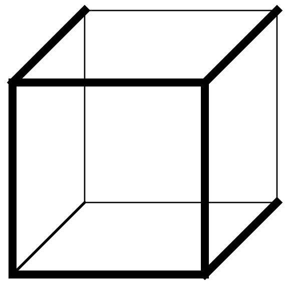
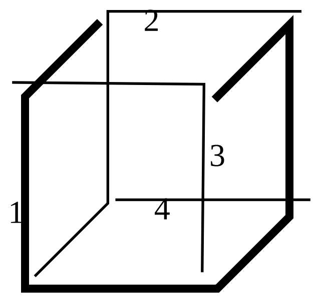
You don’t have to solve them all.

Problem 0: A Wire-Frame Cube is constructed out of idealized wire of infinitesimal thickness run along the edges of the cube and fastened at its corners thus:



It can be assembled out of four pieces of wire as shown below just before they are joined at the corners:



Note that no edge is traversed more than once by any wire or wires. Subject to this constraint, can the wire-frame cube be assembled out of fewer than four pieces of wire? Justify your answer.

Solution 0: NO; the desired construction needs at least four pieces of wire. Here is why: The cube has eight vertices. At each vertex either one wire-end joins a wire’s interior (perhaps of the same piece of wire), or else three wire-ends join. Therefore the pieces of wire must have at least eight ends all told. Each piece has two ends, so there must be at least four pieces of wire.

Problem 1: Mathematicians who disparage ambiguity sensitize themselves to its symptoms so as to detect and correct it. Many other intellectuals remain insensitive. For example, intelligence tests promulgated by *American Mensa*, a self-styled “American High I.Q. Society”, have been notorious for unintended ambiguities that elicit “incorrect” responses from test takers more imaginative and intelligent than the test makers, thus thwarting the tests’ ostensible purpose.

The following three questions, framed by a *Mensa* psychologist, came from the back of a box of *Raisin Bran*. For each question devise answers, as many as you can, all at least as valid as the one answer the psychologist deemed “correct”.

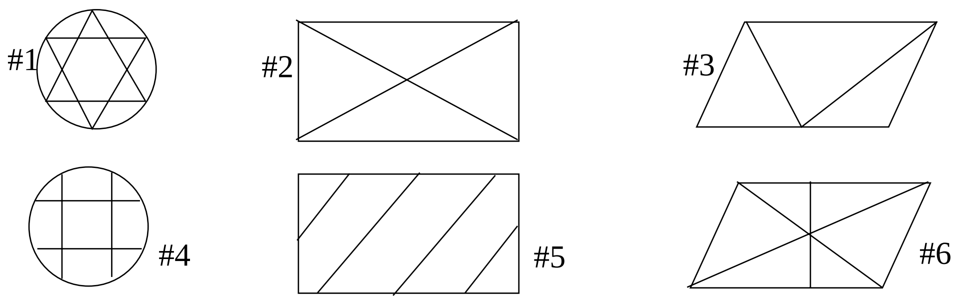
1a: Which one of the following five words doesn't belong with the others, and why?

pail skillet knife suitcase card .

Answers:

card	All the rest have handles. (The expected answer)
"card"	All the rest include the letter "i" .
"card"	All the rest have two or more vowels.
suitcase	None of the others separates its empty interior from its exterior.
"suitcase"	has a 4-bit number 1000 (in binary) of letters; others have 3- .
"suitcase"	has four vowels; all others have one or two.
"suitcase"	is the only word obtained by adjoining two words with no overlap.
"knife"	has a silent consonant "k"; others have no silent consonants.

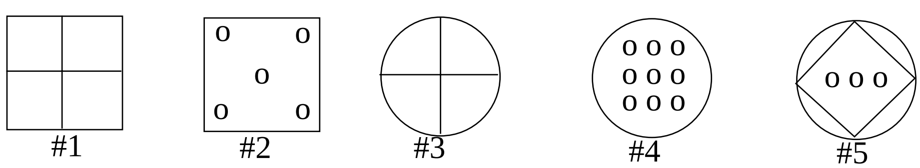
1b: One of the six figures below lacks a characteristic common to the other five figures. Which is that one, and why?



Answers:

#4	is the only one containing no triangles. (The expected answer)
#4	is the only one rotated onto itself by a quarter-turn ($\pi/2$) .
#4	is the only one containing a number of internal regions not divisible by its label.
#5	is the only one containing no intersecting lines strictly inside it.
#6	is the only one containing an intersection of three lines strictly inside it.
#3	is the only one not rotated onto itself by a half-turn (π) .
#3	is the only one none of whose internal regions is congruent to any other.
#2	is the only one with an obtuse triangle in it.
#1	contains more internal regions than any other.
#1	is the only one rotated onto itself by a sixth of a turn ($\pi/3$) .

1c: One of the following five diagrams doesn't fit with the others; which one, and why?



Answers:

#5	is the only one not rotated onto itself by a quarter-turn ($\pi/2$) . (Expected)
#5	is the only one with a square inside a circle.
#1	is the only one without any circle.
#4	is the only one consisting entirely of circles.
#2	is the only one all of which appears on a face of one of a pair of dice.

The design of valid tests requires more knowledge, guile and imagination than the design or production of whatever is being tested, be it hardware, software, medicine, food or education. Especially objectionable are exams reliant exclusively upon multiple-choice questions that will be graded entirely mechanically. Such exams prevent examinees from handling flawed questions intelligently, and incline their examiners to deny that their questions could be flawed.

Similar issues undermine the value of scores from *Teaching Surveys*; I derive more value from what is revealed by students' thoughtful comments. More valuable again are responses to *Exit Surveys* that ask both graduates and drop-outs which instructors most influenced, helped or hindered them, and why.

The Putnam Competition Exam will be set and graded entirely by humans. They can become intolerant of an inconsiderate examinee's illegible handwriting or unintelligible assertions, so rewriting your solution carefully to clarify it can pay off, especially if it exposes a lapse you overlooked earlier. If you think a question is flawed, don't ask the proctor about it; response to such a query is forbidden. Instead either skip the question or else explain what you think the question should have asked and answer that.

Worse than test questions with unintended answers are testing policies with mostly unintended consequences. While purporting to impose "Accountability" upon school teachers, "No Child Left Behind" is actually corroding the American school system by forcing teachers to teach not the subject matter but instead how to psych out the tests. Instead of encouraging students to learn and cultivating each one's talents for leadership, this misguided policy exacerbates the impediments students must overcome to survive the educational system with their native intellectual endowments not abraded. I remember how that worked:

Unlike most of my schoolmates, I enjoy histories and biographies. We were turned off history by the way it was taught in high school. It was not the teacher's fault. I still remember and admire him. But the format of the province-wide History exam forced him to stress non-historical aspects of the syllabus. For instance, if the exit exam asked for the causes of the American Revolution, we had to list 20 causes; neither 19 nor 21 would be deemed correct. All of us disdained the minds that dreamt up those requirements, but I overcame them and even earned a prestigious scholarship. Most of my schoolmates remember about history only how much they disliked it.

Problem 2: Violins produced on the island of Grxcd have become collectors' items since it sank into the sea two centuries ago. All the island's violins were produced by Bropcs or one of his sons, or by Czwyz or one of his sons. Every violin was labelled ostensibly to reveal its maker but, although Bropcs and his sons always labelled their violins truthfully, Czwyz and his sons always labelled their violins with falsehoods. Both families playfully interfered with collectors' attempts to establish provenances for their violins. For example, collectors figured out that a violin labelled " This violin was not made by any son of Bropcs." was made by Bropcs Sr.; can you see why? The most desirable violins are so labelled that a connoisseur can tell that it must have been made by one of the fathers, either Bropcs Sr. or Czwyz Sr., but cannot tell which. How might such a violin be labelled?

