Problem 0: What is Jordan's Canonical Form of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$?

Solution 0: $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Problem 1: Here is what is known about a linear operator $\mathbf{L}$ that maps a vector space to itself: No matrix L that represents $\mathbf{L}$ in any basis can have 0 as its first diagonal element, but this element is 3 in at least one such L . What operator is $\mathbf{L}$? Justify your answer.

Solution 1: We shall see why $\mathbf{L} = 3\mathbf{I}$ where $\mathbf{I}$ is the identity operator. Evidently $\mathbf{L} \ne \mathbf{O}$, so $\mathbf{w}^T\mathbf{L} \ne \mathbf{o}^T$ for some functional $\mathbf{w}^T \ne \mathbf{o}^T$. Suppose now that a vector $\mathbf{v}$ existed satisfying $\mathbf{w}^T\mathbf{v} = 1$ and $\mathbf{w}^T\mathbf{L}\mathbf{v} = 0$; then a basis $\mathbf{B} = [\mathbf{v}, \mathbf{b}_2, \mathbf{b}_3, \ldots]$ could be chosen in which $[\mathbf{b}_2, \mathbf{b}_3, \ldots]$ is a basis for the subspace annihilated by $\mathbf{w}^T$, so that every $\mathbf{w}^T\mathbf{b}_j = 0$, and then $\mathbf{w}^T$ would be the first " row " in the inverse basis $\mathbf{B}^{-1}$, whereupon the matrix $L = \mathbf{B}^{-1}\mathbf{L}\mathbf{B}$ that represents $\mathbf{L}$ in this basis would have $\mathbf{w}^T\mathbf{L}\mathbf{v} = 0$ for its first diagonal element. This can't happen, according to the problem's statement. Therefore no vector $\mathbf{v}$ can ever satisfy both $\mathbf{w}^T\mathbf{v} = 1$ and $\mathbf{w}^T\mathbf{L}\mathbf{v} = 0$; therefore *every* $\mathbf{w}^T\mathbf{L} = \mu\mathbf{w}^T$ for some scalar $\mu = \mu(\mathbf{w}^T) \ne 0$. This implies $L = \mathbf{B}^{-1}\mathbf{L}\mathbf{B}$ is diagonal for *every* basis $\mathbf{B}$. No two diagonal elements can differ without violating the equation $\mathbf{w}^T\mathbf{L} = \mu\mathbf{w}^T$ when $\mathbf{w}^T$ is the difference between their corresponding " rows " in $\mathbf{B}^{-1}$. This makes L a scalar multiple of the identity matrix I , and therefore $\mathbf{L}$ is a scalar multiple of the identity operator $\mathbf{I}$; the scalar turns out to be 3 .

Problem 2: Z is called a " Commutator " just when $Z = XY-YX$ for some matrices X and Y (not determined uniquely by Z). Prove that if Z is a commutator, so is $\overline{Z} = \begin{bmatrix} 0 & r^T \\ c & Z \end{bmatrix}$ for any row r^T and column c .

Solution 2: Suppose $Z = XY-YX$; this equation remains valid if X is replaced by X + ßI for any scalar ß , so we might as well assume X is invertible. Then $\overline{Z} = \overline{X}\,\overline{Y} - \overline{Y}\,\overline{X}$ wherein

$$\overline{X} = \begin{bmatrix} 0 & o^T \\ o & X \end{bmatrix} \text{ and } \overline{Y} = \begin{bmatrix} 0 & -r^TX^{-1} \\ X^{-1}c & Y \end{bmatrix} .$$

Bonus Problem 1/2: Prove that if Trace(Z) = 0 then Z is a commutator.

Solution 1/2: The asserted inference is obviously valid if Z is 1-by-1 or a bigger zero matrix. Therefore assume that Z is a nonzero matrix of dimension bigger than 1 . Our proof goes by induction; we assume the desired inference valid for all matrices of dimension smaller than Z 's with Trace zero. Z cannot be a nonzero scalar multiple of the identity since Trace(Z) = 0 , so Problem 1 implies that some invertible C exists such that $C^{-1}ZC = \begin{bmatrix} 0 & r^T \\ c & B \end{bmatrix}$; observe next that Trace(B) = 0 . Then the induction hypothesis implies that B is a commutator, whereupon Problem 2 implies that $C^{-1}ZC = XY - YX$ is a commutator too, whence so is Z .

