The 1999 W.L. Putnam Competition Exam took place on Sat. 4 Dec. 1999 in two sessions:
Problems A1 - A6 were to be solved during the morning session, 8-11 am;
Problems B1 - B6 were to be solved during the afternoon session, 1-4 pm.

**Problem A1:** Find polynomials \( f(x) \), \( g(x) \) and \( h(x) \), if they exist, such that, for all \( x \),
\[
|f(x)| - |g(x)| + h(x) = \begin{cases} 
-1 & \text{if } x < -1, \\
3x + 2 & \text{if } -1 \leq x \leq 0, \\
-2x + 2 & \text{if } x > 0.
\end{cases}
\]

**Solution A1:** \( f(x) := \frac{3(x+1)}{2} \), \( g(x) := \frac{5x}{2} \) and \( h(x) := \frac{1 - 2x}{2} \) meet the requirements.

This particular solution can be determined most easily by solving three linear equations:
\[
-f + g + h = -1, \quad f + g + h = 3x + 2, \quad f - g + h = -2x + 2.
\]

**Problem A2:** Let \( p(x) \) be a polynomial that is nonnegative for all \( x \) . Prove that, for some \( k \), there are polynomials \( f_1(x), f_2(x), \ldots, f_k(x) \) such that \( p(x) = \sum_{1 \leq j \leq k} (f_j(x))^2 \).

**Solution A2:** Presumably “for all \( x \)” means “for all real scalar \( x \)” since otherwise, if \( x \) could be complex, we would find \( p(x) \) to be a positive constant; and if \( x \) could be a real vector then rational functions \( f_j \) might be needed. See pp. 55-57 and 300-304 in the book *Inequalities* by G.H. Hardy, J.E. Littlewood & G. Pólya, 2d. ed. (1952) for more about that.

Back to our problem: \( p(z) \) must have real coefficients because they can be determined by solving a real system of linear equations with a number of values of \( p(z) \) at real arguments in the right-hand side. Therefore any zeros \( x_k \pm iy_k \) of \( p(z) \) that are not real must come in complex-conjugate pairs. Every real zero \( x_j \) of \( p(z) \) must have even multiplicity lest \( p(z) \) change sign there. (And the degree of \( p(z) \) must be even since otherwise it would take opposite signs at \( z = \pm \infty \).)

Therefore \( p(z) = \prod_j (z-x_j)^2 \prod_k ((z-x_k)^2 + y_k^2) \), whereupon we may rewrite
\[
\prod_k ((z-x_k)^2 + y_k^2) = \prod_k (z-x_k + iy_k)(z-x_k - iy_k) = (R(z) + iI(z))(R(z) - iI(z)) = R(z)^2 + I(z)^2
\]
with the understanding that the factors \( (z-x_j) \) and \( (z-x_k - iy_k) \) need not be all distinct; here polynomial \( R(z) + iI(z) = \prod_k (z-x_k + iy_k) \) broken into real and imaginary parts. Consequently \( k = 2 \), \( f_1(z) := R(z) \prod_j (z-x_j) \), and \( f_2(z) := I(z) \prod_j (z-x_j) \).

**Problem A3:** Consider the power series expansion \( 1/(1 - 2x - x^2) = \sum_{n=0}^\infty a_n x^n \). Prove for each integer \( n \geq 0 \) that there is an integer \( m \) such that \( a_m = a_n^2 + a_{n+1}^2 \).

**Solution A3:** \( m = 2n+2 \). Why? Go from the partial fraction expansion to the power series thus:
\[
\sqrt{8}/(1 - 2x - x^2) = (1+\sqrt{2})/(1 - (1+\sqrt{2})x) - (1-\sqrt{2})/(1 - (1-\sqrt{2})x)
\]
\[
= \sum_{n \geq 0} ((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1})x^n.
\]
Evidently \( a_n = ((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1})/\sqrt{8} \). Aided by the identity \((\sqrt{2}-1)(\sqrt{2}+1) = 1\), we can confirm the formula \( a_{2n} = a_{n-1}^2 + a_n^2 \) with a modest amount of algebraic labor.
Problem A4: Sum the series \( \sum_{m \geq 1} \sum_{n \geq 1} \frac{3^{-m}m^2n}{(3^m n + 3^n m)} \).

