Geometry of Elementary Operations and Subspaces

A continuation of notes titled “Geometry of Elementary Operations”

Matrices Represent Linear Operators:

Let $\mathbf{L}$ be a linear operator that maps a space of vectors $\mathbf{x} = \mathbf{B}x$ to a space of vectors $\mathbf{y} = \mathbf{E}y$ with their respective bases $\mathbf{B}$ and $\mathbf{E}$. Here x is a column vector that represents $\mathbf{x}$ in the basis $\mathbf{B}$ as y represents $\mathbf{y}$ in $\mathbf{E}$. Now, what represents $\mathbf{L}$? Matrix $L := \mathbf{E}^{-1}\mathbf{L}\mathbf{B}$ represents $\mathbf{L}$ with bases $\mathbf{B}$ and $\mathbf{E}$ because $\mathbf{y} = \mathbf{L}\mathbf{x}$ just when $y = \mathbf{E}^{-1}\mathbf{y} = \mathbf{E}^{-1}\mathbf{L}\mathbf{B}x = Lx$.

When bases change, say from $\mathbf{B}$ to $\overline{\mathbf{B}} := \mathbf{B}C$ and $\mathbf{E}$ to $\overline{\mathbf{E}} := \mathbf{E}G$ (where the matrices C and G must be square and invertible, as we have seen), their matrices figure in *changes of coordinates* (representatives) thus:

$$x := \mathbf{B}^{-1}\mathbf{x} \quad \text{and} \quad \overline{x} := \overline{\mathbf{B}}^{-1}\mathbf{x} = \overline{\mathbf{B}}^{-1}\mathbf{B}x = C^{-1}x \qquad \text{represent } \mathbf{x} \,;$$
$$y := \mathbf{E}^{-1}\mathbf{y} \quad \text{and} \quad \overline{y} := \overline{\mathbf{E}}^{-1}\mathbf{y} = \overline{\mathbf{E}}^{-1}\mathbf{E}y = G^{-1}y \qquad \text{represent } \mathbf{y} \,;$$
$$L := \mathbf{E}^{-1}\mathbf{L}\mathbf{B} \quad \text{and} \quad \overline{L} := \overline{\mathbf{E}}^{-1}\mathbf{L}\overline{\mathbf{B}} = \overline{\mathbf{E}}^{-1}\mathbf{E}L\mathbf{B}^{-1}\overline{\mathbf{B}} = G^{-1}LC \qquad \text{represent } \mathbf{L} \,.$$

Evidently $\mathbf{y} = \mathbf{L}\mathbf{x}$ just when $y = Lx$ and $\overline{y} = \overline{L}\,\overline{x}$.

What works for vectors works also for linear functionals; these are just linear maps to a 1-dimensional space whose basis need not change:

$$u^T := \mathbf{u}^T\mathbf{B} \quad \text{and} \quad \overline{u}^T := \mathbf{u}^T\overline{\mathbf{B}} = u^T\mathbf{B}^{-1}\overline{\mathbf{B}} = u^TC \qquad \text{represent } \mathbf{u}^T \,;$$
$$v^T := \mathbf{v}^T\mathbf{E} \quad \text{and} \quad \overline{v}^T := \mathbf{v}^T\overline{\mathbf{E}} = v^T\mathbf{E}^{-1}\overline{\mathbf{E}} = v^TG \qquad \text{represent } \mathbf{v}^T \,;$$

so $\mathbf{u}^T\mathbf{x} = u^Tx = \overline{u}^T\overline{x}$ and $\mathbf{v}^T\mathbf{y} = v^Ty = \overline{v}^T\overline{y}$. Now $\mathbf{u}^T = \mathbf{v}^T\mathbf{L}$ just when $u^T = v^TL$ and $\overline{u}^T = \overline{v}^T\overline{L}$. (Confirm the last seven equations.) In all cases the misnamed *coordinate-free* (it should be called “*coordinate-independent* ”) algebra is the same; only the arithmetic changes when bases change. To simplify arithmetic is the principal motivation to change bases.

Do not try to memorize where to put C or G . Or is it C^{-1} or G^{-1} ? Left or right of L ? You can work out these question’s answers more reliably by remembering that a basis is an invertible linear map from a space of column vectors to another vector space, and a change of basis post-multiplies the basis by an invertible matrix.