Solution 2: " Made by Bropcs himself, or by a son of Czwyz." (… among other possibilities)

Problem 3: What are the minimum and maximum numbers of times that Friday the 13th can occur in the same calendar year, and why?

Solution 3: The 13th day of a month falls on Friday at least once and at most thrice in any calendar year. Why? Friday the 13th occurs only in a month whose first day is Sunday. Let's renumber weekdays **mod** 7 , making Sun. ≡ 0 , Mon. ≡ 1 , Tues. ≡ 2 , … , Sat. ≡ 6 **mod** 7 . Suppose Jan. 1 falls on a weekday numbered n **mod** 7 ; on what days of the week will the other months begin? Table 1 answers this question for a calendar year that is not a Leap-year:

Table 1: Non-Leap-Years

Month:	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec..
# days in Month:	31	28	31	30	31	30	31	31	30	31	30	31
# days **mod** 7 :	3	0	3	2	3	2	3	3	2	3	2	3
1st day **mod** 7 :	n	n+3	n+3	n+6	n+1	n+4	n+6	n+2	n+5	n	n+3	n+5

How often in this calendar year does a month begin on any specified day-of-the-week? Table 2 counts these occurrences from the last row of Table 1 :

Table 2: Frequencies

n	n+1	n+2	n+3	n+4	n+5	n+6
2	1	1	3	1	2	2

Therefore, in any calendar year not a leap-year, a month will begin on Sunday at least once (when Jan. 1 falls on a Sat., Fri. or Wed.) and at most thrice (when Jan. 1 falls on Thurs.).

Tables 3 and 4 exhibit the same calculations for a leap-year:

Table 3: Leap-Years

Month:	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec..
# days in Month:	31	29	31	30	31	30	31	31	30	31	30	31
# days **mod** 7	3	1	3	2	3	2	3	3	2	3	2	3
1st day **mod** 7:	n	n+3	n+4	n	n+2	n+5	n	n+3	n+6	n+1	n+4	n+6

Table 4: Frequencies

n	n+1	n+2	n+3	n+4	n+5	n+6
3	1	1	2	2	1	2

Therefore, in any leap-year, a month will begin on Sunday at least once (when Jan. 1 falls on a Sat., Fri., Wed. or Tues.) and at most thrice (when Jan. 1 falls on Sun.). Finished.

Problem 4: This "proof" explains why division by a variable called " y " is always mistaken: Define $f(x, y) := (x + y)^2$ and then substitute $x = u - v$ and $y = u + v$ to determine that

$$\partial f/\partial x = \partial f/\partial y = 2(x+y)\ ,\quad \partial x/\partial v = -1\ ,\quad \text{and}\quad \partial y/\partial v = +1\ .$$

Now the *Chain Rule* implies

$$\partial f/\partial v = (\partial f/\partial x)\cdot(\partial x/\partial v) + (\partial f/\partial y)\cdot(\partial y/\partial v) = 2(x+y)\cdot(-1) + 2(x+y)\cdot(1) = 0\ .$$

But the definition of $f(u, v) = (u + v)^2$ implies also $\partial f/\partial v = 2(u+v) = 2y$. Therefore $y = 0$. This appears to preclude division by a variable named " y ". Where is the flaw in the "proof" ?

Solution 4: Of course, this flawed argument does not prove $y = 0$. The flaw arises from the careless use of the same name for two different functions. One is $f(x, y) := (x + y)^2$. The other is $F(u, v) := f(u-v, u+v) = (2u)^2$. The chain rule delivered not " $\partial f/\partial v$ " but $\partial F/\partial v = 0$. The second appearance of " $\partial f/\partial v$ " refers not to $\partial F/\partial v$ but to what was computed earlier as $\partial f/\partial y$ though now with (u, v) in place of (x, y) . The notation " $\partial f/\partial v$ " is ambiguous; better is to let $f_x(x, y) := \partial f(x, y)/\partial x$ and $f_y(x, y) := \partial f(x, y)/\partial y$ define as functions f_x and f_y the partial derivatives of f , and then invoke the chain rule in the form

$$\partial f(x, y)/\partial v = f_x(x, y)\cdot(\partial x/\partial v) + f_y(x, y)\cdot(\partial y/\partial v) = 2(x+y)\cdot(-1) + 2(x+y)\cdot(1) = 0.$$

We should all try to avoid notational ambiguities. Unfortunately, ambiguity probably cannot be avoided entirely since the set of all worthwhile ideas may well be uncountable whereas the set of all utterances is surely at most countably infinite. Thefore some distinct ideas have to share an utterance that can be disambiguated only by context. Let's try not to make the situation worse.

(This "proof" that $y = 0$ came from the *College Math. Journal* **25** #1 (Jan. 1994) p. 35.)

Problem 5: Theorem?: $n^n - n^2 + n - 1$ is divisible by $(n-1)^3$ for every integer $n > 1$. Proof?: If $n > 2$ factor $n^n - n^2 + n - 1 = (n-1)\cdot P(n)$ where $P(n) := 1 + n^2(1 + n + \dots + n^{n-4} + n^{n-3})$. Since $n^{n-3} \equiv n^{n-4} \equiv \dots \equiv n^2 \equiv n \equiv 1 \ \mathbf{mod}\ (n-1)$, we find $P(n) \equiv 1 + n(n-2) = (n-1)^2 \ \mathbf{mod}\ (n-1)$, and therefore P(n) is divisible by $(n-1)^2$, whence follows the claim in the "Theorem".

How do you reconcile the alleged "proof" with what happens to the "Theorem" when $n = 3$?

Solution 5: The mistake is deducing from " $P(n) \equiv 1 + n(n-2) = (n-1)^2$ " that P(n) is divisible by $(n-1)^2$; actually $P(n) \equiv 0 \ \mathbf{mod}\ (n-1)$ is all that has been proved. Corrected, the theorem should say " $n^n - n^2 + n - 1$ is divisible by $(n-1)^2$ for every integer $n > 1$, and divisible by $(n-1)^3$ when n is *even*." Can you prove it now? Hint: $P(n) = (n-1)\cdot(n^{n-2} + 2n^{n-3} + 3n^{n-4} + \dots + ?\ )$.

Problem 6**:** Theorem?: π^e is rational. Proof?: Observe first that $\log_\pi r$ is irrational for every rational r because otherwise, were $s = \log_\pi r$ rational, we would find $\pi^s = r$ and hence π would be algebraic, contradicting the known transcendence of π . Now suppose, for the sake of argument, that $\pi^e \ne r$ for every rational r . This would mean $e = \log_\pi \pi^e \ne \log_\pi r$ for every rational r , implying that e is not an irrational number because of the first observation. But e is known actually to be irrational. The contradiction establishes that $\pi^e = r$ for some rational r .

Is the foregoing "proof" correct? If not, what is wrong with it?

Solution 6**:** The proof is mistaken because the countable subset of irrational numbers $\log_\pi r$ generates, as r runs through all rational numbers, does not constitute the uncountable set of *all* irrational numbers, to which e belongs, so no contradiction has been exposed. π^e is probably irrational, since thousands of its decimal digits have been computed with no periodicity apparent, but nobody knows for sure yet.

This spoof-proof devised by C. Counts appeared in *The College Mathematics Journal* **24** #3 (May 1993) p. 229. It exemplifies a kind of logical lapse committed all too often, especially by propagators of letters sent to prestigious universities' mathematics departments announcing or containing lengthy alleged proofs of Fermat's Last Theorem, or Goldbach's Conjecture, or the Twin-Primes Conjecture, or a procedure to TrisectAngles, or … . Sometimes such a letter will offer a reward to anyone who exposes a flaw in the alleged proof, but the reward will amount to meager compensation for time spent trying to explain the flaw well enough to persuade its perpetrator.

