The 1999 W.L. Putnam Competition Exam took place on Sat. 4 Dec. 1999 in two sessions:
Problems A1 - A6 were to be solved during the morning session, 8 - 11 am.;
Problems B1 - B6 were to be solved during the afternoon session, 1 - 4 pm.

Problem A1: Find polynomials f(x), g(x) and h(x), if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1, \\ 3x+2 & \text{if } -1 \le x \le 0, \text{ or} \\ -2x+2 & \text{if } x > 0. \end{cases}$$

Solution A1: $f(x) := 3(x+1)/2$, $g(x) := 5x/2$ and $h(x) := (1-2x)/2$ meet the requirements. This particular solution can be determined most easily by solving three linear equations:

$$-f + g + h = -1, \quad f + g + h = 3x + 2, \quad f - g + h = -2x + 2.$$

Problem A2: Let p(x) be a polynomial that is nonnegative for all x. Prove that, for some k, there are polynomials $f_1(x)$, $f_2(x)$, …, $f_k(x)$ such that $p(x) = \sum_{1\le j\le k} (f_j(x))^2$.

Solution A2: Presumably " for all x " means " for all *real scalar* x " since otherwise, if x could be complex, we would find p(x) to be a positive constant; and if x could be a real vector then rational functions f_j might be needed. See pp. 55-57 and 300-304 in the book *Inequalities* by G.H. Hardy, J.E. Littlewood & G. Pólya, 2d. ed. (1952) for more about that. Back to our problem: p(z) must have real coefficients because they can be determined by solving a real system of linear equations with a number of values of p(z) at real arguments in the right-hand side. Therefore any zeros $x_k \pm \imath y_k$ of p(z) that are not real must come in complex-conjugate pairs. Every real zero x_j of p(z) must have even multiplicity lest p(z) change sign there. (And the degree of p(z) must be even since otherwise it would take opposite signs at $z = \pm\infty$.) Therefore $p(z) = \prod_j (z-x_j)^2\cdot\prod_k ((z-x_k)^2 + y_k^2)$, whereupon we may rewrite $\prod_k ((z-x_k)^2 + y_k^2) = \prod_k (z-x_k + \imath y_k)\cdot(z-x_k - \imath y_k) = (R(z) + \imath I(z))\cdot(R(z) - \imath I(z)) = R(z)^2 + I(z)^2$ with the understanding that the factors $(z-x_j)$ and $(z-x_k - \imath y_k)$ need not be all distinct; here polynomial $R(z) + \imath I(z) = \prod_k (z-x_k + \imath y_k)$ broken into real and imaginary parts. Consequently $k = 2$, $f_1(z) := R(z)\cdot\prod_j (z-x_j)$, and $f_2(z) := I(z)\cdot\prod_j (z-x_j)$.

Problem A3: Consider the power series expansion $1/(1 - 2x - x^2) = \sum_{n\ge0} a_n x^n$. Prove for each integer $n \ge 0$ that there is an integer m such that $a_m = a_n^2 + a_{n+1}^2$.

Solution A3: $m = 2n+2$. Why? Go from the partial fraction expansion to the power series thus:

$$\sqrt{8}/(1 - 2x - x^2) = (1+\sqrt{2})/(1 - (1+\sqrt{2})x) - (1-\sqrt{2})/(1 - (1-\sqrt{2})x)$$
$$= \sum_{n\ge0} \left((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1} \right)\cdot x^n.$$

Evidently $a_n = \left((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1} \right)/\sqrt{8}$. Aided by the identity $(\sqrt{2}-1)(\sqrt{2}+1) = 1$, we can confirm the formula $a_{2n} = a_{n-1}^2 + a_n^2$ with a modest amount of algebraic labor.

Problem A4: Sum the series $\sum_{m\ge1} \sum_{n\ge1} 3^{-m}m^2n/(3^mn + 3^nm)$.

Solution A4: Let S denote this sum. It is unchanged by the exchange of m and n , so
$2S = \sum_{m\ge1} \sum_{n\ge1} (3^{-m}m^2n + 3^{-n}n^2m)/(3^mn + 3^nm) = \sum_{m\ge1} \sum_{n\ge1} 3^{-m-n}mn = T(1/3)^2$ where
$T(x) := \sum_{n\ge1} m\cdot x^m = x\cdot d\big(1/(1-x)\big)/dx = x/(1-x)^2$. Therefore $T(1/3) = 3/4$ and $S = 9/32$.

