Written solutions for these problems were to be handed in for grading on Tues. 28 Nov. 2000.

A real symmetric matrix $H = H^T$ is called “Positive Definite” just when $x^T H x > 0$ for every nonzero vector $x$ of the same dimension as $H$. Positive definite matrices have applications to Mechanics, Statistics, Psychology, Differential Geometry and many other areas.

1. Explain why the inverse $H^{-1}$ exists and is positive definite.

**Solution:** $H$ cannot be singular since “$Hx = o$” implies “$x^T H x = 0$”, whence “$x = o$”. Therefore $H^{-1}$ exists. And $H^{-1 T} = H^{-1}$ is symmetric since $I = I^T = (H^{-1} H)^T = HH^{-1 T}$. Finally, $H^{-1}$ is positive definite because $x H^{-1} x = (H^{-1} x)^T H (H^{-1} x) > 0$ whenever $x \neq o$.

A second proof exploits the factorization of any real symmetric $H = QAQ^T$ in which $Q$ is an Orthogonal (so $Q^T = Q^{-1}$) matrix of eigenvectors and $\Lambda$ is a real diagonal matrix of $H$’s eigenvalues. These are the Stationary Values (including maximum and minimum) of the quotient $x^T H x / x^T x$ as $x$ runs through all nonzero real vectors of $H$’s dimension. When $H$ is positive definite too that quotient is positive, and so are $H$’s eigenvalues; then likewise for $H^{-1} = QA^{-1} Q^T$. Choleski factorization of $H = U^T U$ provides a third proof: $H^{-1} = U^{-1} U^{-1 T}$.

2. Explain why the equation $Y^2 = H$, though it may have many solutions $Y$, has just one symmetric positive definite solution $Y$; it is called the positive definite square root of $H$ and written $Y = \sqrt{H}$.

**Solution:** There are several ways to construct $Y := \sqrt{H}$. One way exploits the eigenvector-value factorization $H = QAQ^T$ already mentioned above; $Y := \sqrt{\Lambda} Q^T$ in which $\sqrt{\Lambda}$ is computed elementwise. A second way sets $Y := (2/\pi) \int_0^\infty (1 + \xi^2 H^{-1})^{-1} d\xi$, confirmed by using the same factorization. A third way sets $Y_0 := I$ and runs an iteration $Y_{n+1} := (Y_n + H \cdot Y_{n-1})/2$ for $n = 0, 1, 2, \ldots$ which can be proved to converge to the limit $Y = \sqrt{H}$ most easily by using that factorization again. If (as seems unlikely) a general formula exists to compute $\sqrt{H}$ using only finitely many rational arithmetic operations and square roots, nobody has found it. No matter how $\sqrt{H}$ may be constructed, we come now to the task of proving it unique.

Suppose $V$ and $Y$ are both real symmetric and positive definite, and $V^2 = Y^2 = H$. Then $x^T (V+Y)(V-Y)x = x^T (VV-YY)x = 0$ for all $x$. To infer that $V = Y$ let $\mu$ be an extreme (maximum or minimum) value of the quotient $x^T (V-Y)x / x^T x$ as $x$ runs over all nonzero vectors $x$. Then setting the quotient’s derivative to zero yields $(V-Y)x = \mu x$; each extreme value $\mu$ must be an eigenvalue of $V-Y$ with an eigenvector $x$ for which we find that $0 = x^T (V+Y)(V-Y)x = \mu x^T (V+Y)x$. Since $V$ and $Y$ are positive definite, $x^T (V+Y)x > 0$; therefore every extreme value $\mu = 0$, whence $x^T (V-Y)x = 0$ for all $x$, which implies $V-Y = o$ thus: $4z^T (V-Y)x = (x+z)^T (V-Y)(x+z) - (x-z)^T (V-Y)(x-z) = 0$ for all $x$ and $z$. 
The existence and uniqueness of a positive definite matrix’s positive definite square root is worth noting because square roots of a matrix are not so simple in general, not even for 2-by-2 matrices. For instance, \[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\] has no square root; \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] has just two; \[
\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\] has four; and \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] has infinitely many. Can you see why?