Problem 7: Theorem?: If $1+2+3+\ldots+n = n(n+1)/2$ for all positive integers n then $n = 1$.
Proof?: If $n \ge 2$ then replace n in the theorem's true hypothesis by $n-1$ to get the equation

$$1 + 2 + 3 + \ldots + (n-1) = (n-1)n/2 .$$

Add 1 to both sides of the equation, producing

$$1 + 2 + 3 + \ldots + n = (n-1)n/2 + 1 .$$

Invoke the theorem's hypothesis again to turn this equation into $n(n+1)/2 = (n-1)n/2 + 1$, whose only finite root is $n = 1$; thus the theorem is confirmed. Where did this "proof" go wrong?

Solution 7: "Add 1 to both sides of the equation, producing …" actually produces "$1 + 2 + 3 + \ldots (n-2) + (n-1) + 1$" instead of "$1 + 2 + 3 + \ldots (n-2) + (n-1) + n$" .

This silly spoof was produced by R. Euler for *The College Mathematics Journal* **24** #3 (May 1993) p. 229.

Problem 8: Two candidates stand for election to a parliamentary seat in ancient Braczia. Each candidate votes for himself by placing a ballot in his one of two big glass bowls. Then, in turn, each of another 10000 Braczian voters places his ballot in the bowl of his choice. But because so many voters like to vote for a winner, the probability is $m/(m+n)$ that the next ballot will go into a bowl containing m ballots already when the other bowl contains n ballots. Choose a bowl before the voting starts; what is the probability that fewer than a quarter of the 10000 ballots cast will go into that bowl?

Solution 8: The requested probability is $1/4.0004$. Here is why:

The following random process is probabilistically equivalent to the balloting. Construct a long horizontal trough much longer then long enough to hold 10000 glass marbles. Throw a red marble into the trough, and then throw in 10000 white marbles one at a time at random, each as likely to fall into any position as into any other between or beyond previously thrown marbles.

When there are $m-1$ white marbles to the left and $n-1$ to the right of the red marble, the next white marble will fall to the left of the red with probability $m/(m+n)$, with probability $n/(m+n)$ to the right, thus mimicking the next Braczian ballot's deposit into the left or right bowl. After the 10000th white marble has been thrown into the trough it will hold 10001 marbles in some order as likely as any other. Therefore the red marble is as likely to sit in one position as in any other of 10001 positions among the marbles. There are 2500 positions with fewer than 2500 white marbles to the left, say, of the red marble. Therefore the probability that fewer than a quarter of the white marbles lie to the left of the red marble is $2500/10001$, so this is the requested probability.

This problem was supplied by Computer Science Prof. Umesh Vazirani.

Problem 9: Every member of the Braczian parliament serves on at least one of its committees, of which there are more than there are members of parliament. Every committee has just three members; no two committees have the same membership. Explain why parliament has at least 5 members, and then why at least one pair of committees must share exactly one member.

Solution 9: Here is David Blackston's solution of a problem adapted from the 1979 U.S. Math. Olympiad. Let M be the number of members of parliament. $M \geq 5$ because otherwise the maximium possible number of distinct three-member committees, namely $M(M-1)(M-2)/6$, would not exceed M . To simplify the explanation, discard any member who serves on only one committee, and discard that committee too even if it shared just one member with another; the diminished parliament still has more committees than members, and another pair of committees that have just one member in common will be shown to exist. In other words, we can assume with no loss of generality that $M \geq 5$ and that every member serves on at least two committees.

For argument's sake *Assume* further that no two committees share just one member; this *Assumption* will be shown to lead to a contradiction:

Since at least 3(M+1) committee seats have to be filled by M members, at least one member must serve on four or more committees; let A be such a member, and let { A, B, C } and { A, B, D } be two of his four or more committees.

Can { A, C, X } be a third? If so, X could not be B (because committees are distinguishable) nor different from D (because of the *Assumption*); therefore three of A's committees would have to be { A, B, C }, { A, B, D } and { A, C, D } , and a fourth would have to include some other member E . But, no matter who served as the third member of this fourth committee, it would violate the *Assumption* by sharing only member A with some other of his committees:

Table 5: Four Committees on which A would serve

Committee 1	A	B	C	—	—	—
Committee 2	A	B	—	D	—	—
Committee 3	A	—	C	D	—	—
Committee 4	A	?(3)	?(2)	?(1)	E	?(1,2,3)

(The last row's parentheses show which committee(s) would share exactly one member with committee 4 if it included the member listed above in the corresponding column.)

Therefore the *Assumption* would imply that A can serve on no second committee with C nor, for similar reasons, with D . Instead, B must serve on each of A's committees and, by symmetrical reasoning, A must serve on each of B's committees:

Table 6: Four Committees on which A could serve

Committee 1	A	B	C	—	—	—
Committee 2	A	B	—	D	—	—
Committee 3	A	B	—	—	E	—
Committee 4	A	B	—	—	—	F

This would leave no other committee for C to serve on without violating the *Assumption*; but all members who, like C , can serve on only one committee were presumed to have been discarded at the beginning. Therefore the *Assumption* must be inconsistent, and the desired explanation is complete.

Alternative Solution 9: This gem is due to Robert Mena. Suppose parliament has M members and N committees; N > M > 0 . Let M-by-N arrray C have, in its row #i and column #j ,

$c_{ij} := 1$ just when parliament member #i belongs to committee #j ; otherwise $c_{ij} := 0$.

Every column of C has exactly three 1's because every committee has three members. Next let N-by-N matrix $H := C' \cdot C$. Here C' is C's transpose. In row #i and column #j of H we find

$h_{ij} =$ the number of members belonging to both committees #i and #j .

Every diagonal element $h_{jj} = 3$. Every off-diagonal element $h_{ij} < 3$ because no two committees have the same members. Therefore H 's off-diagonal elements are all drawn from {0, 1, 2} . The problem is to prove that some $h_{ij} = 1$. The proof goes by contradiction:

For argument's sake suppose no element of H were 1 . Then $H \equiv I \textbf{ mod } 2$; here I is the N-by-

N identity matrix. Consequently $\det(H) \equiv 1$ **mod** 2 . But $\det(H) = 0$ because the rank of a matrix product cannot exceed the rank of any factor, so $\mathrm{rank}(H) \le \mathrm{rank}(C) \le M < N$.

Can you see why the assumption that committees outnumber members is essential for the problem's conclusion and not just the proofs?

Problem 10: Protagoras, who taught Law at his school in ancient Greece, accepted a penniless but bright student on condition that the student pay all of the school's fee after winning his first case at law. The student agreed to this contract but took no law cases after graduating. Instead he made a lot of money dealing in real estate. After a while Protagoras sued him for payment of the school's fee, arguing …

- "If the court rules in my favor it must compel this former student to pay me.
  If I lose this case, the student will have won his first case and the court must enforce
  the contract to which he agreed, and compel him to pay me."

The former student refused to pay, arguing …

- "If I win this case the court has sided with me, so I need not pay the fee.
  If I lose this case the contract between Protagoras and me does not oblige me to pay."

This is one of the earliest paradoxes on record. How can the court best resolve it?

Solution 10: Protagoras loses, and must pay the court's costs, after which he can sue again and win. Then the court will enforce the contract and compel the former student to pay the fee owed plus the court's costs. Turning one lawsuit into two eliminates the paradox and sets a profitable precedent for courts to cope with similar contracts, should any be brought before the court. This resolution does not violate prohibitions of *Double Jeopardy*; they apply only to criminal cases brought by the state. Neither do laws against *Barratry* apply to the second lawsuit in so far as the contract renders it inevitable under the circumstances.

Why did the former student choose to represent himself? Had he hired a lawyer instead. the court would surely have ruled in his favor, but then he would have had to pay his lawyer.