Problem A5: Prove that there is a constant C such that, if p(x) is a polynomial of degree 1999, then $|p(0)| \le C \int_{-1}^{1} |p(x)|dx$.

Solution A5: For any integers $n \ge 0$ and $m \ge 0$ let $Ø_{n,m}$ be the *infimum* (greatest lower bound) of $\int_{-1}^{1} |x^m p(x)/p(0)|\,dx$ taken over all polynomials p(x) of degree at most n with $p(0) \ne 0$. Any constant $C \ge 1/Ø_{1999,0}$ will solve the given problem provided $Ø_{1999,0} > 0$, and this proviso will follow from our proof, by induction on n , that *every* $Ø_{n,m} > 0$.

Let us rewrite $p(x)/p(0) = \prod_k (1 - x/z_k)$ where $z_1, z_2, \dots, z_k, \dots, z_n$ are all n zeros of p(x) ; every $z_k \ne 0$; if any $z_k = \infty$ it merely diminishes the degree of p below n . The infimum we seek is $Ø_{n,m} := \inf_{\mathbf{z}} \int_{-1}^{1} (\prod_k |1 - x/z_k|)\cdot|x|^m dx$ over all n-tuples $\mathbf{z} := (z_1, z_2, \dots, z_k, \dots, z_n)$ of nonzero complex numbers. Actually, since $|1 - x/z| \ge |1 - x/(1/\mathrm{Re}(1/z))|$, the search for $Ø_{n,m}$ can be restricted to n-tuples of real numbers z_k . The restriction $z_k^2 \le 1$ can be imposed too, thus restricting p to polynomials with n zeros all real and nonzero between ±1 inclusive, because while $z_k^2 \ge 1$ the factor $|1 - x/z_k| = 1 - x/z_k$ renders the integral in question a linear function of $1/z_k$ minimized at either $z_k = 1$ or $z_k = -1$. The restrictions don't change $Ø_{n,m}$.

Now define $F_{n,m}(z_1, z_2, \dots, z_k, \dots, z_n) := \int_{-1}^{1} (\prod_k |1 - x/z_k|)\cdot|x|^m dx$ for integers $n \ge 0$ and $m \ge 0$, so that $Ø_{n,m} := \inf_{\mathbf{z}} F_{n,m}(\mathbf{z})$ over all real n-tuples $\mathbf{z} = (z_1, z_2, \dots, z_k, \dots, z_n)$ which we might as well constrain thus: $0 < z_k^2 \le 1$. This $F_{n,m}$ is a continuous function of its n nonzero arguments z_k ; and since it is unaltered by their permutation they shall henceforth be assumed sorted thus: $z_1^2 \ge z_2^2 \ge \dots \ge z_k^2 \ge \dots \ge z_n^2 > 0$. And $Ø_{n,m}$ is a nonincreasing function of n since $Ø_{n-1,m} = \inf_{\mathbf{z}} F_{n-1,m}(z_1, z_2, \dots, z_{n-1}) = \inf_{\mathbf{z}} F_{n,m}(\infty, z_1, z_2, \dots, z_{n-1}) \ge Ø_{n,m}$. (Also $Ø_{n,m}$ is a nonincreasing function of m since $F_{n,m+1} \le F_{n,m}$.) A tedious computation establishes that $Ø_{0,m} = Ø_{1,m} = F_{1,m}(\pm1) = 2/(m+1) > Ø_{2,m} = F_{2,m}(-2^{-2/(m+3)}, 2^{-2/(m+3)}) = (2^{1+2/(m+3)} - 2)/(m+1)$.
This validates the induction hypothesis from which we shall infer next that every $Ø_{n,m} > 0$:

For some $N \ge 2$ and every $m \ge 0$ we find $0 < Ø_{N-1,m} = \min_{\mathbf{z}} F_{N-1,m}(\mathbf{z})$ minimized over all $N-1$-tuples $\mathbf{z} = (z_1, z_2, \dots, z_{N-1})$, and a minimizing $\mathbf{z}$ has $1 \ge z_1^2 \ge z_2^2 \ge \dots \ge z_{N-1}^2 > 0$.