Here is an example to show how changes of bases can simplify arithmetic. A given arbitrary possibly rectangular matrix L represents some linear map $\mathbf{L}$ from one vector space to another if apt bases are used in those spaces. Let G^{-1} be any product of elementary row operations that puts L into Reduced Row-Echelon Form $G^{-1}L$, and let C be any product of elementary column operations that puts $G^{-1}L$ into its Reduced Column-Echelon Form $\overline{L} = G^{-1}LC$. As we have seen in notes “The Reduced Row-Echelon Form is Unique”, this last Reduced Echelon Form $\overline{L}$ is determined uniquely by the starting matrix L , regardless of how the elementary operations get to $\overline{L}$. But $\overline{L}$ must now consist of an identity matrix in its upper left corner, and zeros in all later rows and columns if there are any. Since elementary row operations leave row-rank unchanged, and similarly for column-rank, we see that these ranks are the same, namely the dimension ρ of that identity matrix. Now re-interpret G and C as matrices of two changes of basis, one basis in each of the vector spaces connected by whatever linear operator $\mathbf{L}$ is represented by L ; since $\overline{L} = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$, $\mathbf{L}$ maps the first ρ changed basis vectors in one Affine vector space to the first ρ changed basis vectors in the other.

Thus we conclude that every matrix $L = G\overline{L}C^{-1}$ can be factored into a product whose two outer factors are invertible and whose third inner factor $\overline{L}$ is an identity matrix with perhaps some rows and/or columns of zeros appended to make $\overline{L}$ have the same dimensions as L , whose rank ρ is the dimension of the identity matrix. For any such fixed $\overline{L} = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$, the family of matrices $L = G\overline{L}C^{-1}$ generated as G and C sweep through all invertible matrices of appropriate dimensions is a family called *Equivalent* matrices; they all represent the same abstract linear operator **L** but in different coordinate systems. They all have the same rank ρ , which we might as well define to be Rank(**L**) . What else have they in common?

Dimensions that are Invariants of Equivalent Matrices:

All that Equivalent matrices have in common are their dimensions and their rank ρ . These numbers are the dimensions of three important vector spaces associated with that abstract linear operator **L** . Let us name them. One space is the *Domain*(**L**) , the space of vectors upon which **L** operates. Another is the *Range-space* (an ambiguous phrase best not used) or *Target-space*(**L**) into or onto which **L** throws its results. If $\mathbf{y} = \mathbf{Lx}$ then **x** must come from Domain(**L**) and **y** from Target-space(**L**) . As **x** runs through all of Domain(**L**) , $\mathbf{y} = \mathbf{Lx}$ sweeps out a third vector space *Range*(**L**) . (Why is it a vector space?) Range(**L**) need not fill Target-space(**L**) but may be a proper subspace. (This is why “Target-space” is a better phrase than “Range-space” when they must be distinguished from “Range”.)

Let L be any of the *Equivalent* matrices that represent **L** ; then

Dimension(Domain(**L**)) = Count(Columns(L)) ,
Dimension(Target-space(**L**)) = Count(Rows(L)) , and
Dimension(Range(**L**)) = Rank(**L**) = Rank(L) = ρ .

The last equation comes from the Equivalent matrix $\overline{L} = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$, which tells us that Range(**L**) has a basis with ρ vectors. This basis cannot be chosen uniquely even though Range(**L**) is fully determined by **L** . This basis of Range(**L**) is the image of the first ρ basis vectors in some basis of Domain(**L**) ; those first ρ basis vectors span a subspace of Domain(**L**) that need *not* be determined uniquely by **L** if its rank ρ is less than the dimension of its domain.

To think otherwise is a mistake made by many students; but adding to each of the first ρ basis vectors any linear combinations of subsequent basis vectors (from Nullspace(**L**)) yields a new basis whose first ρ vectors, now spanning another subspace of Domain(**L**) , are mapped by **L** upon the same basis of Range(**L**) .