Solution A4: Let \( S \) denote this sum. It is unchanged by the exchange of \( m \) and \( n \), so
\[
2S = \sum_{m \geq 1} \sum_{n \geq 1} \left( \frac{3^{-m}m^2n}{(3^m n + 3^n m)} \right) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{3^{-m}m^2n}{3^{m+n}} = \frac{T(1/3)^2}{3} \]
where
\[
T(x) := \sum_{n \geq 1} \frac{x^n}{n} = \frac{1}{1-x} \frac{d}{dx} \frac{1}{1-x} = \frac{x}{(1-x)^2} . \]
Therefore \( T(1/3) = 3/4 \) and \( S = 9/32 \).

Problem A5: Prove that there is a constant \( C \) such that, if \( p(x) \) is a polynomial of degree 1999, then
\[
|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx .
\]

Solution A5: For any integers \( n \geq 0 \) and \( m \geq 0 \) let \( \Phi_{n,m} \) be the infimum (greatest lower bound) of \( \int_{-1}^{1} |p(x)/p(0)| \, dx \) taken over all polynomials \( p(x) \) of degree at most \( n \) with \( p(0) \neq 0 \). Any constant \( C \geq 1/\Phi_{1999,0} \) will solve the given problem provided \( \Phi_{1999,0} > 0 \), and this proviso will follow from our proof, by induction on \( n \), that every \( \Phi_{n,m} > 0 \).

Let us rewrite \( p(x)/p(0) = \prod_{k} (1-x/z_k) \) where \( z_1, z_2, \ldots, z_k, \ldots, z_n \) are all \( n \) zeros of \( p(x) \); every \( z_k \neq 0 \); if any \( z_k = \infty \) it merely diminishes the degree of \( p \) below \( n \). The infimum we seek is \( \Phi_{n,m} := \inf_{z} \int_{-1}^{1} (\prod_{k} |1-x/z_k|) |x|^m \, dx \) over all \( n \)-tuples \( z := (z_1, z_2, \ldots, z_k, \ldots, z_n) \) of nonzero complex numbers. Actually, since \( |1-x/z| \geq |1-x/(1/\text{Re}(1/z))| \), the search for \( \Phi_{n,m} \) can be restricted to \( n \)-tuples of real numbers \( z_k \). The restriction \( z_k^2 \leq 1 \) can be imposed too, thus restricting \( p \) to polynomials with \( n \) zeros all real and nonzero between \( \pm 1 \) inclusive, because while \( z_k^2 \geq 1 \) the factor \( |1-x/z_k| = 1-x/z_k \) renders the integral in question a linear function of \( 1/z_k \) minimized at either \( z_k = 1 \) or \( z_k = -1 \). The restrictions don’t change \( \Phi_{n,m} \).

Now define \( F_{n,m}(z_1, z_2, \ldots, z_k, \ldots, z_n) := \int_{-1}^{1} (\prod_{k} |1-x/z_k|) |x|^m \, dx \) for integers \( n \geq 0 \) and \( m \geq 0 \), so that \( \Phi_{n,m} := \inf_{z} F_{n,m}(z) \) over all real \( n \)-tuples \( z := (z_1, z_2, \ldots, z_k, \ldots, z_n) \) which we might as well constrain thus: \( 0 < z_k^2 \leq 1 \). This \( F_{n,m} \) is a continuous function of its \( n \) nonzero arguments \( z_k \); and since it is unaltered by their permutation they shall henceforth be assumed sorted thus: \( z_1^2 \geq z_2^2 \geq \ldots \geq z_k^2 \geq \ldots \geq z_n^2 > 0 \). And \( \Phi_{n,m} \) is a nonincreasing function of \( n \) since \( \Phi_{n-1,m} = \inf_{z} F_{n-1,m}(z_1, z_2, \ldots, z_{n-1}) = \inf_{z} F_{n,m}(\infty, z_1, z_2, \ldots, z_{n-1}) \geq \Phi_{n,m} \). (Also \( \Phi_{n,m} \) is a nonincreasing function of \( m \) since \( F_{n,m+1} \leq F_{n,m} \).) A tedious computation establishes that \( \Phi_{0,m} = \Phi_{1,m} = F_{1,m}(\pm 1) = 2/(m+1) > \Phi_{2,m} = F_{2,m}(2^{-2/(m+3)}, 2^{-2/(m+3)}) = (2+2/(m+3) - 2)/(m+1) \). This validates the induction hypothesis from which we shall infer next that every \( \Phi_{n,m} > 0 \).