Problem 11: Several gentlemen seated at a round table in a restaurant are reputed each to be a proficient jewel thief though none has been caught nor convicted. They are all suspects in the theft of a fabulous diamond, the *Blue Napoleon*, from a museum. Evidence at the scene has convinced the police that only one thief took the diamond but, knowing nothing more, the police have placed all these gentlemen under surveillance. These, surmising correctly that a microphone has been planted at their table, have said nothing about the theft though they are curious to know *whether* one among them stole the diamond. However, to protect the thief from betrayal, they wish not to know *who* stole it. By prearrangement, each gentleman flips a coin and, using a menu or a napkin, hides it from everyone except himself and his neighbor on the right at the table. Then each gentleman winks one eye or the other (the policeman watching could see only some of them do it) for a few seconds after which, from the expressions of satisfaction on their faces, the policeman watching has inferred that they know whether one among them stole the diamond but, if one did, not who did it.

How did they communicate just that information?

Solution 11: In principle each gentleman could communicate two bits of information by winking one eye, both or neither. Thus each gentleman could reveal the status, heads or tails, of both his coin and his right-hand neighbor's; that information would be useless unless the thief, if he is among them, alters his two bits. But then discrepancies with his two neighbors would identify him. A less risky policy is to communicate just one bit per gentleman. Here is one way to do it:

Each gentleman sees how his coin and his neighbor's on the right have turned up, heads or tails, either the same or different. If different he winks his left eye; if the same he winks his right eye; but if he stole the diamond he does the opposite. The number of pairs of adjacent coins that differ must be even; do you see why? Consequently the gentlemen know that the thief is one among them if they see an odd number of left eyes winked; but then they cannot know for sure who the thief is. *Honor among Thieves* precludes that any gentleman would wink the wrong eye.

The foregoing communications protocol was suggested by David Chaum in 1988. Note that "several" exceeds two.

Problem 12: A computerized algebra system has supplied a simple but incorrect closed-form formula for

$$\int_0^{\arctan(t)} \sqrt{2\cdot\tan(x)}\,dx \;=\; \arctan(\sqrt{2}t/(1-t)) - \log(\,(t + \sqrt{2}t + 1)/\sqrt{(1 + t^2)}\,) \quad \text{for all } t \ge 0\,.$$

What's wrong with it? Supply a similarly simple but correct closed-form formula.

Solution 12: The derivative of the formula shown above does "simplify" to $\sqrt{2t}/(1 + t^2)$, which is the derivative of the integral, but the integral is a continuous function of all $t \ge 0$ whereas the formula is discontinuous; it jumps by $-\pi$ as t increases through 1. The formula is wrong. Computerized algebra systems like *Mathematica*, *Maple* and *Derive* are fallible, alas.

Several correct formulas for the integral $J(t) := \int_0^{\arctan(t)} \sqrt{2\cdot\tan(x)}\,dx = \int_0^t \sqrt{2\tau}\,d\tau/(1 + \tau^2)$ have been coaxed out of different computerized algebra systems:

$J(t) = \pi/2 - \arcsin((1-t)/\sqrt{(1+t^2)}) - \log((t+\sqrt{2t}+1)/(1+t^2))$ from Maple V r3 on a Mac.

$J(t) = \pi - \arccos((t-1)/\sqrt{(1+t^2)}) - \log((t+\sqrt{2t}+1)/(1+t^2))$ from Maple 7.

$J(t) = \arctan(\sqrt{2t}-1) + \arctan(\sqrt{2t}+1) + \log((t-\sqrt{2t}+1)/(t+\sqrt{2t}+1))/2$
from Derive 4.11, Macsyma 2.3, Maple 7 & 11, Mathematica 4.2.

The incorrect formula can be derived from this last formula by applying the trigonometric identity

$$\arctan(y) + \arctan(z) = \arctan(\,(y+z)/(1 - y\cdot z)\,) \quad \text{provided } y\cdot z < 1$$

and ignoring the *proviso*. Can you see how this identity must be changed when $y\cdot z \ge 1$?

Here is a derivation *by hand* of the last correct formula: Substitute $\tau := 2w^2$ into $J(t)$ to get

$$\begin{aligned}\int \sqrt{2\tau}\,d\tau/(1 + \tau^2) &= 8\int w^2\,dw/(1 + 4w^4) = 32\int w^2\,dw/\big(((2w+1)^2 + 1)\cdot((2w-1)^2 + 1)\big)\\ &= \int (4w-2 + 2)\,dw/((2w-1)^2 + 1) - \int (4w+2 - 2)\,dw/((2w+1)^2 + 1)\\ &= \log((2w-1)^2 + 1)/2 + \arctan(2w-1) - \log((2w+1)^2 + 1)/2 + \arctan(2w+1)\\ &= \log\big((t-\sqrt{2t}+1)/(t+\sqrt{2t}+1)\big)/2 + \arctan(\sqrt{2t}-1) + \arctan(\sqrt{2t}+1)\,.\end{aligned}$$

END.

Problem 13: Let $g(s) := 8s^3 - 4s^2 - 4s + 1$; why is $g(\sin(\pi/14)) = 0$?
Hint: First let $s^2 = (1-c)/2$ to confirm that $g(\sin(\pi/14))\cdot g(-\sin(\pi/14)) = g(\cos(\pi/7))$.

Solution 13: Observe that $\sin(\pi/14) = \cos(3\pi/7)$. If $g(\cos(3\pi/7)) = 0$ then $g(\cos(\pi/7)) = 0$ too, apparently. To create a polynomial that vanishes at $\cos(k\pi/7)$ for small odd integers k set

$$f(z) := (z^7 + 1)/(z^4 + z^3) = z^3 + z^{-3} - z^2 - z^{-2} + z + z^{-1} - 1 = (z+z^{-1})^3 - (z+z^{-1})^2 - 2(z+z^{-1}) + 1 ,$$

and then substitute $x := (z+z^{-1})/2$, so $z = x \pm \sqrt{(x^2-1)}$, to get $f(z) = f\big(x \pm \sqrt{(x^2-1)}\big) = g(x)$. Now, $f(z) = 0$ just when $z = \exp(k\pi\imath/7) = \cos(k\pi/7) \pm \imath\cdot\sin(k\pi/7)$ for $k = \pm1, \pm3$ or ±5 ; and then $x = \cos(k\pi/7)$ to make $g(x) = 0$. Therefore

$$\begin{aligned} g(s) &= 8\big(s - \cos(\pi/7)\big)\big(s - \cos(3\pi/7)\big)\big(s - \cos(5\pi/7)\big) \\ &= 8\big(s - \sin(5\pi/14)\big)\big(s - \sin(\pi/14)\big)\big(s + \sin(3\pi/14)\big) . \end{aligned}$$

End of answer.

Maple 11 simplified $g(\sin(\pi/14))$ to zero but other automated algebra software didn't.

Problem 14: In Euclidean 3-space a *Lattice-Point* is one whose three Cartesian coordinates are all integers. What is the least integer n with the property that, no matter which n lattice points be chosen, at least one line segment whose end-points are both among the n chosen lattice points contains in its interior another lattice point (chosen or not)? Why?

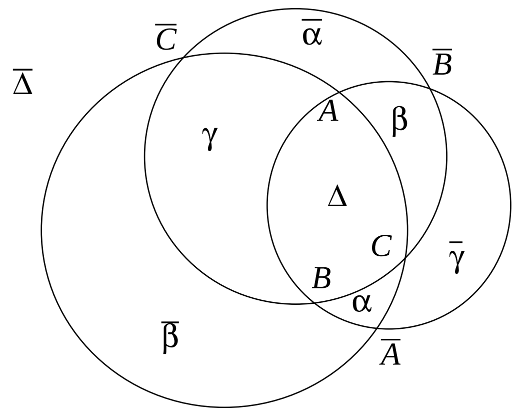
Solution 14: $n = 9$. To see why $n > 8$, examine a cube's eight vertices (x, y, z) with every coordinate either 0 or 1 ; no lattice-point can lie between two of these vertices. To see why $n = 9$, consider nine lattice-points' coordinates (x, y, z) **mod** 2 . Only eight possibilities exist, so some two chosen points have coordinates with respectively matching parities (even or odd), and the midpoint between these two chosen points is a lattice-point.

