Problem 1: Our *Omnipotent and Merciful Emperor* is resolved to punish his advisors. There are two dozen of them and, according to our *Mighty and Magnanimous Emperor*, they all seem obsessed with advising him that he cannot or should not do whatever he intends to do. Every one of them has been collected by military escorts and commanded to protect National Security by maintaining utter secrecy, communicating nothing to anyone including each other. Before being ushered into the Hall of Heros, each one of them passed through a dark antechamber where a hat was placed unseen upon his or her head. Now standing together, the advisors face a big sign:

Whoever violates the commands below dies immediately.
Do not touch nor remove the hat from your head.
Emit no sound nor sign nor signal until our
Glorious and Gracious Emperor addresses you in turn.
Each of you will be asked the color of the hat on your head.
Reply only with one of the following three colors:
1: "Red" 2: "White" 3: "Blue"
Any other reply incurs your death immediately.
A correct reply earns you your freedom immediately.
An incorrect reply or none sends you to prison immediately.

Each advisor can see the colors of only all other advisors' hats. Their colors appear to have been chosen at random, some red, some white, the rest blue. If each advisor replies at random, about two thirds of them will land in prison, as our *Supreme and Solicitous Emperor* must expect. But there is a strategy that will earn freedom for at least most of the advisors, and they have found and adopted it unanimously with no need to discuss it.

Slowly our *Magnificent and Beneficent Emperor* is ascending the podium in the Hall of Heros. What strategy must all advisors have adopted before being addressed?

Solution 1: They have adopted arithmetic **modulo** 3 and have attached the big sign's numbers to the colors: 1 for Red, 2 for White, $0 \equiv 3$ **mod** 3 for Blue. The first advisor asked his or her hat's color replies with the color attached to the sum **mod** 3 of all the other advisors' hats' colors' numbers. Then this first advisor departs the Hall of Heros more likely for prison than to freedom. *Each* remaining advisor can now determine his or her hat's color from the difference between the number of the color just announced and the sum **mod** 3 of every *other* remaining advisor's hat's color's number.

Problem 2: Suppose H is an n-by-n real symmetric ($H = H^T$) matrix, $\mathbf{v}$ is a real column n-vector, and that $H^{k+1}\cdot\mathbf{v} = \mathbf{o}$ for some unknown integer $k \ge 1$. Prove that $H\cdot\mathbf{v} = \mathbf{o}$ too.

Proof 2: Define $\|\mathbf{z}\|^2 := \mathbf{z}^T\cdot\mathbf{z}$ for every real n-vector $\mathbf{z}$ and observe that the sum-of-squares $\|\mathbf{z}\|^2 = 0$ if and only if $\mathbf{z} = \mathbf{o}$. If $k = 2m+1 \ge 1$ is odd, $0 = \mathbf{v}^T\cdot H^{k+1}\cdot\mathbf{v} = \mathbf{v}^T\cdot H^{2m+2}\cdot\mathbf{v} = \|H^{m+1}\cdot\mathbf{v}\|^2$ so $H^{m+1}\cdot\mathbf{v} = \mathbf{o}$ and $m < k$. If $k = 2m \ge 2$ is even, $0 = \mathbf{v}^T\cdot H\cdot H^{k+1}\cdot\mathbf{v} = \mathbf{v}^T\cdot H^{2m+2}\cdot\mathbf{v} = \|H^{m+1}\cdot\mathbf{v}\|^2$ so $H^{m+1}\cdot\mathbf{v} = \mathbf{o}$ and $m < k$ again. Replace k by m and repeat the process until $k = 0$.

An alternative proof changes coordinates to an orthonormal basis consisting of the eigenvectors of H. Doing so replaces H by a diagonal matrix Λ of H's eigenvalues, and turns $\mathbf{v}$ into some vector $\mathbf{u}$ that satisfies $\Lambda^{k+1}\cdot\mathbf{u} = \mathbf{o}$. Now it is obvious that every element of $\Lambda\cdot\mathbf{u}$ vanishes too, and after reverting to the original coordinates we find $H\cdot\mathbf{v} = \mathbf{o}$ too, as claimed.

This alternative proof explains why the problem's conclusion $H\cdot\mathbf{v} = \mathbf{o}$ might not follow from its hypothesis $H^{k+1}\cdot\mathbf{v} = \mathbf{o}$ if H is an arbitrary nonsymmetric matrix; it might be nondiagonalizable.

This Problem 2 was generalized from the Fall 2007 Prelim. Exam for Math. Grad Students.

Problem 3: Exhibit *all* infinitely differentiable real functions $f(x)$ that satisfy $f''(x) = -f(x)$ and $f(2x) = 2\cdot f(x)\cdot f'(x)$ for all real x. How do you know that you have exhibited all of them?