To propel this hypothesis from N to N+1 we observe that $Ø_{N,m} = \inf_{\mathbf{z}} F_{N,m}(\mathbf{z})$ is the limit to which $F_{N,m}(\mathbf{z})$ descends as $\mathbf{z}$ moves through an infinite sequence of constrained N-tuples at each of which $F_{N,m}(\mathbf{z}) - Ø_{N,m}$ is at least, say, twice as big as at $\mathbf{z}$'s successor. We constrain each N-tuple $\mathbf{z} = (z_1, z_2, \dots, z_N)$ in the infinite sequence to satisfy $1 \ge z_1^2 \ge z_2^2 \ge \dots \ge z_N^2 > 0$, a constraint which has been shown above to be compatible with the infimum. This constraint restricts the infinite sequence's N-tuples **z** to a set whose closure, the N-cube in which $-1 \le \mathbf{z} \le 1$ elementwise, is *compact* (closed, bounded and finite-dimensional). Therefore the infinite sequence contains an infinite subsequence convergent to some limit N-tuple $\bar{\mathbf{z}}$ in the N-cube. $F_{N,m}(\mathbf{z}) \to Ø_{N,m}$ from above as $\mathbf{z} \to \bar{\mathbf{z}}$ through the subsequence. The elements of limit N-tuple $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_N)$ satisfy the weakened constraint $1 \ge \bar{z}_1^2 \ge \bar{z}_2^2 \ge \dots \ge \bar{z}_N^2 \ge 0$, but in fact $\bar{z}_N^2 > 0$, as shall now be proved indirectly.

Were $\bar{z}_N^2 = 0$, a nonnegative integer $L < N$ could be found with $1 \ge \bar{z}_1^2 \ge \bar{z}_2^2 \ge \dots \ge \bar{z}_L^2 > 0$ but $\bar{z}_{L+1} = \bar{z}_{L+2} = \dots = \bar{z}_N = 0$. This would imply that $|\prod_{L<j\le N} z_j| \to 0$ as $\mathbf{z} \to \bar{\mathbf{z}}$ although

$$|\textstyle\prod_{L<j\le N} z_j|\cdot F_{N,m}(z_1, z_2, \dots, z_L, z_{L+1}, \dots, z_N) = \int_{-1}^{1} (\prod_{1\le k\le L} |1 - x/z_k|)\cdot(\prod_{L<j\le N} |z_j - x|)\cdot|x|^m dx$$
$$\to \int_{-1}^{1} (\textstyle\prod_{1\le k\le L} |1 - x/\bar{z}_k|)\cdot|x|^{m+N-L}dx \quad \text{as} \quad \mathbf{z} \to \bar{z}$$
$$= F_{L,m+N-L}(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_L) \ \ge\ Ø_{L,m+N-L} > 0\,,$$

which would imply that $F_{N,m}(\mathbf{z}) \to +\infty$ as $\mathbf{z} \to \bar{\mathbf{z}}$ instead of $F_{N,m}(\mathbf{z}) \to Ø_{N,m} \le 2$. This contradiction explains why $L = N$ and $\bar{z}_N^2 > 0$, and therefore $F_{N,m}(\mathbf{z})$ is continuous in the neighborhood of $\mathbf{z} = \bar{\mathbf{z}}$; consequently $F_{N,m}(\mathbf{z}) \to F_{N,m}(\bar{\mathbf{z}}) > 0$ as $\mathbf{z} \to \bar{\mathbf{z}}$, which confirms that $0 < Ø_{N,m} = \min_z F_{N,m}(\mathbf{z}) = F_{N,m}(\bar{\mathbf{z}})$ as claimed. So ends a rather long proof.

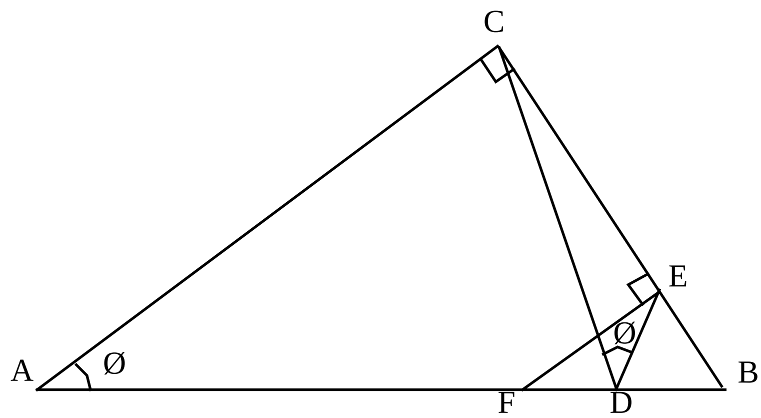
As Putnam problems go this seems excessively long, and we still don't know the least value $1/Ø_{1999,0}$ of C ..