For some \( N \geq 2 \), and every \( m \geq 0 \), we find \( 0 < \Phi_{N-1,m} = \min_{z} F_{N-1,m}(z) \) minimized over all \( N-1 \)-tuples \( z := (z_1, z_2, \ldots, z_{N-1}) \), and a minimizing \( z \) has \( 1 \geq z_1^2 \geq z_2^2 \geq \ldots \geq z_{N-1}^2 > 0 \).

To propel this hypothesis from \( N \) to \( N+1 \), we observe that \( \Phi_{N,m} = \inf_{z} F_{N,m}(z) \) is the limit to which \( F_{N,m}(z) \) descends as \( z \) moves through an infinite sequence of constrained \( N \)-tuples at each of which \( F_{N,m}(z) - \Phi_{N,m} \) is at least, say, twice as big as at \( z \)’s successor. We constrain each \( N \)-tuple \( z := (z_1, z_2, \ldots, z_N) \) in the infinite sequence to satisfy \( 1 \geq z_1^2 \geq z_2^2 \geq \ldots \geq z_N^2 > 0 \).
a constraint which has been shown above to be compatible with the infimum. This constraint restricts the infinite sequence’s N-tuples \( z \) to a set whose closure, the N-cube in which \(-1 \leq z \leq 1\) elementwise, is \textit{compact} (closed, bounded and finite-dimensional). Therefore the infinite sequence contains an infinite subsequence convergent to some limit N-tuple \( \bar{z} \) in the N-cube. \( F_{N,m}(z) \rightleftharpoons \emptyset_{N,m} \) from above as \( z \to \bar{z} \) through the subsequence. The elements of limit N-tuple \( \bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) \) satisfy the weakened constraint \( 1 \geq \bar{z}_1^2 \geq \bar{z}_2^2 \geq \ldots \geq \bar{z}_N^2 \geq 0 \), but in fact \( \bar{z}_N^2 > 0 \), as shall now be proved indirectly.

Were \( \bar{z}_N^2 = 0 \), a nonnegative integer \( L < N \) could be found with \( 1 \geq \bar{z}_1^2 \geq \bar{z}_2^2 \geq \ldots \geq \bar{z}_L^2 > 0 \) but \( \bar{z}_{L+1} = \bar{z}_{L+2} = \ldots = \bar{z}_N = 0 \). This would imply that \( \left| \sum_{L<j \leq N} z_j \right| \rightleftharpoons 0 \) as \( z \to \bar{z} \) although \( \left| \sum_{L<j \leq N} z_j \right| F_{N,m}(z_1, z_2, \ldots, z_L, z_{L+1}, \ldots, z_N) = \int_{-1}^{1} (\sum_{1<k \leq L} |1 - x/z_k|) \cdot (\sum_{L<j \leq N} |z_j - x|) \cdot |x|^m \, dx \rightleftharpoons \int_{-1}^{1} (\sum_{1<k \leq L} |1 - x/z_k|) \cdot |x|^{m+N-L} \, dx \) as \( z \to \bar{z} \) \( = F_{L,m+N-L}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_L) \geq \emptyset_{L,m+N-L} > 0 \), which would imply that \( F_{N,m}(z) \rightleftharpoons +\infty \) as \( z \to \bar{z} \) instead of \( F_{N,m}(z) \rightleftharpoons \emptyset_{N,m} \leq 2 \). This contradiction explains why \( L = N \) and \( \bar{z}_N^2 > 0 \), and therefore \( F_{N,m}(z) \) is continuous in the neighborhood of \( z = \bar{z} \); consequently \( F_{N,m}(z) \rightleftharpoons F_{N,m}(\bar{z}) > 0 \) as \( z \to \bar{z} \), which confirms that \( 0 < \emptyset_{N,m} = \min_x F_{N,m}(z) = F_{N,m}(\bar{z}) \) as claimed. So ends a rather long proof.