The subspace of Domain(**L**) determined uniquely by **L** is its *Kernel* or *Nullspace*, consisting of all vectors **z** that satisfy $\mathbf{Lz} = \mathbf{o}$. (Why is it a vector space?) Looking at $\overline{L}$ tells us

Nullity(**L**) := Dimension(Nullspace(**L**)) = Count(Columns($\overline{L}$)) – Rank(**L**)

for any matrix L that represents **L** . In other words (and this is IMPORTANT),

$$\text{Rank}(\mathbf{L}) + \text{Nullity}(\mathbf{L}) = \text{Dimension}(\text{ Domain}(\mathbf{L})) .$$

Every linear operator **L** operates in two directions. We have just looked at one; operator **L** maps vectors in its domain to vectors in its range. Now look at **L** another way; it maps linear functionals $\mathbf{v}^T$ that act upon Target-space(**L**) linearly to linear functionals $\mathbf{u}^T = \mathbf{v}^T\mathbf{L}$ acting upon Domain(**L**) . This space of linear functionals $\mathbf{v}^T$, dual to Target-space(**L**) , is called

Codomain(**L**) ; as $\mathbf{v}^T$ runs through all of it, $\mathbf{v}^T\mathbf{L}$ sweeps out a subspace of the space dual to Domain(**L**) . This subspace can be called *Corange*(**L**) . The subspace of Codomain(**L**) swept out by solutions $\mathbf{w}^T$ of $\mathbf{w}^T\mathbf{L} = \mathbf{o}^T$ is *Cokernel*(**L**) . You may have seen some of these spaces before in texts where **L** = L is a matrix that maps one space of column vectors to another, and then Range(**L**) is the column-space of L and its row-space is Corange(**L**) .

Exercise 0: The foregoing plethora of (sub)spaces and names for them sound more complicated than they are; describe all eight of them when $\mathbf{L} = \overline{L} = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$. These eight (sub)spaces are …
Target-space, Range, Nullspace, Domain, Codomain, Cokernel, Corange, Dual of Domain.

Exercise 1: Explain why the rank of a matrix product cannot exceed the rank of any factor.

Exercise 2: Every linear operator **L** can be written as a sum $\mathbf{L} = \mathbf{c}_1\mathbf{r}^T{}_1 + \mathbf{c}_2\mathbf{r}^T{}_2 + \ldots + \mathbf{c}_k\mathbf{r}^T{}_k$ of *dyads* $\mathbf{c}\mathbf{r}^T$ (linear operators of rank 1) in infinitely many ways; here each $\mathbf{c}_j$ is drawn from Target-space(**L**) and each $\mathbf{r}^T{}_j$ from the dual of Domain(**L**) . Show that Rank(**L**) is the minimum possible number k of terms in the sum, and exhibit a way to achieve this minimum.
This minimum, called *Tensor Rank*, is an alternative way to define Rank(**L**) ; it can be generalized from linear operators to multi-linear operators, but nobody knows how to compute *Tensor Rank* for multi-linear operators.

Exercise 3: The linear operator whose matrix is $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ maps a plane to a line in the plane; why? The linear operator whose matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ maps a 3-space to a line in the plane; why?
Similarly describe the effects of operators whose matrices are ...

$$A := \begin{bmatrix} 0 & 1 & 5 \\ 2 & 0 & 4 \\ 0 & 7 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 4 & 5 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad C := \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 4 & 6 \end{bmatrix}, \quad D := \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 6 & 7 & 8 \end{bmatrix}, \quad E := \begin{bmatrix} 4 & 5 & 6 & 0 \\ 1 & 2 & 3 & 1 \\ 6 & 7 & 8 & 9 \end{bmatrix}.$$

(The next four exercises were supplied by Prof. B. N. Parlett and A. Hernandez.)

Exercise 4: An operator **L** is represented by a 3-by-3 matrix. The set of all solutions **p** of **Lp** = **o** sweeps out a plane ***P*** through the origin **o** . Vectors **b** := **Lu** and **c** := **Lv** are nonzero. Describe the set ***X*** of all solutions **x** of **Lx** = **b** , and the set ***Y*** of all solutions **y** of **Ly** = **b**+**c** . What is Dimension(Range(**L**)) ?

In the next three exercises, *V* is some vector space of high dimension, and b, c, d, ..., x, y, z are nonzero vectors in it.

Exercise 5: *W* is a subset of *V* containing b, c, and 3c + 5d , but not d nor e – 2d ; determine whether *W* can possibly be a subspace of *V* . Give your reasoning. Do likewise for *U* , a subset of *V* containing 5c and 3d – 2b but not b nor d .