Problem A6: The sequence $\{a_n\}_{n\ge1}$ is defined by $a_1 := 1$, $a_2 := 2$, $a_3 := 24$, and
$$a_n := \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$
for $n \ge 4$. Show that every a_n is an integer multiple of n.

Solution A6: Substitute $r_n := a_n/a_{n-1}$ into the given recurrence to obtain $r_2 = 2$, $r_3 = 12$ and $r_n = 6r_{n-1} - 8r_{n-3}$. This is a linear recurrence whose characteristic polynomial $r^2 - 6r + 8$ factors into $(r-2)(r-4)$; from this soon follows that $r_n = 4^{n-1} - 2^{n-1} = (2^{n-1} - 1)2^{n-1}$ and then $a_n = \prod_{2\le k\le n} r^k = \prod_{2\le k\le n} (2^{k-1} - 1)2^{k-1} = 2^{(n-1)n/2}\cdot\prod_{2\le k\le n} (2^{k-1} - 1)$. Next we shall prove that n divides this a_n by showing that every prime power that divides n divides a_n too.

If n is divisible by 2^m for some $m > 0$ then $m \le n-1$ (because $2^n \ge n+1$), and therefore a_n is divisible by 2^m too. If n is divisible by p^m for some odd prime p and integer $m > 0$, then again $m \le n-1$ (because $p^n > 2^n \ge n+1$), and each of the m integers $k = p, p^2, \dots, p^m$ appears among the consecutive integers $k = 2, 3, \dots, n$. Now we appeal to Fermat's "little" theorem to the effect that $L^{p-1} - 1$ is divisible by prime p whenever integer L is not divisible by p. Apply this for $L = 2^{(k-1)/(p-1)}$ whenever k is a power of p to infer that then $2^{k-1} - 1$ is divisible by p. Therefore, $a_n = 2^{(n-1)n/2}\cdot\prod_{2\le k\le n} (2^{k-1} - 1)$ has $n-1$ odd factors $(2^{k-1} - 1)$ of which at least m are divisible by p, so p^m divides a_n too. Since every prime power that divides n divides a_n too, n must divide a_n. End of proof.

Problem B1: Right triangle ABC has its right angle at C and $\angle BAC = Ø$; the point D is chosen on AB so that $|AC| = |AD| = 1$; The point E is chosen on BC so that $\angle CDE = Ø$. The perpendicular to BC at E meets AB at F . Evaluate lim |EF| as $Ø \to 0$. [Here "|PQ|" denotes the length of the line segment PQ .]



Solution B1: $|EF| \to 1/3$. To see why, observe that EF || CA , so ΔEFB ||| ΔCAB , whence

$$|EF| = |EF|/|CA| = |EB|/|CB| = 1 - |CE|/|CB| = 1 - |CE|/\tan(Ø)$$
$$= 1 - (|CD|\cdot\sin(Ø)/\sin(\angle CED))/\tan(Ø) \quad \text{from the sine law applied to } \Delta CED$$
$$= 1 - \cos(Ø)\cdot|CD|/\sin(\angle DEB) .$$

Since ΔCAD is isosceles, $|CD| = 2\cdot\sin(Ø/2)$ and $\angle CDA = (\pi - Ø)/2$, whereupon we find $\angle DEB = \angle ADE - \angle ABE = (\pi - Ø)/2 + Ø - (\pi/2 - Ø) = 3Ø/2$ and then

$$|EF| = 1 - 2\cdot\cos(Ø)\cdot\sin(Ø/2)/\sin(3Ø/2) = 1 - (\sin(3Ø/2) - \sin(Ø/2))/\sin(3Ø/2)$$
$$= \sin(Ø/2)/\sin(3Ø/2) \to 1/3 \quad \text{as} \quad Ø \to 0 , \text{ as claimed.}$$

An alternative proof, routine but tedious and error-prone, begins by identifying point A with 0 in the complex plane and points B, C, … with complex numbers b, c, … respectively, so that length $|PQ| = |p-q|$. Then $c = e^{\imath Ø}$ where $\imath = \sqrt{-1}$, and $d = 1$, and $b = 1/\cos(Ø)$, *etc*.