3. Show that if \( H \), \( M \) and \( H-M \) are symmetric and positive definite, so is \( M^{-1} - H^{-1} \).

**Solution:** Verify the identity \( M^{-1} - H^{-1} = H^{-1}(H-M)H^{-1} + H^{-1}(H-M)M^{-1}(H-M)H^{-1} \), and then observe that it expresses \( M^{-1} - H^{-1} \) as a sum of positive definite matrices, which must also be positive definite. Another shorter identity \( M^{-1} - H^{-1} = M^{-1}((H-M)^{-1} + M^{-1})^{-1} M^{-1} \) works too but is trickier to verify. A third proof uses \( Y := \sqrt{H} \) and \( X := \sqrt{(Y^{-1}MY^{-1})} \) to deduce that \( I - X^2 = Y^{-1}(I - X^2)Y^{-1} \) is positive definite, whence the same conclusion follows for \( M^{-1} - H^{-1} = Y^{-1}(X^2 - I)Y^{-1} = (XY)^{-1}(I - X^2)(XY)^{-1T} \).

4. Show that if \( H \), \( M \) and \( H-M \) are symmetric and positive definite, so is \( \sqrt{H} - \sqrt{M} \).

**Solution:** One proof uses \( \sqrt{H} = (2/\pi) \int_0^\infty (I + \xi^2H^{-1})^{-1} d\xi \) and \( \sqrt{M} = (2/\pi) \int_0^\infty (I + \xi^2M^{-1})^{-1} d\xi \) from problem 2’s solution as starting points. For every \( \xi > 0 \) apply problem 3 twice to infer that first \( (I + \xi^2M^{-1}) - (I + \xi^2H^{-1}) \) and then \( (I + \xi^2H^{-1})^{-1} - (I + \xi^2M^{-1})^{-1} \) are positive definite; then integrate to deduce the same for \( \sqrt{H} - \sqrt{M} \). Another proof parallels the uniqueness proof of problem 2: Let the minimum of the quotient \( z^T(\sqrt{H} - \sqrt{M})z/z^Tz \) be \( \mu \), the least eigenvalue of \( \sqrt{H} - \sqrt{M} \), so that \( (\sqrt{H} - \sqrt{M})z = \mu z \) for an eigenvector \( z \neq 0 \). Then \( \sqrt{M}z = (\sqrt{H} - \mu I)z \), whence \( 0 < z^T(H-M)z = z^T(H - (\sqrt{H} - \mu I)^2)z = 2\mu z^T\sqrt{H}z - \mu^2 z^Tz \), and therefore \( \mu > 0 \).

5. Show that \( H \), \( M \) and \( H-M \) can be symmetric and positive definite when \( H^2 - M^2 \) is not.

**Solution:** The fact that a real symmetric matrix is positive definite if and only if it and its every principal submatrix have positive determinants helps to reveal the following 2-by-2 examples:

\[
\begin{pmatrix} 2 & 2 \\ 2 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \quad \text{are positive definite but} \quad H^2 - M^2 = \begin{pmatrix} 7 & 22 \\ 22 & 69 \end{pmatrix} \quad \text{isn’t.}
\]

\[
\begin{pmatrix} 3 & 3 \\ 3 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \quad \text{are positive definite but} \quad H^2 - M^2 = \begin{pmatrix} 17 & 33 \\ 33 & 64 \end{pmatrix} \quad \text{isn’t.}
\]

\[
\begin{pmatrix} 8 & 5 \\ 5 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 7 & 5 \\ 5 & 4 \end{pmatrix} \quad \text{are positive definite but} \quad H^2 - M^2 = \begin{pmatrix} 88 & 85 \\ 85 & 81 \end{pmatrix} \quad \text{isn’t.}
\]