Exercise 6: *Q* is spanned by {c, x, y, z} ; here x and y are linearly independent and span a subspace *U* that also contains y + 3z but not y + 2c . What is Dimension(*Q*) , and why ?

Exercise 7: E and F are subspaces of V. F is spanned by {b, e, f} ; and [c, d, f] is a basis for E, which also contains b . However, the spans of {c, d} and of {b, f} intersect only in the zero vector. Explain whether [b, e, f] is a basis for F.

Intersections, Sums and Annihilators of Subspaces:

Let $\boldsymbol{E}$ and $\boldsymbol{F}$ be proper subspaces of a vector space $\boldsymbol{B}$. ("Proper" means neither **o** nor $\boldsymbol{B}$.) Bases **E** for $\boldsymbol{E}$ and **F** for $\boldsymbol{F}$ consist of spanning "rows" of linearly independent vectors drawn from $\boldsymbol{B}$ but not necessarily from a given basis **B** of $\boldsymbol{B}$. Still, some bases must be related; $\mathbf{E} = \mathbf{B}E$ and $\mathbf{F} = \mathbf{B}F$ for rectangular matrices E and F with as many columns as the dimensions of subspaces $\boldsymbol{E}$ and $\boldsymbol{F}$ respectively, and as many rows as the dimension of $\boldsymbol{B}$.

Why must the columns of E be linearly independent, and likewise those of F , but maybe not those of [E, F] ?

Let the *sum* of subspaces $\boldsymbol{E}$ and $\boldsymbol{F}$ be denoted by $\boldsymbol{E} + \boldsymbol{F}$; it consists of all sums $\mathbf{e} + \mathbf{f}$ of vectors **e** drawn from $\boldsymbol{E}$ and **f** drawn from $\boldsymbol{F}$. Note that $\boldsymbol{E} + \boldsymbol{F}$ is a vector space. (Why?)

Don't confuse the sum with the *union* $\boldsymbol{E} \cup \boldsymbol{F}$ of two subspaces, which consists of all vectors that belong to at least one of $\boldsymbol{E}$ and $\boldsymbol{F}$; it need not be a vector space at all; can you see why by providing a suitable example?

Let the *intersection* of subspaces $\boldsymbol{E}$ and $\boldsymbol{F}$ be denoted by $\boldsymbol{E} \cap \boldsymbol{F}$; it consists of all vectors that belong both to $\boldsymbol{E}$ and to $\boldsymbol{F}$, and is a subspace of $\boldsymbol{B}$ too. (Why?) It may be just { **o** } .

Given E and F , can we compute a basis for $\boldsymbol{E} \cap \boldsymbol{F}$? It's a bit tricky. First assemble matrix [E, F] and reduce it to its echelon form $G^{-1}[E, F]C = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$ by pre- and post-multiplication by *invertible* square matrices G^{-1} and C . Next partition $C =: \begin{bmatrix} H & J \\ K & L \end{bmatrix}$ conformably to obtain $G^{-1}(EH + FK) = \begin{bmatrix} I \\ O \end{bmatrix}$ and $G^{-1}(EJ + FL) = \begin{bmatrix} O \\ O \end{bmatrix}$. Then $\mathbf{E}H + \mathbf{F}K$ is a basis for $\boldsymbol{E} + \boldsymbol{F}$; and $\mathbf{E}J = -\mathbf{F}L$ is a basis for $\boldsymbol{E} \cap \boldsymbol{F}$ unless it is { **o** } , in which case J and L are empty matrices. But the assertions in the last sentence are unobvious; can you prove them?