Problem B2: Let P(x) be a polynomial of degree n such that $P(x) = Q(x)P''(x)$ where Q(x) is a quadratic polynomial and P"(x) is the second derivative of P(x) . Show that if P(x) has at least two distinct zeros then it must have n distinct zeros, each either real or complex.

Solution B2: Two proofs come to mind. The first begins by supposing P(x) has at least one multiple zero z of multiplicity $m \ge 2$, and then translates the x-origin to z to simplify the Taylor series $P(x) = \sum_{m\le j\le n} a_j x^j/j!$ with $a_m a_n \ne 0$. Then $P''(x) = \sum_{m\le j\le n} a_j x^{j-2}/(j-2)!$ has at $z = 0$ a zero of multiplicity $m-2$, implying first that $z = 0$ is a double zero of Q(x) and then that $Q(x) = x^2/((n-1)n)$ in order to satisfy the identity $P(x) = Q(x)\cdot P''(x)$ first near $x = 0$ and secondly near $x = \infty$. But then this identity implies the identity

$$0 = (n-1)nP(x) - x^2P''(x) = \sum_{m\le j\le n} ((n-1)n - (j-1)j)a_j x^j/j! = \sum_{m\le j\le n} (n-j)(n+j-1)a_j x^j/j! ,$$

whence follows $a_j = 0$ for all $j < n$, leaving P(x) with one zero 0 of revealed multiplicity $m = n \ge 2$. Thus has the contrapositive of the desired result been proved: if P(x) has fewer than n distinct zeros, implying that at least one zero has multiplicity $m \ge 2$, then P(x) must have just one zero of multiplicity $m = n$. This is merely another way to state the desired result.

The second proof begins with the logarithmic derivative $P'(x)/P(x) = \sum_z m_z/(x-z)$ summed over all distinct zeros z each of multiplicity $m_z \ge 1$; of course $\sum_z m_z = n$. Differentiate again to get first $P''(x)/P(x) - (P'(x)/P(x))^2 = -\sum_z m_z/(x-z)^2$ and then a formula for the quadratic

$$Q(x) = P(x)/P''(x) = 1/\big((\sum_z m_z/(x-z))^2 - \sum_z m_z/(x-z)^2 \big) .$$

We shall deduce from it the desired result, namely that either every $m_z = 1$ or else $m_z = n$ for just one zero z , by considering the possibility that some zero t has multiplicity $m_t \ge 2$. This possibility and the formula together imply $Q(x)/(x-t)^2 \to 1/\big((m_t-1)m_t\big)$ as $x \to t$, implying that the quadratic $Q(x) = (x-t)^2/\big((m_t-1)m_t\big)$; but because $\sum_z m_z = n$ the formula also implies $Q(x)/(x-t)^2 \to 1/\big((n-1)n\big)$ as $x \to \infty$, whence the quadratic $Q(x) = (x-t)^2/\big((n-1)n\big)$. The only way to reconcile these two expressions for Q is to infer that $m_t = n$ whenever $m_t \ge 2$, *QED.*

(Of course, it is legitimate to wonder whether there are any polynomials $P(x)$ of degree $n > 2$ with quadratic P/P'' and more than one distinct zero. There are lots of them; $x^3 \pm x$ and $x^4 \pm 6x^2 + 5$ and $7x^5 \pm 10x^3 + 3x$ are the simplest examples.)

Problem B3: Let $A := \{ (x, y) : 0 \le x, y < 1 \}$. For (x, y) in A let $S(x, y) := \sum\sum x^m y^n$ summed over all pairs (m, n) of positive integers satisfying $1/2 \le m/n \le 2$. Evaluate $\lim (1 - xy^2)(1 - x^2y)S(x, y)$ as (x, y) approaches $(1, 1)$ from within A .