This proof is shorter than was found by K. Shoda (1936) *Japan J. Math.* **13** 361-5, and again by A.A. Albert and B. Muckenhoupt (1957) *Michigan Math. J.* **4** 1-3.

Problem 3: For any two vectors **u** and **v** in Euclidean 3-space $\boldsymbol{E}^3$, an anti-commutative non-associative *Cross-Product* $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$ is defined as follows from the column vectors u, v and u×v that represent their respective vectors $\mathbf{u} = \mathbf{B}u$ etc. in *any* orthonormal basis **B** :

$$\text{If} \quad u = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad \text{then} \quad u\times v := \begin{bmatrix} \beta\zeta - \gamma\eta \\ \gamma\xi - \alpha\zeta \\ \alpha\eta - \beta\xi \end{bmatrix} .$$

u×**v** does depend upon basis **B** . Changing to a new orthonormal basis $\mathbf{B}Q^{-1}$ where $Q^{-1} = Q^T$ changes the respective representatives of **u**, **v** and **u**×**v** from u, v and u×v to Qu, Qv and $(Qu)\times(Qv) = Q(u\times v)\cdot\det(Q)$, thereby changing **u**×**v** by a factor det(Q) .

What values can det(Q) take?

Let $\mathbb{X}(\mathbf{u})$ be the linear operator acting upon $\boldsymbol{E}^3$ that maps **v** to $\mathbb{X}(\mathbf{u})\mathbf{v} = \mathbf{u}\times\mathbf{v}$; in that basis **B** some matrix $X(u)$ represents $\mathbb{X}(\mathbf{u})$. To what does the foregoing change of basis change the operator $\mathbb{X}(\mathbf{u})$? After that prove $\mathbb{X}(\mathbf{u})^2 = \mathbf{u}\mathbf{u}^T - \mathbf{u}^T\mathbf{u}\mathbf{I}$ regardless of orthonormal basis, and then find the eigenvalues of $\mathbb{X}(\mathbf{u})$.

Solution 3: In that orthonormal basis **B** , $\mathbb{X}(\mathbf{u})$ has the matrix $X(u) = X\left(\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}\right) := \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}$, whence $X(u)^2 = uu^T - u^Tu I$, but this by itself does *not* prove $\mathbb{X}(\mathbf{u})^2 = \mathbf{u}\mathbf{u}^T - \mathbf{u}^T\mathbf{u}\mathbf{I}$ since we have not yet established how $\mathbb{X}(\mathbf{u})$ changes when the basis changes. First we observe that $\det(Q) = \det(Q^T) = \det(Q^{-1}) = 1/\det(Q)$ so $\det(Q) = \pm 1$. Changing old basis **B** to new basis $\mathbf{B}Q^{-1}$ changes $\text{old}(\mathbb{X}(\mathbf{u}))\mathbf{v} = \text{old}(\mathbf{u}\times\mathbf{v})$ to $\text{new}(\mathbb{X}(\mathbf{u}))\mathbf{v} = \text{new}(\mathbf{u}\times\mathbf{v}) = \text{old}(\mathbf{u}\times\mathbf{v})\det(Q)$ so that $\text{old}(\mathbb{X}(\mathbf{u}))$ changes to $\text{new}(\mathbb{X}(\mathbf{u})) = \text{old}(\mathbb{X}(\mathbf{u}))\det(Q)$. Therefore $\mathbb{X}(\mathbf{u})^2 = \mathbf{u}\mathbf{u}^T - \mathbf{u}^T\mathbf{u}\mathbf{I}$ for every orthonormal basis. Though Problem 3 didn't ask, $X(u)$ changes to $X(Qu) = QX(u)Q^T\det(Q)$.

There is another way to verify that $\mathbb{X}(\mathbf{u})^2 = \mathbf{u}\mathbf{u}^T - \mathbf{u}^T\mathbf{u}\mathbf{I}$; use the so-called *triple cross-product* formula $\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\mathbf{w}^T\mathbf{v} - \mathbf{v}\mathbf{w}^T\mathbf{u}$ and set $\mathbf{w} = \mathbf{u}$.

