in order to exhibit a contradiction:

Consider first the set of all circular disks with integer radii and with centers at points with integer coordinates. There are countably infinitely many such disks, and every copy of "T" lies inside some of them. Since every union of countably many countable sets is countable, some such disks would have to contain uncountably many copies of "T" . Let D be one such disk.

The copies of "T" inside D shall be classified by size as follows: Each copy of "T" will be characterized by four real parameters— two for the coordinates of the T's crossing, one for the direction in which the copy's tail points, and one for the multiplier μ by which this copy is a magnification (if $\mu \ge 1$) or reduction (if $\mu < 1$) of some standardized T . In other words, figuratively speaking, each copy will be expressed as $\mathbf{t} + \mu \cdot e^{\imath\theta} \cdot \mathrm{T}$ where $\mathbf{t}$ is a translation and $e^{\imath\theta}$ a rotation. The multipliers μ will be collected into *Binades* each of the form $2^n \le \mu < 2^{n+1}$ for an integer n . Each copy of "T" inside D has its multiplier μ in one such binade. If, as is alleged in the previous paragraph, D contained uncountably many copies of "T" , then some binade(s) would have to contain multipliers μ for infinitely many copies of "T" ; otherwise D could contain at most countably many copies of "T" . For definiteness suppose the binade $1 \le \mu < 2$ contained infinitely many such multipliers (not necessarily all distinct), so that D would contain infinitely many copies of "T" of sizes determined by multipliers μ between 1 and 2 . This turns out to be contradictory;

in fact D can contain only finitely many copies of "T" of those sizes.

Here is why:

The binade's copies $\mathbf{t} + \mu \cdot e^{\imath\theta} \cdot \mathrm{T}$ of "T" in D are identified by parameters constrained to satisfy

$$\mathbf{t} \text{ lies in } D\,,\quad -\pi \le \theta < \pi\,,\quad \text{and}\quad 1 \le \mu < 2\,. \qquad (*)$$

Any three copies $\mathbf{t}_j + \mu_j \cdot \exp(\imath\theta_j) \cdot \mathrm{T}$ for j = 1, 2, 3 are further constrained by the requirement that no two copies intersect. Define a function $f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \theta_1, \theta_2, \theta_3, \mu_1, \mu_2, \mu_3)$ to be the diameter of the smallest circle that contains the three crossing-points $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ of any three copies of "T" . This f is a continuous function of twelve real arguments (including two per $\mathbf{t}_j$) , and as they vary subject to the foregoing constraints, namely (*) and nonintersection of the three copies of "T" , so f varies over some interval of nonnegative real numbers. Set ß := *inf f* , the largest number no larger than any number in that interval. We must show next that ß > 0 .

As an *infimum*, ß must be the limit of an infinite sequence of values of f evaluated at arguments that satisfy all the constraints. Because all the arguments are drawn from a *Bounded* and *Closed* region in a 12-dimensional space (*closed* by replacing the inequalities $\theta < \pi$ and $\mu < 2$ in (*) by $\theta \le \pi$ and $\mu \le 2$ resp.), its *Compactness* implies that there must be a subsequence of those arguments convergent to a limit-point $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \theta_1, \theta_2, \theta_3, \mu_1, \mu_2, \mu_3)$ inside or on the boundary of the region. This limit-point determines three copies of "T" , none more than twice as big as any other, none crossing another though they may touch. For instance, one "⊥" could lie atop another "T" with their crossing-points coincident. This would have to happen if ß = 0 ; but then there would be no way to position the third copy of "T" so that its crossing-point would coincide with the other two. Therefore ß > 0 , as was claimed at the end of the last paragraph. We conclude that …

> Every circle that has diameter less than ß can contain at most two crossing-points of nonintersecting copies of "T" whose multipliers μ lie between 1 and 2 .

Some finite number of circular disks of diameter less than ß can be arranged to cover disk D . None of these covering disks can contain more than two crossing-points of D's nonintersecting copies of "T" with multipliers in $1 \le \mu < 2$, and yet every such crossing-point lies in at least one covering disk. Therefore D can contain only finitely many such copies of "T" . End of proof.

The foregoing proof takes for granted that every copy of "T" has its tail (|) and cross-bar (—) in the same proportions as has the standardized "T" , so that the copy's size is determined by one multiplier μ . The proof becomes a little more complicated if copies of "T" are allowed to have different proportions but bounded so that no copy's shape can come arbitrarily close to " | " nor to "—" .

Learn about continuity, limits, countable and uncountable sets, "infimum", "closed", "compact" (the Heine-Borel theorem), and coverings in a course on *Real Variables* (Math. 104) taught from a text like C.C. Pugh's lively new book *Real Mathematical Analysis* (2002, Springer-Verlag, New York).