Problem 15: Over 2250 years ago Archimedes proved that the (surface of the) sphere of radius 1 has area 4π . This area is divided into two hemispheres of equal area 2π by any *Great Circle*, the intersection of the sphere with a plane through its center. Two great circles intersect in *Antipodal* points on the sphere and divide its area into four *Lunes*. Three nonconcurrent great circles divide the sphere into eight *Spherical Triangles*, which come in four pairs of antipodal triangles of equal areas. Let Δ be one of those triangles and also its area; and let A, B and C be that triangle's vertices and also the radian angles at those vertices. Prove that Area $\Delta = A + B + C - \pi$. (The proof is short and needs no trigonometry.)

Solution15: Begin the proof by observing that the area of each lune between two great circles is proportional to the angle at its two antipodal vertices. If this angle is measured in radians it must be half the lune's area, which falls strictly between 0 and 2π .

Let $\overline{\Delta}$ be the spherical triangle antipodal to Δ. Because each of these triangles is the other's reflection through the sphere's center, they have the same area $\overline{\Delta} = \Delta$; moreover the angles at antipodal vertices $\overline{A}$, $\overline{B}$ and $\overline{C}$ are the same respectively as at A, B and C.

Now, to obtain names for all eight spherical triangles and their vertices, we need a picture. To this end, *Stereographic Projection* from a point strictly inside $\overline{\Delta}$ onto the plane tangent to the sphere at the antipodal point inside Δ projects each great circle onto a circle in the plane. It projects seven of the spherical triangles onto curvilinear triangles in the plane; the eighth, $\overline{\Delta}$, is projected onto the plane outside all the other seven curvilinear triangles. Here is how such a projection might look:



Names α, β and γ have been given to three spherical triangles and their areas each of which is complementary to Δ in a lune; for example, *colunar* triangles Δ and α form a lune whose vertices are A and $\overline{A}$, and whose area is $2A$. The $\overline{\text{overstrikes}}$ mark antipodal images equal in angle or area. The three lunes that overlap Δ have areas

$2A = \alpha + \Delta$, $2B = \beta + \Delta$ and $2C = \gamma + \Delta$.

Add these together and note that antipodal area $\overline{\gamma} = \gamma$, and note too that $\alpha + \beta + \overline{\gamma} + \Delta = 2\pi$ is the area of a hemisphere, to complete the proof of

Girard's Theorem (1629): Area $\Delta = A + B + C - \pi$.

The last expression has come to be known as the triangle's *Spherical Excess* because the area of a spherical triangle on the unit sphere is the amount by which the sum of the spherical triangle's angles exceeds the sum π of every plane triangle's angles.

Problem 16: The *Disordered Inequality* is worth knowing: It says if $x_0 \le x_1 \le x_2 \le \ldots \le x_N$ and $y_0 \le y_1 \le y_2 \le \ldots \le y_N$, and if $\pi(\ldots)$ is any permutation of the integers $0, 1, 2, 3, \ldots, N$, then $\sum_k x_k \cdot y_{N-k} \le \sum_k x_k \cdot y_{\pi(k)} \le \sum_k x_k \cdot y_k$. Prove it.

Proof 16: The first inequality is an application of the second after replacing every y_k by $-y_k$, and the second is proved by *undoing the permutation* through a sequence of swaps thus: Whenever some $j < k$ but $\pi(j) > \pi(k)$, change $\pi(\ldots)$ by swapping $\pi(j)$ and $\pi(k)$. Doing so replaces the middle sum's terms $x_j \cdot y_{\pi(j)} + x_k \cdot y_{\pi(k)}$ by $x_j \cdot y_{\pi(k)} + x_k \cdot y_{\pi(j)}$ and increases that sum, if it changes, because $(x_j \cdot y_{\pi(k)} + x_k \cdot y_{\pi(j)}) - (x_j \cdot y_{\pi(j)} + x_k \cdot y_{\pi(k)}) = (x_k - x_j) \cdot (y_{\pi(j)} - y_{\pi(k)}) \ge 0$. Repeating such swaps ultimately *Bubble-sorts* the permutation $\pi(\ldots)$ into increasing order, turning it into the identity and increasing the middle $\sum\ldots$ to the second $\sum\ldots$ unless they were equal initially. Equality between these $\sum$'s can occur only if $\pi(\ldots)$ reorders only equal x's or equal y's.

Problem 17: Generally "$\int_o^H e^{-x}\cdot f(x)\cdot dx = \int_o^h f(x)\cdot dx$ for some h in $0 < h < H$" is obvious when $f(x)$ does not reverse sign in $0 < x < H$. (Can you see why?) Otherwise, if $f(x)$ may reverse sign, the quoted equation and inequality deserve an explanation. Supply one.

Solution 17: Let $F(x) := \int_o^x f(\xi)\cdot d\xi$ for x in $0 \le x \le H$. This $F(x)$ is continuous even if $f(x)$ is not, and $F(0) = 0$. Integration by parts yields

$$\int_o^H e^{-x}\cdot f(x)\cdot dx = \int_o^H e^{-x}\cdot dF(x) = e^{-H}\cdot F(H) - 0 - \int_o^H F(x)\cdot d(e^{-x}) = e^{-H}\cdot F(H) + \int_o^H e^{-x}\cdot F(x)\cdot dx$$
$$= \left(e^{-H}\cdot F(H) + \int_o^H e^{-x}\cdot F(x)\cdot dx\right)/\left(e^{-H} + \int_o^H e^{-x}\cdot dx\right),$$

which is a positively weighted average of values taken by $F(x)$ on the interval $0 \le x \le H$. Therefore $\int_o^H e^{-x}\cdot f(x)\cdot dx = F(h)$ for some h strictly inside that interval, as claimed.

The quoted equation and inequality were asserted as if obvious in the official solution of a Putnam Exam's problem.

Problem 18: Explain why no continuous real-valued function on a real interval (finite or infinite) can take every value in the function's range exactly twice.

Solution 18: Suppose for the sake of argument by contradiction that $f(x)$ were real, continuous and took every value in its range exactly twice for $a \le x \le b$. Necessarily $a < b$; we allow $a = -\infty$ and/or $b = +\infty$, and we allow either or both of $f(a)$ and $f(b)$ to be infinite. Let $\overline{f}$ be the maximum value taken (exactly twice) by f, and say $\overline{f} = f(\overline{x}) = f(\overline{y})$ with $a \le \overline{x} < \overline{y} \le b$. We allow $\overline{f}$ to be infinite. Similarly let the minimum value taken (exactly twice) of f be $\underline{f} = f(\underline{x}) = f(\underline{y})$ with $a \le \underline{x} < \underline{y} \le b$. Necessarily $\underline{f} < \overline{f}$. Consider how the pairs $(\underline{x}, \underline{y})$ and $(\overline{x}, \overline{y})$ could be situated relative to each other. Only three possibilities could arise:

- Neither element of either pair separates the other pair; say $\underline{x} < \underline{y} < \overline{x} < \overline{y}$.
- One pair straddles the other; say $\underline{x} < \overline{x} < \overline{y} < \underline{y}$.
- The pairs interlace; say $\underline{x} < \overline{x} < \underline{y} < \overline{y}$.