Since $\mathbb{X}(\mathbf{u})\mathbf{u} = \mathbf{u}\times\mathbf{u} = \mathbf{o}$, zero is an eigenvalue of $\mathbb{X}(\mathbf{u})$ and $\mathbb{X}(\mathbf{u})^3 = \mathbb{X}(\mathbf{u})\mathbb{X}(\mathbf{u})^2 = -\mathbf{u}^T\mathbf{u}\mathbb{X}(\mathbf{u})$, which implies that at least one of $\pm\imath\sqrt{\mathbf{u}^T\mathbf{u}}$ is an eigenvalue too. ($\imath = \sqrt{(-1)}$.) Since $\mathbb{X}(\mathbf{u})$ is real its complex eigenvalues come in complex-conjugate pairs, so they include both of $\pm\imath\sqrt{\mathbf{u}^T\mathbf{u}}$. In short, the three eigenvalues of $\mathbb{X}(\mathbf{u})$ are 0 and $\pm\imath\sqrt{\mathbf{u}^T\mathbf{u}}$.

Problem 4 (continued from Problem 3): Assume what Problem 3 asked to be proved, and that $\mathbf{u}^T\mathbf{u} = 1$ and μ is a real scalar. Obtain a short formula for $\exp(\mu\,\mathbb{X}(\mathbf{u}))$ as a polynomial in $\mathbb{X}(\mathbf{u})$ and $\cos(\mu)$ and $\sin(\mu)$, and then explain why $\exp(\mu\,\mathbb{X}(\mathbf{u}))$ is the operator that rotates a vector about axis $\mathbf{u}$ through an angle μ . (Recall that $\exp(\imath\mu) = \cos(\mu) + \imath\cdot\sin(\mu)$ if $\imath = \sqrt{(-1)}$.)

Solution 4: Abbreviate $\mathbb{X}(\mathbf{u}) = \mathbf{X}$, and deduce that $\mathbf{X}^3 = -\mathbf{X}$ as was done in the last paragraph of Solution 3. Therefore $\mathbf{X}^{2n+1} = (-1)^n\mathbf{X}$. Then the series for the exponential can be split into

$$\begin{aligned}\exp(\mu\mathbf{X}) &= I + \textstyle\sum_{n>0} \mu^n\mathbf{X}^n/n! \\ &= I + \textstyle\sum_{n\ge0} \mu^{2n+1}\mathbf{X}^{2n+1}/(2n+1)! + \sum_{n>0} \mu^{2n}\mathbf{X}^{2n}/(2n)! \\ &= I + \textstyle\sum_{n\ge0} \mu^{2n+1}(-1)^n\mathbf{X}/(2n+1)! - \sum_{n>0} \mu^{2n}(-1)^n\mathbf{X}^2/(2n)! \\ &= I + \mathbf{X}\sin(\mu) + \mathbf{X}^2(1-\cos(\mu)) ,\end{aligned}$$

which exhibits the desired polynomial. Confirm that $\exp(\mu\mathbf{X})^{-1} = \exp(-\mu\mathbf{X}) = \exp(\mu\mathbf{X})^T$ since $\mathbf{X}^T = -\mathbf{X}$. Therefore $\exp(\mu\mathbf{X})$ is an orthogonal operator that preserves Euclidean length for all μ . Next choose an arbitrary nonzero $\mathbf{v}$ and let $\mathbf{w}(\mu) := \exp(\mu\mathbf{X})\mathbf{v}$ for all real μ . Now, $\mathbf{u}^T\mathbf{w}(\mu) = \mathbf{u}^T\mathbf{v}$ because $\mathbf{u}^T\mathbf{X} = \mathbf{o}^T$, and $\mathbf{w}(\mu)^T\mathbf{w}(\mu) = \mathbf{v}^T\mathbf{v}$, which implies that the cosine of the angle between $\mathbf{u}$ and $\mathbf{w}(\mu)$ stays the same as between $\mathbf{u}$ and $\mathbf{v}$ for all μ . Therefore the angle between $\mathbf{u}$ and $\mathbf{w}(\mu)$ stays constant as $\mathbf{w}(\mu)$ moves; it must be revolving about the axis $\mathbf{u}$ at some rate determinable by differentiation: $d\mathbf{w}/d\mu = \mathbf{X}\mathbf{w} = \mathbf{u}\times\mathbf{w}$ stays orthogonal to both $\mathbf{u}$ and $\mathbf{w}(\mu)$ and has constant squared magnitude $\mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w} = -\mathbf{v}^T\mathbf{X}^2\mathbf{v}$. This means $\mathbf{w}(\mu)$ revolves with constant angular velocity performing one revolution when μ increases by 2π . Therefore μ is the angle of revolution through which $\exp(\mu\mathbb{X}(\mathbf{u}))$ rotates any vector about the axis $\mathbf{u}$.

