Problem Set 1 : Problems about our Axioms for a Vector Space

Three questions deserve to be asked about any axiom system:

Are they consistent? They are worse than useless if they are mutually contradictory. Consistency is proved, if it can be proved, by exhibiting a *model*, a set that satisfies the axioms.

Are they categorical? If the axioms admit different models that are too different, not much can be inferred from the axioms.

Are they parsimonious? If some axioms can be inferred from others, perhaps redundant axioms should be renamed as lemmas or theorems to enhance the ostensible elegance of the axioms that are left.

Questions like these three inspired the following four problems. Though posed as questions that could be answered “Yes” or “No”, they require explanatory answers.

1. Can the set of all rows of six positive real numbers be a vector space over the reals?

Yes; it is the image of a space of real 6-vectors under the map $\Omega = \log$ and its inverse $f = \exp$. For any two of these rows x and y define $x \oplus y := x \cdot y$ elementwise, so $x \ominus y = x/y$ elementwise, and $ß \circ x := x^{ß}$ elementwise for every row x and scalar ß .

2. Is there a set of entities $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ that satisfies all but one of our axioms specified for a vector space over a field, that one being the axiom “ $(-1) \circ \mathbf{x} = -\mathbf{x}$ ” ?

No; the assertion “ $(-1) \circ \mathbf{x} = -\mathbf{x}$ ” is redundant among our axioms because it can be inferred from $-\mathbf{x} \oplus \mathbf{o} = -\mathbf{x} \oplus 0 \circ \mathbf{x} = -\mathbf{x} \oplus (1 + (-1)) \circ \mathbf{x} = -\mathbf{x} \oplus \mathbf{x} \oplus (-1) \circ \mathbf{x} = \mathbf{o} \oplus (-1) \circ \mathbf{x}$.

3. Is there a set of entities $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ that satisfies all but two of our axioms specified for a vector space over a field, those two being the axioms “ $1 \circ \mathbf{x} = \mathbf{x}$ ” and “ $(-1) \circ \mathbf{x} = -\mathbf{x}$ ” ?

Yes; start with a subset of vectors from a vector space but redefine $ß \circ \mathbf{x} = \mathbf{o}$ for every scalar ß . The resulting set is a kind of lattice containing only linear combinations of integer multiples of the vectors in the subset; for instance, $\mathbf{x} + \mathbf{x} + \mathbf{x} + \mathbf{y} + \mathbf{y} = 3\mathbf{x} + 2\mathbf{y}$ but $3 \circ \mathbf{x} + 2 \circ \mathbf{y} = \mathbf{o}$.

4. Is there a set of entities $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ that satisfies all but one of our axioms specified for a vector space over a field, that one being the axiom “ $\mathbf{x} \oplus \mathbf{y} = \mathbf{y} \oplus \mathbf{x}$ ” for commutative vector addition ?

No;

$$(\mathbf{x} \oplus \mathbf{y}) \ominus (\mathbf{y} \oplus \mathbf{x}) = (\mathbf{x} \oplus \mathbf{y}) \oplus (-1) \circ (\mathbf{y} \oplus \mathbf{x}) = (\mathbf{x} \oplus \mathbf{y}) \oplus ((-\mathbf{y}) \oplus (-\mathbf{x})) = \mathbf{x} \oplus (\mathbf{y} \oplus (-\mathbf{y})) \oplus (-\mathbf{x}) = \mathbf{o} .$$