In the first two cases we could locate $\underline{z}$ where $f(\underline{z}) := \min_{\overline{x}<z<\overline{y}} f(z)$, so that $\underline{f} < f(\underline{z}) < \overline{f}$; then values of f between $f(\underline{z})$ and $\overline{f}$ would have to be taken at least *three* times, at least once in each of the intervals separating $\underline{y} < \overline{x} < \underline{z} < \overline{y}$ in the first case, $\underline{x} < \overline{x} < \underline{z} < \overline{y}$ in the second. In the third case, values of f between $\overline{f}$ and $\underline{f}$ would have to be taken at least *three* times, at least once in each of the intervals separated by $\underline{x} < \overline{x} < \underline{y} < \overline{y}$. *Three* > two. END of proof.

Problem 19: Suppose infix operator # acting upon members of a set S satisfies the identities

Idempotent: x#x = x for every x in S , and

Circular: (x#y)#z = (y#z)#x for all x, y and z in S .

Show why # must be Commutative (x#y = y#x) and Associative ((x#y)#z = x#(y#z)) too.

Solution 19: Repeated application of the Circular identity produces nothing new:

Circular: (x#y)#z = (y#z)#x = (z#x)#y for all x, y and z in S .

The second equation is a rewrite of the first with circularly permuted names of variables, but it also reminds us that both directions of circular permutation are allowed. This indicates that an expression with at least four operands will be needed to generate anything new. And the operands will have to be repeated to elicit the Commutative relation. This will be attacked first because afterwards Commutativity will imply Associativity easily as follows:

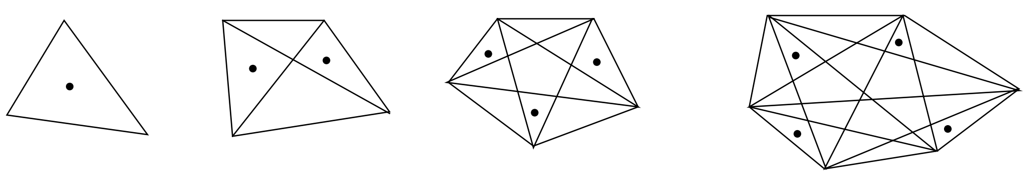
(x#y)#z = (y#x)#z = (z#y)#x = (y#z)#x = x#(y#z) .

Commutativity is elicited as follows:

x#y = (x#y)#(x#y) = ((x#y)#x)#y	… after I. and then reversed C.
= ((x#x)#y)#y = (x#y)#y	… after reversed C. and then I.
= (y#y)#x = y#x	… after C. and then I. END.

This was problem B1 on the 1971 Putnam Exam. It said "binary operation" instead of "infix operator".

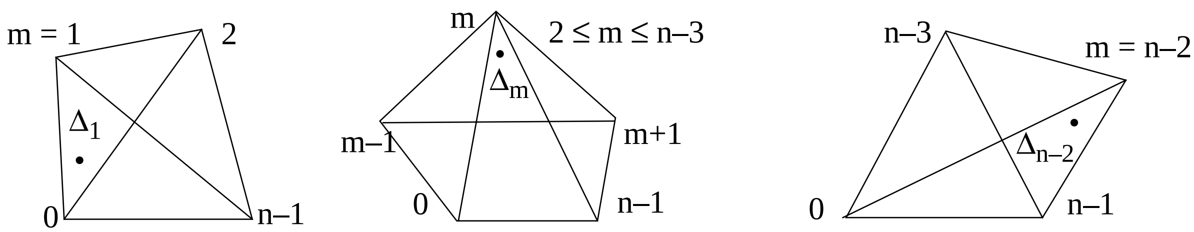
Problem 20: Let P be a given convex polygon with n > 2 sides. Show how to find a set S of n–2 points inside P with this property: Every three different vertices of P are the vertices of a triangle that has strictly inside it exactly one of the points of S . (This is hard.)



Solution 20: A solution is easy when n = 3 or n = 4 but is already challenging when n = 5 . As n increases, the number $^nC_3 = n\cdot(n-1)\cdot(n-2)/6$ of triangles in question grows rapidly, and the number of solutions increases too. Hereunder is one solution:

Assume n > 4 to simplify the exposition. Number P's vertices 0, 1, 2, …, n–1 in consecutive order around P . Let Δ(i, j, k) be the triangle whose vertices are numbered i, j and k ; if these are distinct, Δ(i, j, k) must be a non-degenerate triangle because P is convex. Let £(i, j) be the line segment joining distinct vertices numbered i and j .

For each m = 1, 2, 3, …, n–2 the triangles Δ(0, m, n–1) and Δ(m–1, m, m+1) intersect in a non-degenerate triangle Δ_m because the vertices numbered 0, m–1, m, m+1, n–1 are the vertices of either a convex pentagon, if 2 ≤ m ≤ n–3 , or else a convex quadrilateral, so £(m–1, m+1) cuts through Δ(0, m, n–1) and separates vertex m and the interior of Δ_m from the base £(0, n–1) of Δ(0, m, n–1) as shown here:



Choose any point s_m strictly inside Δ_m to construct set $S := \{s_1, s_2, s_3, \ldots, s_{n-2}\}$. This was the easy part of the solution. The hard part is proving that S so constructed satisfies the requirements of the problem, namely that every non-degenerate $\Delta(i, j, k)$ contains exactly one point of S .

Nothing is lost by assuming that $0 \le i < j < k \le n-1$. Then s_j must lie inside $\Delta(i, j, k)$ because $\Delta(i, j, k) \supset \Delta(0, j, n-1) \cap \Delta(i, j, k) \supseteq \Delta(0, j, n-1) \cap \Delta(j-1, j, j+1) = \Delta_j \supset s_j$. What remains to be proved is that no other point s_m of S lies in $\Delta(i, j, k)$. There are two cases to consider:

- When $0 < m \le i$ or $k \le m < n-1$, the point s_m inside $\Delta(0, m, n-1)$ is separated from the interior of $\Delta(i, j, k)$ by its base £(i, k) , so s_m cannot lie in $\Delta(i, j, k)$.
- When $i < m < j$ or $j < m < k$, the point s_m inside $\Delta(i, m, j)$ or $\Delta(j, m, k)$ respectively is separated from the interior of $\Delta(i, j, k)$ by a side £(i, j) or £(j, k) respectively.

These cases exclude from $\Delta(i, j, k)$ all points s_m for $1 \le m \le n-2$ except s_j . END of proof.

This problem was posed in 2002 by Andor Lucács and Szilard András in Romania.

Problem 21: Why is $n = 1$ the only positive integer n for which $4^n + n^4$ is a prime?

Solution 21: When n is even so is $4^n + n^4$, so only odd $n = 2k+1 \ge 3$ need be considered. Then $4^n + n^4 = (2^n + n^2)^2 - (2^{k+1} \cdot n)^2 = (2^n + n^2 + 2^{k+1} \cdot n) \cdot (2^n + n^2 - 2^{k+1} \cdot n)$. The smaller factor is $2^n + n^2 - 2^{k+1} \cdot n = (2^k - n)^2 + 4^k$, which is at least 5 , so $4^n + n^4$ cannot be a prime.

Problem 22: Given integer $N > 0$ your task is to partition set $S := \{0, 1, 2, \ldots, 2N-2, 2N-1\}$ into two disjoint subsets $X := \{x_1, x_2, x_3, \ldots, x_N\}$ and $Y := \{y_1, y_2, y_3, \ldots, y_N\}$ in such a way that every value $i + x_i - y_i$, for $i = 1, 2, 3, \ldots$ and N , is an integer multiple (perhaps 0) of $2N$. Show that your task is feasible for $N = 1, 4$ and 5 , but not for $N = 54$.

Solution 22:

A Partition for N = 4 | A Partition for N = 5

i	x_i	y_i	$i + x_i - y_i$		i	x_i	y_i	$i + x_i - y_i$
1	0	1	0		1	0	1	0
2	3	5	0		2	3	5	0
3	4	7	0		3	6	9	0
4	2	6	0		4	4	8	0
					5	2	7	0

For $N = 4$ only 32 out of $8! = 40320$ partitions are eligible, and among these only six have every value $i + x_i - y_i = 0$. For $N = 5$ only 80 out of 3628800 partitions are eligible, and among these only ten have every value $i + x_i - y_i = 0$. I do not know an efficient way to cope with larger integers N except that the task is impossible if N is congruent to 2 or 3 mod 4 .