Solution 3: There are two solutions $f(x) = \sin(x)$ and $f(x) = 0$ for all real x. What follows explains why no possibilities exist other than those two.

Consider first how $f(x)$ must behave in any open interval $\mathbb{X}$ in which $f(x)$ never vanishes. Let $t(x) := f'(x)/f(x)$ there and observe that $t'(x) = f''(x)/f(x) - t(x)^2 = -(1 + t(x)^2)$. An integration reveals that $\operatorname{arccot}(t) - ß = -\int dt/(1 + t^2) = \int dx = x$ for some constant ß not yet determined, so $f'(x)/f(x) = \cot(x+ß)$. Integrate again: $\log(f) - \text{const.} = \int df/f = \int \cot(x+ß)\cdot dx = \log(\sin(x+ß))$ whence $f(x) = \mu\cdot\sin(x+ß)$ for some constants $\mu \neq 0$ and ß not yet determined and all x in $\mathbb{X}$.

Before concluding that $f(x) := \mu\cdot\sin(x+ß)$ at all real x we must rule out the possibility that some other solution $F(x)$ of the same differential equation $F'' = -F$ agrees with $f(x)$ at all x in $\mathbb{X}$ but differs from this definition of f elsewhere. Let $h(x) := F(x) - f(x)$; this difference satisfies the same differential equation $h'' = -h$ everywhere but $h(x) = 0$ at all x in $\mathbb{X}$. The differential equation implies that $(h(x)^2 + h'(x)^2)' = 2\cdot(h(x) + h''(x))\cdot h'(x) = 0$ everywhere and therefore that $h(x)^2 + h'(x)^2$ is the same constant everywhere; this constant must be 0 as it is at x in $\mathbb{X}$.

Thus we conclude that $f(x) = \mu\cdot\sin(x+ß)$ at all real x for some constants $\mu \neq 0$ and ß not yet determined. The same conclusion could have been drawn directly from the general theory of the existence and uniqueness of solutions of differential equations taught in some Math. courses.

What remains to be done is to determine the constants by applying the problem's given constraint $f(2x) = 2\cdot f(x)\cdot f'(x)$. This implies that $\mu\cdot\sin(2x+ß) = 2\cdot\mu^2\cdot\sin(x+ß)\cdot\cos(x+ß) = \mu^2\cdot\sin(2x+2ß)$, whence $\sin(2x+ß) = \mu\cdot\sin(2x+2ß)$ at all x if $\mu \neq 0$. First set $x := \pi/4 - ß/2$ to infer that $1 = \mu\cdot\cos(ß)$, and then set $x := \pi/4 - ß$ to infer that $\cos(ß) = \mu$. These inferences imply that $\cos^2(ß) = 1$, so $ß = n\cdot\pi$ for some integer n, and then $f(x) = (-1)^n\cdot\sin(x + n\cdot\pi) = \sin(x)$ as claimed.

This Problem 3 was derived from a complex version on the Fall 2007 Prelim. Exam for Math. Grad Students.

Problem 4: Suppose real sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ satisfy $0 \le a_{n+1} \le a_n + b_n$ for all $n \ge 1$, and that $\sum_{n\ge1} b_n$ converges. Prove that $\lim_{n\to\infty} a_n$ exists and is finite.

Proof 4: The sequence $\{a_n\}_{n\ge1}$ will be proved *Bounded*, from which will follow the existence of at least one *Cluster-point* of the sequence. Then this point will be proved unique.

But first some definitions for students who have not yet taken a course on *Real Variables* like Math. 104, or for students who have forgotten it:

- Most mathematicians exclude ∞ from the set of *real* numbers; we too will do so hereunder. All mathematicians exclude 0 from the set of *positive* numbers; we too will do so hereunder.
- The real sequence $\{a_n\}_{n\ge1}$ is called "Bounded" just when numbers L and U exist satisfying $L \le a_n \le U$ for all $n \ge 1$; we need know only that L and U exist, not their values.
- A *Neighborhood* of a real number r is the set of all numbers x satisfying $-b < x - r < c$ for some *positive* numbers b and c. Usually we care about only tiny numbers b and c.
- If *the Limit* L of a real sequence $\{a_n\}_{n\ge1}$ exists, every neighborhood of L contains all but at most finitely many of the members of the sequence. Consequently *the* limit L is unique.
- If a real sequence $\{a_n\}_{n\ge1}$ "has" *Limit-points* (plural) or *Cluster-points* or *Points of accumulation*, none need be a member of the sequence, but every neighborhood of each cluster-point must contain infinitely many members of the sequence. The term "Limit" is so easily confused with "Limit-point" that the latter term will not be used hereunder.
- Every bounded sequence has at least one cluster-point (do you see why?), and if just one then it is the sequence's limit. Conversely, a sequence convergent to its limit is bounded.
- Do not confuse an infinite *Sequence* $\{b_n\}_{n\ge1}$ with the infinite *Series* $\sum_{n\ge1} b_n$ that generates an infinite sequence $\{s_n\}_{n\ge0}$ of *Partial Sums* $s_N := \sum_{1\le n\le N} b_n$ whose limit, if it exists, is the sum of the series. Otherwise the series has no sum or *Diverges* to an infinite sum.

