Only Commutators Have Trace Zero

Prof. W. Kahan
Mathematics Dept. #3840
University of California
Berkeley CA 94720-3840

Abstract: This note proves an old theorem in an elementary, succinct and perspicuous way derived from a similarity devised to equalize all the diagonal elements of a matrix.

Introduction: Square matrix Z is called a " Commutator " just when Z = XY–YX for some matrices X and Y (not determined uniquely by Z); then $\mathrm{Trace}(Z) := \sum_i z_{ii} = 0$ because Trace(XY) = Trace(YX) for all matrices X and Y both of whose products XY and YX are square. Conversely, according to an unobvious old theorem, if Trace(Z) = 0 then Z must be a commutator. This theorem has been proved in considerable generality; for instance see proofs by K. Shoda (1936) *Japan J. Math.* **13** 361-5, and by A.A. Albert and B. Muckenhoupt (1957) *Michigan Math. J.* **4** 1-3. Presented below is a shorter proof extracted from my lecture notes.

The shorter proof came to light during the investigation of another old theorem to the effect that, for each square matrix Z , there exist invertible matrices C such that all the diagonal elements of $C^{-1}ZC$ are the same. They are all zeros if Trace(Z) = 0 which, in this context, is easy to arrange by subtracting Trace(Z)/Dimension(Z) from every diagonal element of Z . The construction of the similarity $C^{-1}ZC$ was reduced to a finite sequence of steps each derived from a similarity $B^{-1}ZB$ that injected another zero into the diagonal, starting with the first diagonal element. Thus the investigation swirled around two questions:

How easily can B be chosen to put zero into the first diagonal element of $B^{-1}ZB$?
If this can be done easily, what good does it accomplish?

Lemma 1: If Z is a commutator, so is $\overline{Z} = \begin{bmatrix} 0 & r^T \\ c & Z \end{bmatrix}$ for every row r^T and column c of the same dimension as Z .

Proof 1: Suppose Z = XY–YX ; this equation remains valid after X is replaced by X + ßI for any scalar ß , so we might as well assume X is invertible. Then $\overline{Z} = \overline{X}\,\overline{Y} - \overline{Y}\,\overline{X}$ wherein $\overline{X} := \begin{bmatrix} 0 & o^T \\ o & X \end{bmatrix}$ and $\overline{Y} := \begin{bmatrix} 0 & -r^T X^{-1} \\ X^{-1}c & Y \end{bmatrix}$. End of Proof 1.

(Later we'll see why the converse of Lemma 1 is true too: For any r^T and c , if $\overline{Z}$ is a commutator so is Z because they have the same zero Trace.)

Lemma 2: Suppose no matrix $B^{-1}SB$ similar to a given square matrix S can have 0 as its first diagonal element no matter how matrix B is chosen so long as it is invertible. Then S must be a nonzero scalar multiple of the identity matrix I .

Proof 2: Evidently $S \neq O$, so some nonzero row w^T exists for which $w^TS \neq o^T$. Suppose now, for the sake of argument, that a column v existed satisfying $w^Tv = 1$ and $w^TSv = 0$. Then an invertible matrix $B = [v, b_2, b_3, \ldots]$ could be chosen in which the latter columns $[b_2, b_3, \ldots]$ constituted a basis for the subspace of columns annihilated by w^T ; every $w^Tb_j = 0$. This w^T would be the first row in the inverse B^{-1} , whereupon the matrix $B^{-1}SB$ would have $w^TSv = 0$ for its first diagonal element. But this element can't vanish, according to the lemma's hypothesis. Therefore no vector v can ever satisfy both $w^Tv = 1$ and $w^TSv = 0$; therefore $w^TS = \mu w^T$ for some scalar $\mu \neq 0$. This persists no matter how w^T is chosen; in fact *every* row w^T must satisfy either $w^TS = o^T$ or $w^TS = \mu w^T$ for some scalar $\mu = \mu(w^T) \neq 0$. Therefore $B^{-1}SB$ is diagonal for *every* invertible matrix B . Moreover no two diagonal elements of $B^{-1}SB$ can differ without violating the equation $w^TS = \mu w^T$ when w^T is the difference between their corresponding rows in B^{-1} . This makes S a nonzero scalar multiple of the identity matrix I . End of Proof 2. (It may be the only novelty in this note.)

We shall apply Lemma 2 in its *contrapositive* form: Unless S is a nonzero scalar multiple of the identity, invertible matrices B exist for which the first diagonal element of $B^{-1}SB$ is zero. (Don't confuse this with the *converse* of Lemma 2; it says that if S is a nonzero scalar multiple of I then no diagonal element of $B^{-1}SB$ can vanish, which is obviously true too.)