As Putnam problems go this seems excessively long, and we still don’t know the least value \( 1/\emptyset_{1999,0} \) of \( C .. \)

**Problem A6:** The sequence \( \{a_n\}_{n\geq1} \) is defined by \( a_1 := 1 \), \( a_2 := 2 \), \( a_3 := 24 \), and
\[
 a_n := \frac{6a_{n-1}^2 - 8a_{n-1}a_{n-3} - 8a_{n-3}^2}{a_{n-2}a_{n-3}} \quad \text{for } n \geq 4.
\]
Show that every \( a_n \) is an integer multiple of \( n \).

**Solution A6:** Substitute \( r_n := a_n/a_{n-1} \) into the given recurrence to obtain \( r_2 = 2 \), \( r_3 = 12 \) and \( r_n = 6r_{n-1} - 8r_{n-3} \). This is a linear recurrence whose characteristic polynomial \( r^2 - 6r + 8 \) factors into \( (r-2)(r-4) \); from this soon follows that \( r_n = 4^{n-1} - 2^{n-1} = 2^{n-1} - 1 \) \( 2^{n-1} \) and then
\[
 a_n = \prod_{2 \leq k \leq n} r^k = \prod_{2 \leq k \leq n} (2^{k-1} - 1)^{2^{k-2}} = 2^{(n-1)n/2} \prod_{2 \leq k \leq n} (2^{k-1} - 1) .
\]
Next we shall prove that \( n \) divides this \( a_n \) by showing that every prime power that divides \( n \) divides \( a_n \) too.

If \( n \) is divisible by \( 2^m \) for some \( m > 0 \) then \( m \leq n-1 \) (because \( 2^n \geq n+1 \)), and therefore \( a_n \) is divisible by \( 2^m \) too. If \( n \) is divisible by \( p^m \) for some odd prime \( p \) and integer \( m > 0 \), then again \( m \leq n-1 \) (because \( p^n > 2^n \geq n+1 \)), and each of the \( m \) integers \( k = p, p^2, \ldots, p^m \) appears among the consecutive integers \( k = 2, 3, \ldots, n \). Now we appeal to Fermat’s “little” theorem to the effect that \( L^{p^m-1} \) is divisible by prime \( p \) whenever integer \( L \) is not divisible by \( p \). Apply this for \( L = 2^{k-1}/(p-1) \) whenever \( k \) is a power of \( p \) to infer that then \( 2^{k-1} - 1 \) is divisible by \( p \). Therefore, \( a_n = 2^{(n-1)n/2} \prod_{2 \leq k \leq n} (2^{k-1} - 1) \) has \( n-1 \) odd factors \( 2^{k-1} - 1 \) of which at least \( m \) are divisible by \( p \), so \( p^m \) divides \( a_n \) too. Since every prime power that divides \( n \) divides \( a_n \) too, \( n \) must divide \( a_n \). End of proof.
Problem B1: Right triangle ABC has its right angle at C and \( \angle BAC = \emptyset \); the point D is chosen on AB so that \(|AC| = |AD| = 1\); The point E is chosen on BC so that \( \angle CDE = \emptyset \). The perpendicular to BC at E meets AB at F. Evaluate \( \lim_{\emptyset \to 0} |EF| \) as \( \emptyset \to 0 \). [Here "|PQ|" denotes the length of the line segment PQ.]