Back to the proof. Let $s_\infty := \sum_{n\ge1} b_n$ be the sum of this given convergent series. For any integers $N > M \ge 1$ the problem's hypotheses imply that $0 \le a_N \le a_M + s_M - s_N$ because

$$0 \le a_N \le a_{N-1} + b_{N-1} \le a_{N-2} + b_{N-2} + b_{N-1} \le \dots \le a_M + b_M + b_{M+1} + \dots + b_{N-2} + b_{N-1} = a_N \le a_M + s_M - s_N .$$

In particular $0 \le a_N \le a_1 + s_1 - s_N \to a_1 + s_1 - s_\infty$ as $N \to \infty$, so the sequence $\{a_n\}_{n\ge1}$ must be bounded and consequently have at least one cluster-point. How many of these are there?

Let each of x and $y > x$ be such a cluster-point if there be more than one of them. This would mean that for every $\varepsilon > 0$ infinitely many sequence members a_M satisfy $|a_M - x| < \varepsilon$ and infinitely many members a_K satisfy $|a_K - y| < \varepsilon$. Now choose any positive $\varepsilon < (y - x)/8$. Next choose M big enough that $|a_M - x| < \varepsilon$ and $|s_M - s_\infty| < \varepsilon$; this would be feasible since $s_N \to s_\infty$ as $N \to \infty$. Next choose $K > M$ and big enough that $|a_K - y| < \varepsilon$ and $|s_K - s_\infty| < \varepsilon$. These choices would imply an impossible inequality

$$0 < y - x = y - a_K + a_K - a_M + a_M - x < \varepsilon + s_K - s_M + \varepsilon < 4\varepsilon < (y-x)/2 .$$

The contradiction implies the sequence $\{a_n\}_{n\ge1}$ has just one cluster-point, its limit, as claimed.

This Problem 4 was taken from the Fall 2007 Prelim. Exam for Math. Grad Students.

Problem 5: Given is an ellipse E neither a circle nor degenerate (a straight line segment). Let P be the largest of the perimeters of triangles inscribed in E . How many inscribed triangles have maximal perimeter P ? At least two do since E is centrally symmetric. Are there more? Why?

The known solution for this problem requires algebra so onerous that it must be computerized. Can you do better?

Solution 5: There are infinitely many; every point of E is the vertex of an inscribed triangle with maximal perimeter P . This was surmised by Paul Penning on p.219 of *Niew Archief voor Wiskunde* 5/7 #3 (Sept. 2006), but without saying how to prove it. Subsequently his surmise was found to be a special case of 19th century theorems by Chasles and by Poncelet:

> Let E and Ë be *Confocal Ellipses* (they share the same two foci) so situated that there is an n-sided polygon inscribed in (with all vertices on) E and circumscribed around (with every edge touching) Ë . Then for each point e on E (or each tangent t to Ë) there is an n-sided polygon inscribed in E and circumscribed around Ë with one vertex at e (or an edge lying on t resp.). Every such n-sided polygon has the same perimeter, and this perimeter is maximal among n-sided polygons inscribed in E , and minimal among n-sided polygons circumscribed about Ê . Any two edges incident at such a polygon's vertex on E make equal angles with the tangent to E tat the vertex.

Long geometrical proofs of the foregoing can be found starting on p. 243 in M. Berger's book *Geometry II* (Springer Verlag, Berlin, 1987) and in George Lion's paper "Variational Aspects of Poncelet's Theorem", *Geometriae Dedicata* **52** (1994) pp. 105-118, and references cited therein.

A shorter proof, still too long, of Penning's surmise was extracted from lengthy polynomial manipulations performed by MAPLE, a computerized algebra system:

With $t := \tan(\theta/2)$ in mind, define $C(t) := (1-t^2)/(1+t^2) = \cos(\theta)$ and $S(t) := 2t/(1+t^2) = \sin(\theta)$. Then as t runs from $-\infty$ to $+\infty$ the ellipse whose equation is $x^2 + y^2/b^2 = 1$ is traversed by the point $[x, y] = E(t, b) := [C(t), b\cdot S(t)]$. We assume $0 < b \neq 1$ to ensure that this ellipse is not a circle nor degenerate, and then every ellipse's shape is obtained from some such value of b . Of course, the size and orientation of these ellipses is irrelevant to our problem. Next, for any real s and t define $L(s,t, b) := \| E(s, b) - E(t, b) \| \equiv L(t,s, b) \equiv L(-s,-t, b) \equiv L(1/s,1/t, b)$ to be the Euclidean length of the line segment joining the points E(s, b) and E(t, b) . For any real distinct parameters x, y, z the expression $P(x,y,z, b) := L(x,y, b) + L(y,z, b) + L(z,x, b)$ is the perimeter of a nondegenerate triangle whose vertices lie on the ellipse. With a view to choosing parameters that maximize P(x,y,z, b) we define $L^\dagger(x,y, b) := (1/2)\cdot(1+x^2)\cdot\partial L(x,y, b)/\partial x$. Every choice of parameters {x, y, z} that maximizes perimeter P(x,y,z, b) must satisfy