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Here are proofs of those assertions. First $\mathbf{E}J + \mathbf{F}L = \mathbf{B}(EJ + FL) = \mathbf{B}O$ if it is not empty, so $\mathbf{E}J = -\mathbf{F}L$. If z satisfies $Jz = o$ then it also satisfies $o = (EJ + FL)z = FLz$, and then $Lz = o$ too because the columns of F are linearly independent; since the last columns of matrix C are independent (else it wouldn't be invertible), $z = o$ too. Therefore the columns of J (and similarly L) must be linearly independent if not empty, so $\mathbf{E}J = -\mathbf{F}L$ is a basis for a nonzero subspace of $\boldsymbol{E} \cap \boldsymbol{F}$ if not all of it. To show that none of it is left out we must solve equation $\mathbf{E}Jx = \mathbf{E}u$ for x whenever $\mathbf{E}u = -\mathbf{F}v$ lies in $\boldsymbol{E} \cap \boldsymbol{F}$. The solution can be expressed in terms of a conformable partition of $C^{-1} =: \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$; here $C^{-1}C = I = CC^{-1}$ implies that MH+NK , PJ+QL , HM+JP and KN+LQ are identity matrices, and MJ+NL , PH+QK , HN+JQ and KM+LP are zero matrices of perhaps diverse sizes. Now a little algebra (*Do it!*) suffices to confirm that $x := Pu + Qv$ is the desired solution, so $\mathbf{E}J = -\mathbf{F}L$ does span all of $\boldsymbol{E} \cap \boldsymbol{F}$.

$\boldsymbol{E} + \boldsymbol{F}$ is next. If $(\mathbf{E}H + \mathbf{F}K)z = \mathbf{o}$ then z satisfies $o = [I, O]G^{-1}(EH + FK)z = z$ too, so that $\mathbf{E}H + \mathbf{F}K$ must be a basis for a subspace of $\boldsymbol{E} + \boldsymbol{F}$ if not all of it. To show none of it is left out we need only solve equation $(\mathbf{E}H + \mathbf{F}K)y = (\mathbf{E}s + \mathbf{F}t)$ for y . The solution is $y := Ms + Nt$.

Do the algebra to confirm this formula for y . The formulas for x and y above are fragile numerically, easily broken by rounding errors. Robust formulas are more subtle.

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A by-product of the proof, obtained by counting columns of the echelon form, is the formula

$$\text{Dimension}(\boldsymbol{E} \cap \boldsymbol{F}) + \text{Dimension}(\boldsymbol{E} + \boldsymbol{F}) = \text{Dimension}(\boldsymbol{E}) + \text{Dimension}(\boldsymbol{F}) .$$

This IMPORTANT formula deserves a proof simpler than the computation above; can you find a simpler proof?

Here it is. Let $\mathbf{D}$ be any basis for $\boldsymbol{E} \cap \boldsymbol{F}$. If not already a basis for $\boldsymbol{E}$, $\mathbf{D}$ can be augmented to form a basis $[\mathbf{D}, \overline{\mathbf{E}}]$ for $\boldsymbol{E}$. Likewise $[\mathbf{D}, \overline{\mathbf{F}}]$ forms a basis for $\boldsymbol{F}$. Certainly $\boldsymbol{E} + \boldsymbol{F}$ is spanned by the elements of $[\mathbf{D}, \overline{\mathbf{E}}, \overline{\mathbf{F}}]$. It is a basis too if its elements are linearly independent. Suppose $\mathbf{D}\mathrm{d} - \overline{\mathbf{E}}\mathrm{e} - \overline{\mathbf{F}}\mathrm{f} = \mathbf{o}$. This says that $\overline{\mathbf{F}}\mathrm{f} = \mathbf{D}\mathrm{d} - \overline{\mathbf{E}}\mathrm{e}$ lies in $\boldsymbol{F}$ and in $\boldsymbol{E}$, so it lies in $\boldsymbol{E} \cap \boldsymbol{F}$. Therefore $\overline{\mathbf{F}}\mathrm{f} = \mathbf{D}\bar{\mathrm{d}} = \mathbf{D}\mathrm{d} - \overline{\mathbf{E}}\mathrm{e}$ for some $\bar{\mathrm{d}}$. Then $\mathbf{D}\bar{\mathrm{d}} - \overline{\mathbf{F}}\mathrm{f} = \mathbf{o}$ is the zero vector in $\boldsymbol{F}$, so $\bar{\mathrm{d}} = \mathrm{o}$ and $\mathrm{f} = \mathrm{o}$ (perhaps of different dimensions). $\mathbf{D}(\mathrm{d}-\bar{\mathrm{d}}) - \overline{\mathbf{E}}\mathrm{e} = \mathbf{o}$ implies $\mathrm{d} - \bar{\mathrm{d}} = \mathrm{o}$ and $\mathrm{e} = \mathrm{o}$ similarly. Therefore the elements of $[\mathbf{D}, \overline{\mathbf{E}}, \overline{\mathbf{F}}]$ really are linearly independent; they do form a basis of $\boldsymbol{E} + \boldsymbol{F}$. Counting elements confirms the IMPORTANT formula above.