Solution B1: \(|EF| \to 1/3\). To see why, observe that \( EF \parallel CA \), so \( \Delta EFB \parallel \Delta CAB \), whence
\[
|EF| = |EF|/|CA| = |EB|/|CB| = 1 - |CE|/|CB| = 1 - |CE|/\tan(\emptyset)
\]
\[
= 1 - ((CD)\cdot\sin(\emptyset)/\sin(\angle CED))/\tan(\emptyset) \quad \text{from the sine law applied to } \Delta CED
\]
\[
= 1 - \cos(\emptyset)\cdot|CD|/\sin(\angle DEB). 
\]
Since \( \Delta CAD \) is isosceles, \(|CD| = 2\cdot\sin(\emptyset/2)\) and \( \angle CDA = (\pi - \emptyset)/2 \), whereupon we find
\[
\angle DEB = \angle ADE - \angle ABE = (\pi - \emptyset)/2 + \emptyset - (\pi/2 - \emptyset) = 3\emptyset/2 
\]
and then
\[
|EF| = 1 - 2\cos(\emptyset)\cdot\sin(\emptyset/2)/\sin(3\emptyset/2) = 1 - \left(\sin(3\emptyset/2) - \sin(\emptyset/2)/\sin(3\emptyset/2)\right)
\]
\[
= \sin(\emptyset/2)/\sin(3\emptyset/2) \to 1/3 \quad \text{as } \emptyset \to 0, \text{ as claimed.}
\]

An alternative proof, routine but tedious and error-prone, begins by identifying point A with 0 in the complex plane and points B, C, … with complex numbers b, c, … respectively, so that length |PQ| = |p−q|. Then \( c = e^{i\emptyset} \) where \( i = \sqrt{-1} \), and \( d = 1 \), and \( b = 1/\cos(\emptyset) \), etc.

Problem B2: Let P(x) be a polynomial of degree n such that \( P(x) = Q(x)P''(x) \) where Q(x) is a quadratic polynomial and \( P''(x) \) is the second derivative of \( P(x) \). Show that if \( P(x) \) has at least two distinct zeros then it must have n distinct zeros, each either real or complex.

Solution B2: Two proofs come to mind. The first begins by supposing \( P(x) \) has at least one multiple zero \( z \) of multiplicity \( m \geq 2 \), and then translates the x-origin to \( z \) to simplify the Taylor series \( P(x) = \sum_{m \leq j \leq n} a_j x^j/j! \) with \( a_m a_n \neq 0 \). Then \( P''(x) = \sum_{m \leq j \leq n} a_j x^{j-2}/(j-2)! \) has at \( z = 0 \) a zero of multiplicity \( m - 2 \), implying first that \( z = 0 \) is a double zero of \( Q(x) \) and then that \( Q(x) = x^2/((n-1)n) \) in order to satisfy the identity \( P(x) = Q(x)\cdot P''(x) \) first near \( x = 0 \) and secondly near \( x = \infty \). But then this identity implies the identity
\[
0 = (n-1)n P(x) - x^2 P''(x) = \sum_{m \leq j \leq n} ((n-1)n - (j-1))a_j x^j/j! = \sum_{m \leq j \leq n} (n-j)(n+j-1)a_j x^j/j!,
\]
whence follows \( a_j = 0 \) for all \( j < n \), leaving \( P(x) \) with one zero 0 of revealed multiplicity \( m = n \geq 2 \). Thus has the contrapositive of the desired result been proved: if \( P(x) \) has fewer than \( n \) distinct zeros, implying that at least one zero has multiplicity \( m \geq 2 \), then \( P(x) \) must have just one zero of multiplicity \( m = n \). This is merely another way to state the desired result.
The second proof begins with the logarithmic derivative \( \frac{P'(x)}{P(x)} = \sum_z \frac{m_z}{x-z} \) summed over all distinct zeros \( z \) each of multiplicity \( m_z \geq 1 \); of course \( \sum_z m_z = n \). Differentiate again to get first \( \frac{P''(x)}{P(x)} - \left( \frac{P'(x)}{P(x)} \right)^2 = -\sum_z \frac{m_z}{(x-z)^2} \) and then a formula for the quadratic

\[
Q(x) = \frac{P(x)}{P''(x)} = \frac{1}{\left( \sum_z \frac{m_z}{(x-z)} \right)^2 - \sum_z \frac{m_z}{(x-z)^2}}.
\]