Problem 5: A *Simplex* in an n-dimensional vector space is the convex hull of any n+1 points that do not lie in any hyperplane of dimension less than n . Each of those points is a *vertex* of the simplex. Opposite every vertex is a *face* of that simplex consisting of the convex hull of the other n vertices. For example, in 3-space the simplex is a tetrahedron and its faces are triangles. Suppose column vectors $x_0, x_1, x_2, \dots, x_n$ are given as the coordinates of the vertices for an Orthonormal basis of an Euclidean space. Exhibit one short formula for the n-dimensional unoriented content (like volume) of the simplex, and another for the (n–1)-dimensional unoriented content (like area) of the face opposite x_0 , whose vertices are at $x_1, x_2, \dots, x_n$. (Unoriented contents are the nonnegative magnitudes of oriented contents.)

Solution 5: The unoriented content of the simplex is $|\det([x_1-x_0, x_2-x_0, x_3-x_0, \ldots, x_n-x_0])|/n!$ because it is a fraction $1/n!$ of the content of a parallelepiped from whose corner at x_0 radiate n edges to all the other vertices of the simplex. The fraction $1/n!$ comes from repeated integrations of the cross-sections parallel to a face of each of successive simplices of lower dimensions.

Let n-by-(n–1) matrix $F = [x_2-x_1, x_3-x_1, x_4-x_1, \ldots, x_n-x_1]$. The (n–1)-dimensional unoriented content of the face opposite x_0 turns out to be $\sqrt{(\det(F^TF))}/(n-1)!$. To see why this is so, let $W = W^T = W^{-1}$ be the elementary orthogonal reflector that swaps the last column of the n-dimensional identity matrix I with a unit vector orthogonal to all n–1 columns of F . This W reflects the given simplex and its faces without altering their unoriented contents. The last row of $R := WF$ is a row of zeros; the columns of R are the vectors emanating from Wx_1 to the other n–1 vertices of the reflected image of the face opposite x_0 . This image is an (n–1)-dimensional simplex whose unoriented content is |det(first n–1 rows of R)|/(n–1)! . This is the same as $\sqrt{(\det(R^TR))}/(n-1)! = \sqrt{(\det(F^TF))}/(n-1)!$.

Problem 6: A *Polar Factorization* $F = QH$ of a given possibly rectangular real matrix F has $Q^TQ = I$ and H *Symmetric Positive Semidefinite* , which means that $H = H^T$ and $x^THx \ge 0$ for every x . It is analogous to the representation of each complex number $f = q \cdot h$ as a product of its magnitude $h = |f|$ by a complex number q of magnitude $|q| = 1$. Which real matrices F have a polar factorization? Why?

Solution 6: If F has more columns than rows it has no polar factorization because Q would have to have more columns than rows and couldn't satisfy $Q^TQ = I$ since the rank of a product (I) can't exceed the rank of a factor (Q). Otherwise, when F has no more columns than rows, $F = QH$ obtainable as follows: Let R be an orthogonal matrix of eigenvectors of $F^TF = RV^2R^T$ so that $R^T = R^{-1}$ and V^2 is a diagonal matrix of eigenvalues of F^TF . We know that $V^2 \ge O$ elementwise because $x^TV^2x = (FRx)^T(FRx) \ge 0$ for every x . Therefore diagonal matrix $V \ge O$ can be obtained from V^2 by taking square roots elementwise. Then $H := RVR^T$ must satisfy $F^TF = H^2$; in other words, H is the symmetric positive semidefinite square root of F^TF .

Life is easy when the columns of F are linearly independent. Then $x^TV^2x = (FRx)^T(FRx) > 0$ for every $x \ne 0$, so that all diagonal elements of V are positive; and $Q = FH^{-1} = FRV^{-1}R^T$. When the columns of F are linearly dependent, some diagonal element(s) of V must vanish, and then the corresponding columns of FR must vanish too because $(FR)^T(FR) = V^2$. Dividing the nonzero columns of FR by the corresponding nonzero diagonal elements of V produces a matrix whose nonzero columns are orthonormal; replace its zero columns by more orthonormal columns, as must be possible because the total number of orthonormal columns will not exceed the number of rows, to obtain finally a matrix P of orthonormal columns that satisfies $P^TP = I$ and $PV = FR$. Now set $Q := PR^T$ to find $Q^TQ = P^TRR^TP = I$ and $QH = PR^TRVR^T = F$.

Incidentally, if non-square F has a polar factorization, F^T doesn't, and *vice-versa*.