Solution B3: The desired limit is 3 because $S(x, y) = (1 + xy + x^2y^2)/\big((1 - xy^2)(1 - x^2y)\big) - 1$. To justify this formula expand the right-hand side's quotient in a power series convergent in A :

$$(1 + xy + x^2y^2)\cdot\sum_{i\ge0} \sum_{j\ge0} x^{i+2j}\cdot y^{2i+j} = \sum_{i\ge0} \sum_{j\ge0} (x^{i+2j}\cdot y^{2i+j} + x^{i+2j+1}\cdot y^{2i+j+1} + x^{i+2j+2}\cdot y^{2i+j+2}) .$$

Except for the one term $x^0y^0 = 1$, all the terms in this power series have the form x^my^n with $m > 0$, $n > 0$ and $1/2 \le m/n \le 2$. What remains to be seen is that every term of this form does appear just once in the series. Terms of this form fall into three disjoint subsets:

0: $m+n \equiv 0$ **mod** 3 . Let $k := (m+n)/3$, $i := n-k$, $j := m-k$; then $i+2j = m$ and $2i+j = n$.
1: $m+n \equiv 1$ **mod** 3 . Let $k := (m+n-1)/3$, $i := n-1-k$, $j := m-1-k$; $i+2j+2 = m$, $2i+j+2 = n$.
2: $m+n \equiv 2$ **mod** 3 . Let $k := (m+n-2)/3$, $i := n-1-k$, $j := m-1-k$; $i+2j+1 = m$, $2i+j+1 = n$.

It seems necessary to check that $i \ge 0$ and $j \ge 0$ for each subset separately:

0: $i = n-k = (2n-m)/3 \ge 0$ because $m/n \le 2$. Similarly $j \ge 0$.
1: $i = n-1-k = (2n-m-2)/3 \ge -2/3$, so integer $i \ge 0$. Similarly $j \ge 0$.
2: $i = n-1-k = (2n-m-1)/3 \ge -1/3$, so integer $i \ge 0$. Similarly $j \ge 0$.

Thus the three subsets' union provides a one-to-one association between all terms x^my^n in the given sum $S(x, y)$, and all terms except the constant term in the power series expansion. This justifies the formula alleged for $S(x, y)$ and confirms that the limit was evaluated correctly.

Problem B4: Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Solution B4: For any x and X, Taylor's formula for $f(X)$ with an integral remainder is $f(X) = f(x) + (X-x)f'(x) + (X-x)^2 f''(x)/2 + \int_x^X (X-t)^2 f'''(t)dt/2$; and if $X < x$ we find, since $f'''(t) > 0$, that $f(X) < f(x) + (X-x)f'(x) + (X-x)^2 f''(x)/2$. Set $X := x - f'(x)/f''(x) < x$ to infer first $0 < f(X) < f(x) - f'(x)^2/f''(x) + (f'(x)^2/f''(x))/2$ and then $f'(x)^2 < 2f(x)f''(x)$.

Similarly $f'(X) = f'(x) + (X-x)f''(x) + \int_x^X (X-t)f'''(t)dt$; now, since $f''' \le f$ and $X-t$ has the same sign as $X-x$ and also dt, we infer $f'(X) \le f'(x) + (X-x)f''(x) + \int_x^X (X-t)f(t)dt$. Integration by parts turns this last inequality into

$$f'(X) \le f'(x) + (X-x)f''(x) + (X-x)^2 f(x)/2 + \int_x^X (X-t)^2 f'(t)dt/2 .$$

Again, if $X < x$ we find, since $f'(t) > 0$, that $f'(X) \le f'(x) + (X-x)f''(x) + (X-x)^2 f(x)/2$; this time set $X := x - f''(x)/f(x) < x$ to infer from $0 < f'(X)$ that $f''(x)^2 < 2f'(x)f(x)$. This combines with the inequality $f'(x)^2 < 2f(x)f''(x)$ inferred above to prove $f'(x) < 2f(x)$, *QED.*

(An example of such a function $f(x)$ is $ß + \exp(\mu\cdot x)$ for any constants $ß \ge 0 < \mu \le 1$. The example $ß + (\mu x + \sqrt{\mu^2 x^2+1})^n$ for $n \ge 2$, $ß \ge 0$ and $\mu \le \sqrt[3]{(5/2)^{5/2}/(n(n+1)^{5/2}(n+2))}$ is less obvious.)

Problem B5: For any integer $n \ge 3$ let $Ø := 2\pi/n$. Evaluate the determinant of the n-by-n matrix $I + A$ where I is the identity matrix and $A = \{a_{jk}\}$ has entries $a_{jk} := \cos((j+k)Ø)$ for all indices j and k.