We shall deduce from it the desired result, namely that either every \( m_z = 1 \) or else \( m_z = n \) for just one zero \( z \), by considering the possibility that some zero \( t \) has multiplicity \( m_t \geq 2 \). This possibility and the formula together imply \( Q(x)/(x-t)^2 \rightarrow 1/((m_t-1)m_t) \) as \( x \rightarrow t \), implying that the quadratic \( Q(x) = (x-t)^2/((m_t-1)m_t) \); but because \( \sum_z m_z = n \) the formula also implies \( Q(x)/(x-t)^2 \rightarrow 1/((n-1)n) \) as \( x \rightarrow \infty \), whence the quadratic \( Q(x) = (x-t)^2/((n-1)n) \). The only way to reconcile these two expressions for \( Q \) is to infer that \( m_t = n \) whenever \( m_t \geq 2 \), QED.

(Of course, it is legitimate to wonder whether there are any polynomials \( P(x) \) of degree \( n > 2 \) with quadratic \( P/P'' \) and more than one distinct zero. There are lots of them; \( x^3 \pm x \) and \( x^4 \pm 6x^2 + 5 \) and \( 7x^5 \pm 10x^3 + 3x \) are the simplest examples.)

**Problem B3:** Let \( A := \{ (x, y) : 0 \leq x, y < 1 \} \). For \( (x, y) \) in \( A \) let \( S(x, y) := \sum \sum x^m y^n \) summed over all pairs \( (m, n) \) of positive integers satisfying \( 1/2 \leq m/n \leq 2 \). Evaluate

\[
\lim (1 - xy^2)(1 - x^2y)S(x, y) \quad \text{as} \quad (x, y) \quad \text{approaches} \quad (1, 1) \quad \text{from within} \quad A.
\]

**Solution B3:** The desired limit is 3 because \( S(x, y) = (1 + xy + x^2y^2)/((1 - xy^2)(1 - x^2y)) - 1 \). To justify this formula expand the right-hand side’s quotient in a power series convergent in \( A : \)

\[
(1 + xy + x^2y^2)^{\sum_{i \geq 0} \sum_{j \geq 0} x^{i+2j} y^{2i+j}} = \sum_{i \geq 0} \sum_{j \geq 0} (x^{i+2j} y^{2i+j} + x^{i+2j+1} y^{2i+j+1} + x^{i+2j+2} y^{2i+j+2}).
\]

Except for the one term \( x^0y^0 = 1 \), all the terms in this power series have the form \( x^m y^n \) with \( m > 0 \), \( n > 0 \) and \( 1/2 \leq m/n \leq 2 \). What remains to be seen is that every term of this form does appear just once in the series. Terms of this form fall into three disjoint subsets:

0: \( m+n \equiv 0 \mod 3 \). Let \( k := (m+n)/3 \), \( i := n-k \), \( j := m-k \); then \( i+2j = m \) and \( 2i+j = n \).

1: \( m+n \equiv 1 \mod 3 \). Let \( k := (m+n-1)/3 \), \( i := n-1-k \), \( j := m-1-k \); \( i+2j+2 = m \), \( 2i+j+2 = n \).

2: \( m+n \equiv 2 \mod 3 \). Let \( k := (m+n-2)/3 \), \( i := n-1-k \), \( j := m-1-k \); \( i+2j+1 = m \), \( 2i+j+1 = n \).

It seems necessary to check that \( i \geq 0 \) and \( j \geq 0 \) for each subset separately:

0: \( i = n-k = (2n-m)/3 \geq 0 \) because \( m/n \leq 2 \). Similarly \( j \geq 0 \).

1: \( i = n-1-k = (2n-m-2)/3 \geq -2/3 \), so integer \( i \geq 0 \). Similarly \( j \geq 0 \).

