Students were asked on Fri. 2 Oct. to work out some of these problems aided by their own notes and by any texts but by no other person, and to hand in solutions Mon. morning 5 Oct. 1998.

**Problem 0:** When we see our own images in a mirror, why does it swap Left and Right but not Up and Down?

It doesn’t swap Left and Right; it swaps Forward and Backward.

**Problem 1:** Exhibit two matrices $P$ and $Q$ such that $(PQ)^2 = O \neq (QP)^2$.

Try $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, for which $PQ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O$ and $QP = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

The necessity of $PQ \neq O$ comes from $O \neq (QP)^2 = Q(PQ)P$; but $QP$ must satisfy $(QP)^3 = Q(PQ)^2P = O$.

**Problem 2a:** Obtain an explicit formula for $(I - cr^T)^{-1}$ given $c$ and $r^T$ and that $r^Tc \neq 1$.

$(I - cr^T)^{-1} = I + cr^T/(1-r^Tc)$.

**Problem 2b:** Obtain $(B - cr^T)^{-1}$ explicitly given $B^{-1}$, $c$, $r^T$, and that $r^TB^{-1}c \neq 1$.

$(B - cr^T)^{-1} = B^{-1} + B^{-1}cr^TB^{-1}/(1-r^TB^{-1}c)$.

**Problem 3:** Obtain the $n$-by-$n$ matrix $U$ from the identity by deleting its first row and appending a row of zeros after its last. Obtain $R$ from $2I - (I-U)^{-1}$ by inserting the scalar $\mu$ into its lowest leftmost element. Express the value of $\mu$ for which $R$ is not invertible as a function of $n$, assuming $n > 1$. Hint: Experiment with $n = 2, 3, 4, \ldots$ first.

$\mu = -2^{2-n}$. Here is why: $R = 2I - (I-U)^{-1} + \mu ef^T$ in which $e$ is the column whose last element is 1 and the rest zeros, and $f^T$ is the row whose first element is 1 and the rest zeros. $R$ is not invertible just when $Rx = 0$ for some $x \neq o$. Then $x = -\mu(2I - (I-U)^{-1})ef^Tx \neq 0$, whence follows $f^Tx = -\mu e^T(2I - (I-U)^{-1})ef^Tx \neq 0$, and then $\mu = -1/f^T(2I - (I-U)^{-1})e$. Now, $(2I - (I-U)^{-1})^{-1} = (I-U)(I-2U)^{-1} = I + U + 2U^2 + 4U^3 + 8U^4 + \ldots + 2^{n-2}U^{n-1}$ since $U^n = O$. Consequently $\mu = -1/f^T(2^{n-2}U^{n-1})e = -2^{2-n}$. Determinantal manipulation gives the same result; $0 = \det(R) = \det(R(I-U)) = \det(I-2U+\mu ef^T(I-U)) = \det((I-2U+\mu ef^T(I-U))(I+ff^TU)) = 1 + 2^{n-2}\mu$. 

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Problem 4a: Given two different vectors \( x \) and \( y \) of the same Euclidean length (so \( x^T x = y^T y \neq 0 \)), exhibit an elementary orthogonal reflector \( W = I - (2/c^T c)c c^T \) that swaps them. Choose \( c = x-y \); then \( Wx = y \) and so \( Wy = x \). (Note that this kind of \( W = W^T = W^{-1} \).)

Problem 4b: Prove that every \( n \)-by-\( n \) orthogonal matrix \( Q = (Q^T)^{-1} \) can be expressed as a product of at most \( n \) elementary orthogonal reflectors like \( W \).

“\( Q^T Q = I \)” implies that every column of any orthogonal matrix \( Q \) has the same length 1 as every column of the identity \( I \). Choose reflector \( W_1 \) to swap the first column of \( Q \) with the first column of \( I \). Note that \( W_1 Q \) is still orthogonal, and its first column (and first row) must be the same as \( I \’s \). Choose reflector \( W_2 \) to swap the second column of \( W_1 Q \) with the second column of \( I \). \( W_2 \) leaves the first column of \( W_1 Q \) unchanged because it is orthogonal to the second columns of \( W_1 Q \) and of \( I \). Therefore the first two columns (and first two rows) of \( W_2 W_1 Q \), which is still orthogonal, must be the same as \( I \’s \). Choose reflector \( W_3 \) to swap the third column of \( W_2 W_1 Q \) with the third column of \( I \), and so on. Of course, if a column to be swapped with a column of \( I \) already matches it, a reflector can be skipped. So, premultiplying \( Q \) by at most \( n \) reflectors transforms it into \( I \). Therefore \( Q \) equals the inverse of that product, which is the product of the same reflectors in reverse order.

Problem 5: Two proper subspaces of a vector space are complementary just when their sum is the whole space and their intersection is \( \{ o \} \). Can either determine the other uniquely? Why?