For instance, the task is impossible for $N = 54$ because S has an even number ($N = 54$) of odd members, but the number ($\lfloor(N+1)/2\rfloor = 27$) of odd indices is odd. If a partition were eligible, the number of odd members x_i and y_i with even indices i would have to be even to make their values of $i + x_i - y_i$ even. The number of odd members x_i and y_i with odd indices i would have to be the same as the odd number of odd indices i to make their values of $i + x_i - y_i$ even. Thus, the total number of odd members of S would have to be odd, which it isn't for $N = 54$.

For similar reasons, the task is impossible for small numbers $N = 2, 3, 6$ and 7 . Experiments with these small examples in 1992 may have lead to an insight that Howard Morris generalized to create the previous paragraph.

Problem 23: Can a finite-dimensional Euclidean space contain an uncountable infinitude of disjoint open subsets of the space? Justify your answer.

Solution 23: No, the disjoint open subsets must constitute a countable infinitude. Here is why: Let N be the dimension of the space. Every open subset must contain a ball (a disk if $N = 2$, a solid sphere if $N = 3$, a solid hypersphere if $N > 3$) of positive radius. Since rational numbers are dense among the reals, each ball must contain points $(x_1, x_2, x_3, \ldots, x_N)$ whose coordinates x_j are all rational. Choose one such rational point from each open subset. The rational numbers are countable; see `http://www.cs.berkeley.edu/~wkahan/MathH90/S12Oct07.pdf` for examples of enumerations. Therefore all N-tuples $(x_1, x_2, x_3, \ldots, x_N)$ of rational coordinates are countable, as are all the chosen rational points, no two of which can lie in the same open subset since these are disjoint. Therefore these are countable too.

Prof. G. Caviglia put a problem like this on his Midterm Test for Math. 104.

Problem 24: Find as compact an expression as possible for the limit, as $n \to \infty$, of

$$(n+2)^{n+2}/(n+1)^{n+1} - (n+1)^{n+1}/n^n .$$

Don't make the mistake of assuming that a limit of products always equals the product of limits.

Solution 24: The limit is e. To see why, set $n := -1 + 1/x$ and $f(x) := (1 + x)^{1/x}$, and use the Taylor series for $\ln(1+x)$ to find that $f(x) = \exp(1 - x/2 + x^2/3 - x^3/4 + \dots)$. Consequently $f(x) \longrightarrow e$ and $f'(x) \longrightarrow -e/2$ as $x \longrightarrow 0$. Apply this to $f(x) + f(-x) + (f(x) - f(-x))/x$.

Problem 25: Alice and Bob play a game. Starting with Alice, they select alternate digits for a six-digit decimal number UVWXYZ they construct from left to right. Alice chooses U; Bob chooses V; Alice chooses W; and so on. No digit may be repeated. Alice wins if UVWXYZ is not a prime. Why can Alice always win?

Solution 25: David Blackston supplied a strategy whereby Alice can win every time: First let Alice choose U = 3. If Bob chooses V = 9 then Alice chooses W = 6; otherwise she chooses W = 9. Her final choice for Y is a digit other than 5 that makes U+V+W+X+Y + 1 exactly divisible by 3. That will force Bob to choose for Z a 5, or an even digit, or a 1 or a 7 that makes U+V+W+X+Y+Z (and hence UVWXYZ) exactly divisible by 3. Alice's choice for Y is feasible because of the counts of decimal digits modulo 3. (We say that "N is congruent to n mod 3" if and only if N and n leave equal remainders when divided by 3.)

Four digits, namely 0, 3, 6 and 9, are congruent to 0;
three, namely 1, 4 and 7, are congruent to 1; and
three, namely 2, 5 and 8, are congruent to 2.

Bob's two prior choices will leave at least one digit in each congruence class available for Alice to choose for Y unless Bob has chosen 2 and 8, in which case Y = 7 will make U+V+W+X+Y congruent to 2. Bob must choose Z from a set of five remaining digits among which those congruent to 0 are all even, those congruent to 2 are all even or 5, and those congruent to 1 make UVWXYZ divisible by 3, so he loses.

Swapping digits 3 and 9 provides another winning strategy for Alice. Are there any others?

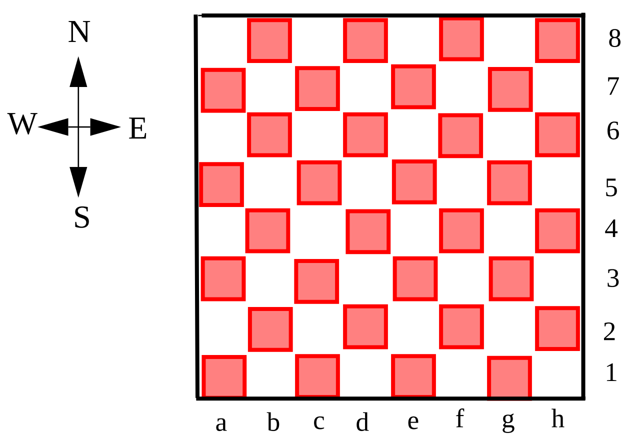
Problem 26: A chessboard is covered completely by 64 dice each of which covers exactly one square. Dice may be turned over, but only if all eight dice in a *Rank* (row), or all eight dice in a *File* (column), are rotated through some integer multiple of a quarter turn ($\pi/2$) about the axis through all eight dice's centers as if they constituted a rigid body. Each die's six faces are numbered with spots in the standard way. Your task is to turn the dice to bring every die's five-spots-face facing upward (away from the board) no matter how the dice faced initially. Then all the dice's exposed faces will look alike. How will you do it?

Solution 26: Let’s number the chessboard’s Ranks 1 - 8 and Files a - h in the customary way as the figure shows, and orient the Files North-South and the Ranks East-West. Our task will be accomplished by alternating two sub-procedures. Sub-procedure East-Face(n) will act upon each of the dice in Rank n, for $1 \le n \le 8$, to align Eastward all their five-spots-faces. Then sub-procedure North-Face(n) will act upon them to align all their five-spots-faces North. Laid out hereunder are the sub-procedures’ actions:

East-Face(n): For z = a, b, …, h in turn, rotate the dice in Rank n and then the dice in File z to align Eastward the five-spot-face of the die in (Rank n, File z) . Doing so will not misalign dice in Rank n previously aligned with five-spots-faces facing East. The amounts of East-Face(n)’s rotations will depend upon the initial orientations of the dice. When the time comes to rotate the die in (Rank n, File z), if its five-spots-face faces North or South rotate its rank through a quarter-turn. Then at most two quarter-turns of the dice in File z will have that face facing East.

North-Face(n): Now every die in Rank n has its five-spots-face facing East. For z = a, b, …, h in turn, rotate File z through a quarter turn to align upwards the five-spots-face of the die in (Rank n, File z) . Doing so will not misalign any five-spots-faces previously aligned to face North. Next, rotate Rank n through a quarter-turn to align all its dice’s five-spot-faces facing North.

To accomplish our task, we invoke East-Face(1), North-Face(1), East-Face(2), North-Face(2), … , East-Face(8), North-Face(8), and then, for n = 1, 2, …, 8 in turn, rotate Rank(n)’s dice through a quarter-turn to align every five-spots-face from Northward to upward, as desired.



“Up” faces the reader.

“Down” faces into the page.

Had we chosen to align all the six-spots-faces upward, instead of the five-spots-faces, could we get all the exposed faces to look alike regardless of their initial orientations?