2: \( i = n-1-k = (2n-m-1)/3 \geq -1/3 \), so integer \( i \geq 0 \). Similarly \( j \geq 0 \).

Thus the three subsets’ union provides a one-to-one association between all terms \( x^m y^n \) in the given sum \( S(x, y) \), and all terms except the constant term in the power series expansion. This justifies the formula alleged for \( S(x, y) \) and confirms that the limit was evaluated correctly.
**Problem B4:** Let \( f \) be a real function with a continuous third derivative such that \( f(x) \), \( f'(x) \), \( f''(x) \) and \( f'''(x) \) are positive for all \( x \). Suppose that \( f'''(x) \leq f(x) \) for all \( x \). Show that \( f'(x) < 2f(x) \) for all \( x \).

**Solution B4:** For any \( x \) and \( X \), Taylor’s formula for \( f(X) \) with an integral remainder is

\[
f(X) = f(x) + (X-x)f'(x) + (X-x)^2 f''(x)/2 + \int_x^X (X-t)^2 f'''(t)dt/2 ;
\]

and if \( X < x \) we find, since \( f'''(t) > 0 \), that \( f(X) < f(x) + (X-x)f'(x) + (X-x)^2 f''(x)/2 \). Set \( X := x - f'(x)/f''(x) < x \) to infer first \( 0 < f(X) < f(x) - f'(x)^2 /f''(x) \) and then \( f'(x)^2 < 2f(x)f''(x) \).

Similarly \( f'(X) = f'(x) + (X-x)f''(x) + \int_x^X (X-t) f'''(t)dt \); now, since \( f''' \leq f \) and \( X-t \) has the same sign as \( X-x \) and also \( dt \), we infer \( f'(X) \leq f'(x) + (X-x)f''(x) + \int_x^X (X-t) f(t)dt \).

Integration by parts turns this last inequality into

\[
f'(X) \leq f'(x) + (X-x)f''(x) + (X-x)^2 f(x)/2 + \int_x^X (X-t)^2 f'(t)dt/2 ;
\]

Again, if \( X < x \) we find, since \( f''(t) > 0 \), that \( f'(X) \leq f'(x) + (X-x)f''(x) + (X-x)^2 f(x)/2 \); this time set \( X := x - f''(x)/f'(x) < x \) to infer from \( 0 < f'(X) \) that \( f''(x)^2 < 2f'(x)f(x) \). This combines with the inequality \( f'(x)^2 < 2f(x)f''(x) \) inferred above to prove \( f'(x) < 2f(x) \), QED.

(An example of such a function \( f(x) \) is \( \beta + \exp(\mu \cdot x) \) for any constants \( \beta \geq 0 < \mu \leq 1 \). The example \( \beta + (\mu x + \sqrt{(\mu^2 x^2 + 1)})^n \) for \( n \geq 2 \), \( \beta \geq 0 \) and \( \mu \leq 3\sqrt{(5/2)^{5/2}/n(n+1)^{5/2}(n+2))} \) is less obvious.)

**Problem B5:** For any integer \( n \geq 3 \) let \( \phi := 2\pi/n \). Evaluate the determinant of the \( n \)-by-\( n \) matrix \( I + A \) where \( I \) is the identity matrix and \( A = \{a_{jk}\} \) has entries \( a_{jk} := \cos((j+k)\phi) \) for all indices \( j \) and \( k \).

**Solution B5:** \[ \det(I+A) = 1 - n^2/4 \]. One neat way to prove this uses a determinantal identity \( \det(I + P\cdot R^T) = \det(I + R^T\cdot P) \) in which \( P \) and \( R \) are matrices of the same dimensions, so that both products \( P\cdot R^T \) and \( R^T\cdot P \) are square though of perhaps different dimensions. Here \( R^T \) is the transpose of \( R \); and the two identity-matrices “\( I \)” may have different dimensions too. To confirm the identity apply the formula \( \det(X\cdot Y) = \det(X)\cdot \det(Y) \) to the triangular factorizations

\[
\begin{bmatrix} 1 & 0 \\ R^T & 1 \end{bmatrix}, \begin{bmatrix} 1 & -P \\ O^T & I + (R^T\cdot P) \end{bmatrix} = \begin{bmatrix} 1 & -P \\ O^T & I \end{bmatrix} = \begin{bmatrix} 1 & -P \\ O^T & I \end{bmatrix} \cdot \begin{bmatrix} 1 + (P\cdot R^T) O \\ O^T & 1 \end{bmatrix} .
\]

