Gaussian Elimination and Triangular Factorization are normally performed with pivotal row exchanges to prevent division by zero and fend against numerical instability. These exchanges are unnecessary when the square matrix $A = \{a_{ij}\}$ to be factored belongs to one of two classes of matrices with sufficiently prominent diagonal elements a_{ii} . For a matrix A in either class, every Schur Complement of A belong to the same class, as we shall prove for the first Schur Complement $S := H - cr^T/\mu$ of μ in the partitioned matrix $A = \begin{bmatrix} \mu & r^T \\ c & H \end{bmatrix}$. More important, we shall prove the Schur Complements of matrices in both classes cannot grow much if at all; how this aids numerical stability by preventing rounding errors in cr^T/μ from swamping the data in H was known to J. von Neumann in 1946 though first explained by J.H. Wilkinson in 1959.

1. Matrices A Dominated by their Diagonals:

A is *Row-Dominated by its Diagonal* when every $\rho_i := |a_{ii}| - \sum_{j\neq i} |a_{ij}| > 0$.

A is *Column-Dominated by its Diagonal* when every $\gamma_j := |a_{jj}| - \sum_{i\neq j} |a_{ij}| > 0$.

We shall prove first that S inherits the same kind of diagonal dominance from A . Column-dominance will be treated since the treatment of row-dominance is very similar. Let S and the sub-arrays of A inherit their subscripts from A , so that $S = \{s_{ij}\}$ and $H = \{h_{ij}\} = \{a_{ij}\}$ for $i > 1$ and $j > 1$, and row $r^T = [r_2, r_3, \dots] = [a_{12}, a_{13}, \dots]$ and similarly for column c . Then $s_{ij} = h_{ij} - c_i r_j/\mu$, and $\gamma_1 = |\mu| - \sum_{i>1} |c_i| > 0$, and $\gamma_j = 2|h_{jj}| - |r_j| - \sum_{i>1} |h_{ij}| > 0$ for $j > 1$. Now

$$|s_{jj}| - \sum_{i\neq j} |s_{ij}| \ \geq\ (|h_{jj}| - |c_j r_j/\mu|) - \Big(\sum_{i>1} (|h_{ij}| + |c_i r_j/\mu|) - (|h_{jj}| + |c_j r_j/\mu|) \Big)$$
$$= \gamma_j + |r_j|(1 - \sum_{i>1} |c_i/\mu|) = \gamma_j + \gamma_1|r_j/\mu| \ \geq\ \gamma_j > 0 .$$

Therefore no column of S is less diagonally dominant than the same column of A . Finally we observe that S can't grow much because …

$$\sum_{i>1} |s_{ij}| \leq \sum_{i>1} |h_{ij}| + \sum_{i>1} |c_i r_j/\mu| = \sum_{i>1} |h_{ij}| + (1 - \gamma_1/|\mu|)|r_j| \leq \sum_i |a_{ij}| .$$

This says no column of the first Schur Complement S has a bigger sum of magnitudes than has the same column of A , whence the same follows for all subsequent Schur Complements.

2. Positive Definite Matrices A :

Real matrix A is *Positive Definite* just when $x^TAx > 0$ for every real nonzero x , and then every principal submatrix of A turns out to be positive definite too; this is proved by setting to zeros the elements of x in rows that do not belong to the chosen submatrix. Therefore $\mu > 0$ and H is positive definite in the partition of A exhibited above. Our first task is to show that the first Schur Complement $S := H - cr^T/\mu$ is positive definite too. Choose any real nonzero row y^T of the same dimension as r^T and from it construct $x^T := [-y^T(c + r),\ 2\mu y^T]$ to find that $0 < x^TAx = 4\mu^2 y^TSy - \mu(y^T(c-r))^2$; evidently $y^TSy > 0$ too, so S must be positive definite as claimed. This implies that every Schur Complement must be positive definite too.

However, Schur Complements may grow unless A is *Symmetric* as well as positive definite real. In this case $A = A^T$, $r = c$, $H = H^T$, and $S = H - cc^T/\mu = S^T$ is also symmetric positive definite real; moreover every diagonal element $s_{ii} = h_{ii} - c_i^2/\mu \leq a_{ii}$. Consequently diagonal elements of successive Schur Complements form positive non-increasing sequences, and off-diagonal elements can't grow much since each $|s_{ij}| < \sqrt{(s_{ii}s_{jj})}$, as can be inferred from $y^TSy > 0$ by setting only two elements of y to nonzero values $\sqrt{s_{jj}}$ and $\pm\sqrt{s_{ii}}$. All this ensures that a *Choleski* factorization $A = U^TU$ with triangular U exists, but that is a story for another day.