Solution B5: $\det(I+A) = 1 - n^2/4$. One neat way to prove this uses a determinantal identity $\det(I + P\cdot R^T) = \det(I + R^T\cdot P)$ in which P and R are matrices of the same dimensions, so that both products $P\cdot R^T$ and $R^T\cdot P$ are square though of perhaps different dimensions. Here R^T is the transpose of R; and the two identity-matrices " I " may have different dimensions too. To confirm the identity apply the formula $\det(X\cdot Y) = \det(X)\cdot\det(Y)$ to the triangular factorizations

$$\begin{bmatrix} I & O \\ R^T & I \end{bmatrix} \cdot \begin{bmatrix} I & -P \\ O^T & I + (R^T \cdot P) \end{bmatrix} = \begin{bmatrix} I & -P \\ R^T & I \end{bmatrix} = \begin{bmatrix} I & -P \\ O^T & I \end{bmatrix} \cdot \begin{bmatrix} I + (P \cdot R^T) & O \\ R^T & I \end{bmatrix} .$$

Now set $\mu := \exp(\imath Ø) \ne \pm 1$ so that $\mu^k = \exp(\imath kØ) = \cos(kØ) + \imath\cdot\sin(kØ)$ for every integer k; in particular $\mu^n = 1$ and the complex conjugate $\bar{\mu} = \exp(-\imath Ø) = 1/\mu$. Let row vector $w^T := [\mu, \mu^2, \mu^3, \dots, \mu^n]$ and, for future reference, compute $\bar{w}^T\cdot w = w^T\cdot\bar{w} = n$ and $w^T\cdot w = \mu^2 + \mu^4 + \mu^6 + \dots + \mu^{2n} = (1-\mu^{2n})/(1-\mu^2) = 0 = \bar{w}^T\cdot\bar{w}$. All this is relevant because, as is easily verified, $A = \mathrm{Re}(w\cdot w^T) = (w\cdot w^T + \bar{w}\cdot\bar{w}^T)/2 = [w, \bar{w}]\cdot[w, \bar{w}]^T/2$. Consequently

$$\det(I+A) = \det(I + [w, \bar{w}]\cdot[w, \bar{w}]^T/2) = \det(I + [w, \bar{w}]^T\cdot[w, \bar{w}]/2), \quad \text{from the identity},$$
$$= (1 + w^T\cdot w/2)(1 + \bar{w}^T\cdot\bar{w}/2) - (w^T\cdot\bar{w})(\bar{w}^T\cdot w)/4 = 1 - n^2/4 \text{ as claimed.}$$

(This is surprising. Rarely does a formula hold for all n-by-n matrices with $n \ge 3$ but not for $n = 2$ nor $n = 1$.)

Problem B6: Let S be a finite set of integers each greater than 1 . Suppose that for each integer n there is some s in S such that GCD(s, n) = 1 or GCD(s, n) = s . Show that S must contain some s and t for which GCD(s, t) is a prime. [Here "GCD(x, y)" denotes the Greatest Common Divisor of x and y .]

Solution B6: (Presumably s = t is allowed since S := { 3 } meets the specifications for S .) This neat solution was suggested by David Blackston, a graduate student of Computer Science.

Let L be the least positive integer such that GCD(s, L) > 1 for every s in S ; this L must exist because it need not exceed the product of all primes each of which divides at least one member of S . But L may be smaller than that product if there exist two primes both of which divide every member of S divisible by either. Anyway, L is a product of primes none of whose squares divides L ; otherwise L would not be the *least* … .

Some t in S must satisfy GCD(t, L) = t > 1 ; choose any prime p that divides t . Since L/p < L , there must be some s in S such that GCD(s, L/p) = 1 . David asserts that GCD(s, t) = p . Let's see why his assertion is true.

This t = GCD(t, L) is a product of a nonempty subset (containing p) selected from the prime factors of L . Since GCD(s, L/p) = 1 , no prime factor of L/p divides s , and therefore no prime factor of t/p divides s ; *i.e.* GCD(s, t/p) = 1 . But GCD(s, L) > 1 , so GCD(s, L) = p ; this means that some positive power of p divides s , and therefore GCD(s, t) = p as claimed.

The foregoing solutions have been posted on the class web-page:

`http://cs.berkeley.edu/~wkahan/MathH90/Putnam99.pdf`