Now set \( \mu := \exp(i\phi) \neq \pm 1 \) so that \( \mu^k = \exp(ik\phi) = \cos(k\phi) + \imath\sin(k\phi) \) for every integer \( k \); in particular \( \mu^n = 1 \) and the complex conjugate \( \overline{\mu} = \exp(-i\phi) = 1/\mu \). Let row vector \( w^T := [\mu, \mu^2, \mu^3, \ldots, \mu^n] \) and, for future reference, compute \( \overline{w^T}\cdot w = w^T\cdot \overline{w} = n \) and \( w^T\cdot \overline{w} = \mu^2 + \mu^4 + \mu^6 + \ldots + \mu^{2n} = (1 - \mu^{2n})/(1 - \mu^2) = 0 = \overline{w^T}\cdot \overline{w} \). All this is relevant because, as is easily verified, \( A = \text{Re}(w\cdot w^T) = (w\cdot w^T + \overline{w^T}\cdot \overline{w})/2 = [w, \overline{w}]\cdot[w, \overline{w}]^T/2 \). Consequently \( \det(I+A) = \det(1 + [w, \overline{w}]\cdot[w, \overline{w}]^T/2) = \det(I + [w, \overline{w}]\cdot[w, \overline{w}]^T/2) \), from the identity \( (1 + w^T\cdot w)(1 + \overline{w}^T\cdot \overline{w})/2 - (w^T\cdot \overline{w})(\overline{w^T}\cdot w)/4 = 1 - n^2/4 \) as claimed.

(This is surprising. Rarely does a formula hold for all \( n \)-by-\( n \) matrices with \( n \geq 3 \) but not for \( n = 2 \) nor \( n = 1 \).)
Problem B6: Let $S$ be a finite set of integers each greater than 1. Suppose that for each integer $n$ there is some $s$ in $S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that $S$ must contain some $s$ and $t$ for which $\gcd(s, t)$ is a prime. [Here “$\gcd(x, y)$” denotes the Greatest Common Divisor of $x$ and $y$.]

Solution B6: (Presumably $s = t$ is allowed since $S := \{3\}$ meets the specifications for $S$.)

This neat solution was suggested by David Blackston, a graduate student of Computer Science.

Let $L$ be the least positive integer such that $\gcd(s, L) > 1$ for every $s$ in $S$; this $L$ must exist because it need not exceed the product of all primes each of which divides at least one member of $S$. But $L$ may be smaller than that product if there exist two primes both of which divide every member of $S$ divisible by either. Anyway, $L$ is a product of primes none of whose squares divides $L$; otherwise $L$ would not be the least ….

Some $t$ in $S$ must satisfy $\gcd(t, L) = t > 1$; choose any prime $p$ that divides $t$. Since $L/p < L$, there must be some $s$ in $S$ such that $\gcd(s, L/p) = 1$. David asserts that $\gcd(s, t) = p$. Let’s see why his assertion is true.

This $t = \gcd(t, L)$ is a product of a nonempty subset (containing $p$) selected from the prime factors of $L$. Since $\gcd(s, L/p) = 1$, no prime factor of $L/p$ divides $s$, and therefore no prime factor of $t/p$ divides $s$; i.e. $\gcd(s, t/p) = 1$. But $\gcd(s, L) > 1$, so $\gcd(s, L) = p$; this means that some positive power of $p$ divides $s$, and therefore $\gcd(s, t) = p$ as claimed.

The foregoing solutions have been posted on the class web-page:

http://cs.berkeley.edu/~wkahan/MathH90/Putnam99.pdf