Angles between Vectors and Planes

In a given *Euclidean* vector space, regardless of its dimension, the *Scalar Product* of vectors $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x'y} = \mathbf{y'x}$, and the length of a vector $\mathbf{x}$ is $\|\mathbf{x}\| := \sqrt{(\mathbf{x'x})}$. Here " $\mathbf{x'}$ " stands for the transpose of $\mathbf{x}$ if it is a column of real numbers x_j ; more generally $\mathbf{x'}$ is the *Linear Functional Dual* to $\mathbf{x}$ in the vector space *Dual* to the given space. Since both spaces are Euclidean they are *Isomorphic*, which means "Indistinguishable" by most eyes. Still, $\mathbf{x'}$ is distinguishable from $\mathbf{x}$, and the distinction will matter. Other notations for " $\mathbf{x'y}$ " are " $\mathbf{x}\bullet\mathbf{y}$ ", " $(\mathbf{y}, \mathbf{x})$ " and " $\langle\mathbf{y}|\,\mathbf{x}\rangle$ "; other notations for " $\|\mathbf{x}\|$ " are " $|\mathbf{x}|$ " and just " x "; but none of these will be used here. Our notation fails to distinguish a vector $\mathbf{x}$ from the point $\mathbf{x}$ reached via displacement by vector $\mathbf{x}$ from an origin $\mathbf{o}$ chosen arbitrarily. Points cannot be added like vectors to get another point; but adding a vector to a point translates it, so the difference between two points is a vector.

The angle between nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ is $\angle(\mathbf{x}, \mathbf{y}) := \arccos(\ \mathbf{x'y}/(\|\mathbf{x}\|\cdot\|\mathbf{y}\|)\ ) = \angle(\mathbf{y}, \mathbf{x})$ and disregards their order, so $0 \le \angle(\mathbf{x}, \mathbf{y}) \le \pi$ and $\sin(\angle(\mathbf{x}, \mathbf{y})) = +\sqrt{(\ 1 - (\mathbf{x'y})^2/(\mathbf{x'x}\cdot\mathbf{y'y})\ )}$. The quantity under the last $\sqrt{}$ sign is nonnegative because of *Cauchy's Inequality* and *Lagrange's Identity* : $0 \le \mathbf{x'x}\cdot\mathbf{y'y} - (\mathbf{x'y})^2 = \det(\ [\mathbf{x}\ \ \mathbf{y}]'\cdot[\mathbf{x}\ \ \mathbf{y}]\ ) = \sum_i \sum_{j>i} \det(\begin{bmatrix} x_i & y_i \\ x_j & y_j \end{bmatrix})^2$.

The equation of a (hyper)plane $\prod$ is $\mathbf{n'x} = \mu$ for some constant scalar μ and functional $\mathbf{n'} \ne \mathbf{o'}$; the vector $\mathbf{n}$ is called "the normal to plane $\prod$". The *Orthogonal Projection* of a point $\mathbf{y}$ onto plane $\prod$ is the point $\mathbf{p} := \mathbf{y} - \mathbf{n}\cdot(\mathbf{n'y}-\mu)/\mathbf{n'n}$ because $\mathbf{p}$ lies in $\prod$ and $\mathbf{y}-\mathbf{p}$ is (anti)parallel to $\mathbf{n}$. The distance from point $\mathbf{y}$ to plane $\prod$ is $\|\mathbf{y} - \prod\| := \min_{\mathbf{x} \text{ in } \prod} \|\mathbf{y}-\mathbf{x}\| = \|\mathbf{y}-\mathbf{p}\| = |\mathbf{n'y}-\mu|/\|\mathbf{n}\|$ because every $\mathbf{x}$ in $\prod$ satisfies $\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{y}-\mathbf{p}\|^2 + \|\mathbf{x}-\mathbf{p}\|^2$. (Check this out after rewriting $\|\mathbf{y}-\mathbf{x}\| = \|(\mathbf{y}-\mathbf{p}) - (\mathbf{x}-\mathbf{p})\|$.) Consequently $\|\mathbf{y} - \prod\| = |\cos(\angle(\mathbf{n}, \mathbf{y}-\mathbf{x}))|\cdot\|\mathbf{y}-\mathbf{x}\|$ for every $\mathbf{x}$ in $\prod$.