Problem 27: Let $Z = -Z^T$ be an N-by-N skew-symmetric matrix whose elements $\zeta_{i,j} = -\zeta_{j,i}$ are real variables. Then $\det(Z) = \det(-Z^T) = (-1)^N\cdot\det(Z)$, so $\det(Z) = 0$ if N is odd. If N is even, $\det(Z)$ is a nonzero polynomial in the elements $\zeta_{i,j}$; for example $\det(Z) = \zeta_{12}{}^2$ if $N = 2$, and $\det(Z) = (\zeta_{12}\cdot\zeta_{34} - \zeta_{13}\cdot\zeta_{24} + \zeta_{14}\cdot\zeta_{23})^2$ if $N = 4$. More generally, prove that …
If N is even, $\det(Z)$ is the square of a polynomial in the elements $\zeta_{i,j}$ with integer coefficients.

Proof 27: This polynomial is called "The Pfaffian" of Z. Nobody knows how to compute it quickly using only additions, subtractions and multiplications (no divisions nor square roots) when N gets huge unless Z is very special, like tridiagonal: For instance, by induction, …

$$\text{if even N-by-N}\quad Z = \begin{bmatrix} 0 & \varsigma_1 & 0 & 0 & 0 & \dots \\ -\varsigma_1 & 0 & \varsigma_2 & 0 & 0 & \dots \\ 0 & -\varsigma_2 & 0 & \varsigma_3 & 0 & \dots \\ 0 & 0 & -\varsigma_3 & 0 & \varsigma_4 & \dots \\ 0 & 0 & 0 & -\varsigma_4 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad\text{then}\quad \det(Z) = (\,\zeta_1\cdot\zeta_3\cdot\zeta_5\cdots\cdot\zeta_{N-1}\,)^2 .$$

In general, the proof that Pfaffians exist proceeds by induction. They have been exhibited above for $N = 2$ and $N = 4$, so for some $n \ge 2$ suppose there exists a polynomial P_n with integer coefficients such that every 2n-by-2n skew-symmetric $Z = -Z^T$ with elements $\zeta_{i,j} = -\zeta_{j,i}$ has $\det(Z) = P_n(\{\zeta_{i,j}\})^2$. Any (2n+2)-by-(2n+2) skew-symmetric $S = -S^T$ can be partitioned thus:

$$S = \begin{bmatrix} \eta J & C^T \\ -C & Z \end{bmatrix} = \begin{bmatrix} I & O^T \\ CJ/\eta & I \end{bmatrix}\cdot\begin{bmatrix} \eta J & C^T \\ O & Z - CJC^T/\eta \end{bmatrix} \quad\text{wherein}\quad Z = -Z^T,\quad J = -J^T = -J^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad\text{and}$$

C is some 2n-by-2 matrix. We must assume temporarily that $\eta \ne 0$. Then, because $Z - CJC^T/\eta = (\eta\cdot Z - CJC^T)/\eta$ is 2n-by-2n and skew-symmetric, the induction hypothesis yields $\det(S) = \det(\eta J)\cdot\det(\eta\cdot Z - CJC^T)/\eta^{2n} = (\,Q(\{\sigma_{i,j}\})/\eta^k\,)^2$ for some nonnegative integer $k \le n-1$ and some polynomial $Q(\{\sigma_{i,j}\})$ in the elements $\sigma_{i,j} = -\sigma_{j,i}$ of S with integer coefficients. This $Q(\{\sigma_{i,j}\})$ is obtained from $P_n(\{\zeta_{i,j}\})$ by substituting the elements of $\eta\cdot Z - CJC^T$ in place of $\{\zeta_{i,j}\}$ respectively, and then factoring out from the result as high a power of η as possible. Because $\det(S)$ is a polynomial in η among other things, $k = 0$ and therefore, as claimed, $Q(\{\sigma_{i,j}\}) =: \pm P_{n+1}(\{\sigma_{i,j}\})$ except possibly if $\eta = 0$. Unless $S = O$, we get rid of this exception by permuting the rows and columns of S symmetrically to put a nonzero $\sigma_{i,j}$ in place of η without changing $\det(S)$. Thus we advance our induction hypothesis from n to n+1.

Actually $P_n(\{\zeta_{i,j}\})$ is a sum of $(2n)!/(2^n\cdot n!) \approx \sqrt{2}\cdot(2n/e)^n$ products of n different $\zeta_{i,j}$'s with coefficients all ±1. The number of products grows very quickly with n. There are 15 products in

$$P_3(\{\zeta_{i,j}\}) = z_{12}\cdot z_{34}\cdot z_{56} - z_{12}\cdot z_{35}\cdot z_{46} + z_{12}\cdot z_{36}\cdot z_{45} - z_{13}\cdot z_{24}\cdot z_{56} + z_{13}\cdot z_{25}\cdot z_{46} - z_{13}\cdot z_{26}\cdot z_{45} + z_{14}\cdot z_{23}\cdot z_{56} - {}$$
$$- z_{14}\cdot z_{25}\cdot z_{36} + z_{14}\cdot z_{26}\cdot z_{35} - z_{15}\cdot z_{23}\cdot z_{46} + z_{15}\cdot z_{24}\cdot z_{36} - z_{15}\cdot z_{26}\cdot z_{34} + z_{16}\cdot z_{23}\cdot z_{45} - z_{16}\cdot z_{24}\cdot z_{35} + z_{16}\cdot z_{25}\cdot z_{34}$$

and 105 products in $P_4(\{\zeta_{i,j}\})$, 945 in $P_5(\{\zeta_{i,j}\})$, 10395 in $P_6(\{\zeta_{i,j}\})$, 135135 in $P_7(\{\zeta_{i,j}\})$, …. Our proof exploits divisions in a *Recursive* procedure (not a *Recurrence*) that can compute the Pfaffian of a 2n-by-2n skew-symmetric numerical (not symbolic) matrix Z in time and (alas) memory proportional to n^3 instead of $(2n/e)^n$. A recursion (not a recurrence) without divisions but taking time proportional to n^4 was found several years ago.

Much to be desired is a recursion (or preferably a recurrence) using only additions, subtractions and multiplications to compute a Pfaffian in time proportional to n^3. Such a procedure would serve to compute also the determinant of an n-by-n matrix B in the same time because det(B) is the Pfaffian of $\begin{bmatrix} O & -B \\ B^T & O \end{bmatrix}$. Perhaps no such procedure exists.

Here are two easy exercises about Pfaffians:

- Show that, if $\zeta_{i,j} := -\zeta_{j,i} := \text{signum}(j-i) = \pm 1$ if $j \neq i$, otherwise 0, then every Pfaffian $P_n(\{\zeta_{,i,j}\}) = 1$.
- Show that, if Z and S are skew-symmetric matrices of the same dimension, then $\sqrt{\det(I + S \cdot Z)}$ is a Pfaffian.

The Pfaffian was named in 1852 by the British algebraist Arthur Cayley in honor of a German mathematician Johann Friedrich Pfaff (1765 - 1825) associated also with the treatment of certain partial differential equations. Pfaffians figure in some combinatorial and graph-theoretic. problems.

The Pfaffian goes unmentioned in most modern texts on Matrix Theory or Linear Algebra. To read about it (with difficulty in an over-condensed notation) see §410 *et seq.* of *A Treatise on the Theory of Determinants by Thomas Muir* revised and enlarged by Wm. H. Metzler, republished in 1960 by Dover, N.Y. W.H. Greub's *Multilinear Algebra* (1967, Springer-Verlag, N.Y.) has a tensor treatment on pp. 176-8. A neater treatment is on pp. 242-3 of P.M. Cohn's *Algebra* vol. 1, 2nd ed. (1982, Wiley, N.Y.) and pp. 235-7 of his *Classic Algebra* (2000, Wiley, N.Y.). To Google "Pfaffian" without being drowned in partial differential equations, include some of the words "determinant", "permanent" and "hafnian".