No. Let \( E \) and \( F \) be bases for complementary subspaces of a vector space for which \([E, F]\) must therefore be a basis. Given any nonzero matrix \( G \) with as many columns as \( E \) has, and with as many rows as \( F \) has columns, we shall show that \([E+FG, F]\) is another basis for the vector space, but \( \text{Range}(E+FG) \neq \text{Range}(E) \); this will confirm that the subspace \( \text{Range}(F) \) cannot determine its complementary subspace uniquely. \([E+FG, F] = [E, F]\begin{bmatrix} 1 & O \\ G & 1 \end{bmatrix}\) is a basis because the last matrix in the product has an inverse obtained by reversing the sign of \( G \). To see why \( \text{Range}(E+FG) \neq \text{Range}(E) \) choose any column \( z \) for which \( Gz \neq o \) and verify that the equation \( Ex = (E+FG)z \) cannot be solved for \( x \) because otherwise \( FGz = E(x-z) \) would be a nonzero vector in the intersection of complementary subspaces \( \text{Range}(F) \) and \( \text{Range}(E) \).

Problem 6: \( S \) and \( T \) are two subspaces of a vector space \( V \), and \( f \) is a real scalar-valued function defined for every vector in \( V \). Moreover, \( f(s) < f(t) \) for every nonzero vector \( s \) in \( S \) and every nonzero vector \( t \) in \( T \). How must Dimension(\( S \)) + Dimension(\( T \)) compare with Dimension(\( V \))?

Subspaces \( S \) and \( T \) can have only the zero vector \( o \) in their intersection, so Dimension(\( S \)) + Dimension(\( T \)) = Dimension(\( \{ o \} \)) + Dimension(\( S + T \)) \( \leq \) Dimension(\( V \)).
Problem 7: Let the cubic polynomial whose value at $\xi$ is
\[ p(\xi) = \pi_0 + 3\pi_1\xi + 3\pi_2\xi^2 + \pi_3\xi^3 \]
be represented by a row-vector $p^T := [\pi_0, \pi_1, \pi_2, \pi_3]$ of its coefficients. For constant $\mu$ let cubic
\[ b(\xi) := p(\xi + \mu) = \beta_0 + 3\beta_1\xi + 3\beta_2\xi^2 + \beta_3\xi^3 \]
be represented by $b^T := [\beta_0, \beta_1, \beta_2, \beta_3]$. Exhibit the matrix $L$ that takes $p^T$ to $b^T = p^T L$. This matrix can be factorized; $L$ is the product of three matrices among whose elements only the numbers $0$, $1$ and $\mu$ appear. Find these factors and thus determine how few scalar multiplications suffice to compute $p^T L$ given $p^T$ and $\mu$.

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
3\mu & 1 & 0 & 0 \\
3\mu^2 & 2\mu & 1 & 0 \\
\mu^3 & \mu^2 & \mu & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\mu & 1 & 0 & 0 \\
0 & \mu & 1 & 0 \\
0 & 0 & \mu & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
\mu & 1 & 0 & 0 \\
0 & \mu & 1 & 0 \\
0 & 0 & \mu & 1
\end{bmatrix},
\]
so six multiplications and additions suffice to compute $p^T L$ without first computing $\mu^2$ and $\mu^3$ which would cost two more multiplications.

Problem 8: Given matrices $E$ and $F$ with the same number of rows but any numbers of columns (and their columns need not be linearly independent), we seek a matrix $S$ whose range is the intersection of Range($E$) and Range($F$). Show how and why $S$ may be constructed if matrices $J$, $L$, $P$, $Q$ and $R$ are found to satisfy $EJ + FL = O$ and $E[JP–I, JQ] = R[E, F]$.

The range of $S := EJ = -FL$ is contained in both Range($E$) and Range($F$), and therefore in their intersection. Any vector $w := Eu = -Fv$ in that intersection can also be found as $w = S(Pu + Qv)$ in Range($S$), implying that Range($S$) contains that intersection, because
\[
S(Pu + Qv) - w = EJ(Pu + Qv) - Eu = E((JP–I)u + JQv) = R(Eu + Fv) = O.
\]
Therefore Range($S$) is the intersection of Range($E$) and Range($F$), as required. (To find $J$, $L$, $P$, $Q$ and $R$, which the problem did not request, see the lecture notes titled “Geometry of Elementary Operations and Subspaces” and set $R := EH[I, O]G^{-1}$.)

Problem 9: Given a matrix $F$ whose target-space is Euclidean, and a vector $g$ in that space but not in Range($F$), explain how to find a vector $r$ perpendicular to Range($F$) such that $g – r$ lies in Range($F$).

Solve the Least-Squares problem that chooses $x$ to minimize $\|Fx - g\|$. Then $r := g - Fx$ because $F^TR = F^Tg - F^TFx = O$. The lecture notes on Least Squares explain why the last equation always has a solution $x$. 