Plane $\prod$ divides the space of vectors $\mathbf{x}$ into two *Half-Spaces* according to the sign of $\mathbf{n'x} - \mu$. The direction of a vector $\mathbf{v} \ne \mathbf{o}$ points into one of these half-spaces according to the sign of $\mathbf{n'v}$ if it is nonzero. The angle $\angle(\mathbf{v}, \prod)$ between $\mathbf{v}$ and $\prod$ is *Complementary* to $\angle(\mathbf{v}, \mathbf{n})$, so $-\pi/2 \le \angle(\mathbf{v}, \prod) := \pi/2 - \angle(\mathbf{v}, \mathbf{n}) = \arcsin(\ \mathbf{n'v}/(\|\mathbf{n}\|\cdot\|\mathbf{v}\|)\ ) \le \pi/2$. If nonzero the sign of $\angle(\mathbf{v}, \prod)$ matches the sign of $\mathbf{n'v}$ to indicate into which half-space $\mathbf{v}$ points. The magnitude of $\angle(\mathbf{v}, \prod)$ solves a minimization problem:

For any point $\mathbf{q}$ in $\prod$, the minimum of $\angle(\mathbf{v}, \mathbf{r})$ over all other points $\mathbf{q}+\mathbf{r}$ in $\prod$ is $|\angle(\mathbf{v}, \prod)|$.

To see why this is so, first simplify the algebra by translating $\mathbf{q}$ to $\mathbf{o}$, thus setting $\mu = 0$ in $\prod$'s equation $\mathbf{n'x} = 0$. Now we have to prove $\angle(\mathbf{v}, \mathbf{r}) \ge |\angle(\mathbf{v}, \prod)|$ whenever $\mathbf{n'r} = 0$ but $\mathbf{r} \ne \mathbf{o}$. To this end let $\mathbf{p} := \mathbf{v} - \mathbf{n}\cdot\mathbf{n'v}/\mathbf{n'n}$ be $\mathbf{v}$'s orthogonal projection onto $\prod$, so that $\mathbf{n'p} = 0$ and $\|\mathbf{v} - \prod\| = \|\mathbf{v}-\mathbf{p}\| = |\mathbf{n'v}|/\|\mathbf{n}\| = \|\mathbf{v}\|\cdot\sin(|\angle(\mathbf{v}, \prod)|)$. Similarly let $\mathbf{s} := \mathbf{r}\cdot\mathbf{r'v}/\mathbf{r'r}$ be the orthogonal projection of $\mathbf{v}$ onto the line £ consisting of all scalar multiples of $\mathbf{r}$, since $\mathbf{r'}(\mathbf{v}-\mathbf{s}) = 0$. And $\|\mathbf{v} - £\| = \|\mathbf{v}-\mathbf{s}\| = \dots = \|\mathbf{v}\|\cdot\sin(\angle(\mathbf{v}, £)) = \|\mathbf{v}\|\cdot\sin(\angle(\mathbf{v}, \mathbf{r}))$. Now, $\|\mathbf{v}-\mathbf{s}\| \ge \|\mathbf{v}-\prod\|$ since $\mathbf{s}$ lies in $\prod$, so $\sin(\angle(\mathbf{v}, \mathbf{r})) \ge \sin(|\angle(\mathbf{v}, \prod)|)$, and therefore $\angle(\mathbf{v}, \mathbf{r}) \ge |\angle(\mathbf{v}, \prod)|$ as claimed.

Angles exemplify the often close analogy between the geometries of three-dimensional and multidimensional Euclidean spaces. But sometimes the analogy fails, as it does in problem #3 issued on 27 Oct. 2003; see http://www.cs.berkeley.edu/~wkahan/MathH90/S27Oct03.pdf . Between two subspaces both of dimensions bigger than 1 there are as many *Principal Angles* as the lesser dimension. They are described in articles by C. Davis & W. Kahan in *Bull. Amer. Math. Soc.* **75** #4 (1969) and *SIAM J. Numer. Anal.* **7** #1 (1970); their computation is discussed in §12.4 of *Matrix Computations* by G.H. Golub & C.F. Van Loan (1989–1996, Johns Hopkins U.P., Baltimore).