Theorem 3: If $\mathrm{Trace}(Z) = 0$ then Z is a commutator.

Proof 3: The theorem is obviously valid if Z is 1-by-1 or a bigger zero matrix. Therefore assume that Z is a nonzero square matrix of dimension bigger than 1 . Our proof goes by induction; we assume the desired inference valid for all matrices of dimensions smaller than Z 's with Trace zero. Because of that zero Trace, Z cannot be a nonzero scalar multiple of I , so Lemma 2 implies that some invertible B exists making $B^{-1}ZB = \begin{bmatrix} 0 & r^T \\ c & K \end{bmatrix}$. Observe next that $\mathrm{Trace}(K) = \mathrm{Trace}(Z) = 0$. The induction hypothesis implies that K is a commutator; then Lemma 1 implies that $B^{-1}ZB = XY - YX$ is a commutator too for some X and Y , whereupon $Z = (BXB^{-1})(BYB^{-1}) - (BYB^{-1})(BXB^{-1})$ must be a commutator too. End of Proof 3.

Corollary 4: For each square matrix Z invertible matrices C exist that make every diagonal element of $C^{-1}ZC$ the same.

Proof 4: This is actually a corollary of Lemma 2. Let $S := Z - I \cdot \mathrm{Trace}(Z)/\mathrm{Dimension}(Z)$ to get $\mathrm{Trace}(S) = 0$. Since S cannot be a nonzero scalar multiple of I , some invertible B must exist to make $B^{-1}SB = \begin{bmatrix} 0 & r^T \\ c & K \end{bmatrix}$. Since $\mathrm{Trace}(K) = \mathrm{Trace}(B^{-1}SB) = \mathrm{Trace}(S) = 0$, this step can be repeated to replace K by a matrix whose every diagonal element is zero (thereby changing c and r^T) thus constructing C so that every diagonal element of $C^{-1}SC$ is zero. End of Proof 4.

Corollary 4 is too easy because too many matrices C meet its requirements. Are any of these computationally convenient? For instance, triangular matrices C would be convenient because their inverses can be computed easily; but no triangular matrix can serve as C in Corollary 4 if Z is diagonal and not a scalar multiple of I . Real orthogonal matrices C and complex unitary matrices C are computationally convenient partly because their inverses are obtained so easily and partly because they do not amplify rounding errors much. Here we are in luck:

Corollary 5: For each square matrix Z unitary matrices $C = (C^H)^{-1}$ exist that make every diagonal element of $C^{-1}ZC$ the same; here C^H is the complex conjugate transpose of C . And if Z is real $C = (C^T)^{-1}$ can be real orthogonal.

Proof 5: Let $S := Z - I \cdot \mathrm{Trace}(Z)/\mathrm{Dimension}(Z)$ again. The *Numerical Range* of S is the set of complex numbers swept out by the *Rayleigh Quotient* v^HSv/v^Hv as v runs through all nonzero complex columns. Digress to *Canad. Math. Bull.* **14** (1971) pp. 245-6 for Chandler Davis' short proof of the Töplitz-Hausdorff theorem which asserts that, when plotted in the complex plane, the numerical range of S constitutes a convex region containing, among other things, all the eigenvalues of S . Since their sum Trace(S) = 0 , zero lies in that convex region. Therefore a column v exists with $v^HSv = 0$ and $v^Hv = 1$. Now set $w^T := v^H$ in the proof of Lemma 2 to determine (not uniquely) a unitary matrix B that makes $B^{-1}SB = \begin{bmatrix} 0 & r^T \\ c & K \end{bmatrix}$; and continue as in the proof of Corollary 4 to build a unitary C that makes every diagonal element of $C^{-1}SC$ zero. If Z is real so is S , and then the Rayleigh Quotient v^TSv/v^Tv runs through the numerical range of $(S+S^T)/2$ as v runs through all nonzero real columns; then B is real orthogonal *etc*. End of Proof 5.

Knowing C exists is one thing; finding C another. To find a real orthogonal C is easy if Z is real, as is S , because when Trace(S) = 0 a nonzero column v satisfying $v^TSv = 0$ can be found with two nonzero elements, corresponding in location to two diagonal elements of S with opposite signs, at scarcely more than the cost of solving a real quadratic equation; this is the crucial step towards finding each of the orthogonal matrices B needed in the corollaries' proofs. But finding a complex unitary C is not so easy when Z and S are complex; a nonzero column v satisfying $v^HSv = 0$ generally requires three nonzero elements. In this complex case a simpler way to find C may be the Jacobi-like iteration described on p. 77 of R.A. Horn and C.R. Johnson's *Matrix Analysis* (1985/7, Cambridge Univ. Press).