Exercise 8: If the dimension of a vector space is less than the sum of the dimensions of two of its subspaces, can their intersection be just $\{ \mathbf{o} \}$? Justify your answer.

Exercise 9: Two proper subspaces of a vector space are *Complementary* just when their sum is the whole space and their intersection $\{ \mathbf{o} \}$. Can either determine the other uniquely? Why?

The *Annihilator* of a subspace $\boldsymbol{E}$ is the set of all linear functionals $\mathbf{w}^T$ that satisfy $\mathbf{w}^T\mathbf{e} = 0$ for every $\mathbf{e}$ in $\boldsymbol{E}$, and is denoted by $\boldsymbol{E}^\perp$. This annihilator is a subspace of the dual space determined uniquely by $\boldsymbol{E}$.

The notation " $\boldsymbol{E}^\perp$ " is a relic from Euclidean spaces, which are their own duals; that is why $\boldsymbol{E}^\perp$ is often called the "orthogonal complement" of $\boldsymbol{E}$ even if it is a subspace of a non-Euclidean space. This terminology can mislead; only in Euclidean spaces are $\boldsymbol{E}$ and $\boldsymbol{E}^\perp$ complementary. "Annihilator" is unmistakable.

A good way to think about subspaces is in terms of their bases. Given a basis $\mathbf{B}$ for a vector space, think of $\mathbf{E} := \mathbf{B}\mathrm{E}$ for some rectangular matrix E with linearly independent columns as the basis for a proper subspace $\boldsymbol{E} := \text{Range}(\mathbf{E})$ whose annihilator is $\boldsymbol{E}^\perp = \text{Cokernel}(\mathbf{E})$. This latter subspace has a basis too consisting of linear combinations of the "rows" of $\mathbf{B}^{-1}$: …

Exercise 10: Confirm important relations $\text{Dimension}(\boldsymbol{E}) + \text{Dimension}(\boldsymbol{E}^\perp) = \text{Dimension}(\mathbf{B})$ and $(\boldsymbol{E}^\perp)^\perp = \boldsymbol{E}$ by augmenting $\mathbf{E}$ to get a basis $[\mathbf{E}, \overline{\mathbf{E}}] = \mathbf{B}[\mathrm{E}, \overline{\mathrm{E}}]$ and partitioning its inverse conformably to get a basis for $\boldsymbol{E}^\perp$.

Exercise 11: Cite " $(\boldsymbol{E}^\perp)^\perp = \boldsymbol{E}$ " for a quick proof of Fredholm's alternative (1) in the notes "The Reduced Row-Echelon Form is Unique". (It works for some infinite-dimensional spaces.)

Exercise 12: Prove that $(\boldsymbol{E} + \boldsymbol{F})^\perp = \boldsymbol{E}^\perp \cap \boldsymbol{F}^\perp$. If $\boldsymbol{E} \cap \boldsymbol{F} \neq \{ \mathbf{o} \}$ must $\boldsymbol{E}^\perp \cap \boldsymbol{F}^\perp \neq \{ \mathbf{o}^T \}$ too? Prove the answer is surely "Yes" if $\text{Dimension}(\boldsymbol{E}) + \text{Dimension}(\boldsymbol{F}) - \text{Dimension}(\boldsymbol{E} \cap \boldsymbol{F})$ is less than the dimension of the whole space, but otherwise surely "No".

Warning about Duals of Duals of Infinite-Dimensional Spaces:

The proof techniques used above are based upon finite-dimensional matrix multiplication; but most of the definitions and inferences make sense for infinite dimensional spaces too. There is one important exception that would go unnoticed because our notation takes it for granted: "A vector space is the dual of its dual." This assertion, obviously true for all finite-dimensional spaces, is false for many infinite-dimensional spaces. It is false for the space of continuous functions on a closed domain, and for the space of all absolutely convergent series, and for the space of all infinite sequences with at most finitely many nonzero terms; these three spaces are each properly contained in the dual of its dual. For infinite-dimensional spaces, Linear Algebra has to be rebuilt from the ground up in a graduate course that takes convergence into account; it lies beyond the syllabus of this course except for occasional warnings like this one.