$$L^\dagger(x,y, b) + L^\dagger(x,z, b) = L^\dagger(y,z, b) + L^\dagger(y,x, b) = L^\dagger(z,x, b) + L^\dagger(z,y, b) = 0 .$$

Distinct roots {x, y, z} of these three equations provide locally maximized perimeters.

The equation $L^\dagger(x,y, b)+L^\dagger(x,z, b) = 0$ implies $L^\dagger(x,y, b)^2 - L^\dagger(x,z, b)^2 = 0$ which simplifies to a polynomial equation $Q(x,y,z, b) = 0$ after irrelevant factors have been discarded. This polynomial turns out to be

$$Q(x,y,z, b) := x^4\cdot b^2\cdot(y+z) + \big(4 - 2b^2\cdot(y\cdot z+ 1)\big)\cdot x^3 + \big(2b^2\cdot(y\cdot z+ 1) - 4y\cdot z\big)\cdot x - b^2\cdot(y+z) .$$

When $P(x,y,z, b)$ is maximized we must have $Q(x,y,z, b) = Q(y,z,x, b) = Q(z,x,y, b) = 0$. As expected, $Q(x,y,z, b) \equiv Q(x,z,y, b)$. Unexpectedly, $Q(x,y,z, b)$ is linear in y and in z separately. Therefore the equation $Q(x,y,z, b) = 0$ is satisfied just when $y = G(x,z, b)$;

$$G(x,z, b) := -\left(b^2\cdot z\cdot x^4 + (4 - 2b^2)\cdot x^3 + 2b^2\cdot x - b^2\cdot z\right)/\left(b^2\cdot x^4 - 2b^2\cdot x^3 + (2b^2 - 4)\cdot z\cdot x - b^2\right) .$$

As expected, $G(x, G(x,y, b), b) \equiv y$ because $Q(x,y,z, b) \equiv Q(x,z,y, b)$.

So, when $P(x,y,z, b)$ is maximized $\{x,y,z\}$ must satisfy $y = G(x,z, b) = G(z,x, b)$ as well as $Q(y,z,x, b) = Q(G(x,z, b), z,x, b) = 0$. We seem to require that x and z satisfy two equations $\left(G(x,z, b) - G(z,x, b) \right)/(x-z) = 0$ and $Q(G(x,z, b), z,x, b)/(x-z) = 0$. However, though derived independently these two equations are not independent. The numerator of the rational expression $\left(G(x,z, b) - G(z,x, b) \right)/(x-z)$ is a polynomial $H(x,z, b)$ too complicated to reproduce here, and the same polynomial turns up as a factor of the numerator of $Q(G(x,z, b), z,x, b)/(x-z)$; this numerator's other factor $(b^2\cdot(x^2 - 1)^2 + 4x^2)^3$ vanishes for no real x.

Thus, the three equations that must be satisfied when perimeter $P(x,y,z, b)$ is maximized do not determine discrete values of $\{x,y,z\}$ but a continuum of values instead. For every real x some root(s) $z = Z(x)$ satisfying $H(x,Z(x), b) \equiv 0$ determine $y = Y(x) := G(x,Z(x), b)$ such that perimeter $P = P(x,Y(x),Z(x), b)$ is maximal and independent of x. Apparently every point of the ellipse is a vertex of an inscribed triangle of maximal perimeter P. This is astonishing.

• • • • •

An alternative solution for this problem might begin with the observation that any inscribed triangle of maximal perimeter P must satisfy a *Reflection Condition* at each vertex: The ellipse's inward normal there bisects the angle between the triangle's sides adjacent to this vertex because otherwise moving it along the ellipse to bring the normal towards bisection lengthens the triangle's perimeter. Therefore the three normals at the vertices must be concurrent. This concurrency can be translated into an algebraic constraint upon the vertices $E(x, b)$, $E(y, b)$ and $E(z, b)$ on the ellipse E, namely that $(x+y+z)/(x\cdot y\cdot z)^2 = 1/x + 1/y + 1/z$, but this alone is not sufficient to ensure maximization of the perimeter. We haven't yet figured out where to go next towards a solution manageable without the computer's aid.