The Reduced Row-Echelon Form is Unique

Any (possibly not square) finite matrix B can be reduced in many ways by a finite sequence of *Elementary Row-Operations* $E_1, E_2, ..., E_m$, each one invertible, to a *Reduced Row-Echelon Form* (RREF) $U := E_m \cdots E_2 \cdot E_1 \cdot B$ characterized by three properties:

1: The first nonzero element in any nonzero row is "1" .
2: Each nonzero row's leading "1" comes in a column whose every other element is "0" .
3: Each such leading "1" comes in a column after every preceeding row's leading zeros.

Here is an example of a matrix U in RREF :

$$U = \begin{pmatrix} 0 & 0 & \mathbf{1} & 2 & 0 & 3 & 0 & 4 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 7 & 0 & 8 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

That B determines its RREF U uniquely will be demonstrated below, even though B does not determine uniquely the sequences of Elementary Row-Operations $E_1, E_2, ..., E_m$ that reduce B to U . In other words, if $V := F_n \cdots F_2 \cdot F_1 \cdot B$ is another RREF of B then, we shall prove, $V = U$ although $F_1, F_2, ..., F_m$ may differ from $E_1, E_2, ..., E_m$.

First confirm that $U = C \cdot V$ and $V = C^{-1} \cdot U$ where

$$C := (E_m \cdots E_2 \cdot E_1) \cdot (F_n \cdots F_2 \cdot F_1)^{-1} = E_m \cdots E_2 \cdot E_1 \cdot F_1^{-1} \cdot F_2^{-1} \cdots F_n^{-1} .$$

Our task is to infer that the two RREFs U and V are the same even though C need not be an identity matrix.

For any integer $j > 0$ let l_j denote the j^{th} column of an identity matrix of whatever size is appropriate, so that $u_j := U \cdot l_j$, $v_j := V \cdot l_j$ and $c_j := C \cdot l_j$ are respectively the j^{th} columns of U , V and C . Let us also note that if $v_j = o$ for some j then $u_j = C \cdot v_j = o$ too; similarly if $u_j = o$ then $v_j = C^{-1} \cdot u_j = o$ too. Therefore, we may simplify our task by striking out columns of zeros from B , U and V ; those columns will have corresponding indices, and striking them out will not invalidate anything said so far.

In the absence of zero columns, we can assume that the RREFs U and V have $u_1 = v_1 = l_1$. Therefore $c_1 = C \cdot l_1 = C \cdot v_1 = u_1 = l_1$ too. And if any column v_j is a scalar multiple of l_1 , say $v_j = \mu_j \cdot l_1$, then $u_j = U \cdot l_j = C \cdot V \cdot l_j = C \cdot v_j = \mu_j \cdot C \cdot l_1 = \mu_j \cdot l_1 = v_j$ too. Similarly, with C^{-1} in place of C , if any column $u_j = \mu_j \cdot l_1$ then $v_j = u_j$ too. Therefore all the columns of U and V that are proportional to l_1 match. The first column not proportional to l_1 is l_2 , and it appears in the same column positions in U and V if it appears at all. Since all columns of U and V that lie between the leading appearances of l_1 and l_2 are the same, striking them out of both U and V does no harm.

The foregoing theme will be extended by induction. Suppose that the first k columns of the RREFs U and V match the first k columns of an identity matrix whose first $k-1$ columns also match those of C , and $U = C \cdot V$. We have just seen that this is the case for $k = 2$ unless

U and V have fewer than k nonzero rows. Then we see $c_k = C\cdot l_k = C\cdot v_k = u_k = l_k$. Now consider any subsequent column v_j (with $j > k$) whose elements beyond the k^{th} all vanish; then $u_j = C\cdot v_j = v_j$ because only the first k columns of C matter to $C\cdot v_j$, and those columns match an identity matrix. Similarly, any subsequent column u_j (with $j > k$) whose elements beyond the k^{th} all vanish must match v_j . Therefore, all columns of U and V that lie between the leading appearances of l_k and l_{k+1} are the same; we may strike them out and continue the induction. The process stops when the nonzero rows are exhausted. Therefore U = V after certain identical columns have been struck out, so U = V after they are restored. End of proof.

Corollary:

The RREF of B is unchanged when it is pre-multiplied by an invertible matrix. The proof is the same as before except for a change in its definition of C .

Uses for the Reduced Row-Echelon Form:

Having proved that every matrix B has its own unique RREF U , we show next how U helps us determine the degrees of freedom available to solutions x of a system " $B\cdot x = y$ " of linear equations. First premultiply by a product H of elementary operations to change B into its RREF $U = H\cdot B$, simultaneously changing y into $z := H\cdot y$, without changing any solutions x ; if $B\cdot x = y$ then $U\cdot x = H\cdot B\cdot x = H\cdot y = z$, and if $U\cdot x = z$ then $B\cdot x = H^{-1}\cdot U\cdot x = H^{-1}\cdot z = y$. Such a solution x can exist just when z has no nonzero element in any row where U has only zeros; then elements of x in rows corresponding to columns of U with a leading "1" are determined from equation " $U\cdot x = z$ " by *back substitution* after all other elements of x have been chosen arbitrarily. The solution x is unique just when none of its elements can be choosen arbitrarily; that will be the case just when every column of U has just one nonzero element and no row has more than one. Otherwise the equation " $U\cdot v = o$ " will have nonzero solutions v that can be added to any solution x of " $B\cdot x = y$ " and still leave $U\cdot(x+v) = U\cdot x = z$ so $B\cdot(x+v) = H^{-1}\cdot U\cdot(x+v) = H^{-1}\cdot U\cdot x = B\cdot x = y$ too.

Fredholm's Alternatives:

Ivar Fredholm (1866-1927) enunciated these to characterize the solvability of integral equations and of *infinite* systems of linear equations without using determinants nor inverses.

1) At least one solution x of " $B\cdot x = y$ " exists if and only if
   every solution w^T of " $w^T\cdot B = o^T$ " also makes $w^T\cdot y = 0$.
2) If a solution x exists, it is unique if and only if
   " $B\cdot v = o$ " has no nonzero solution v .

Proof: We have seen how any nonzero solution v of " $B\cdot v = o$ " can be added to one solution x of " $B\cdot x = y$ " to get another; conversely if " $B\cdot x = y$ " has *different* solutions $x = x_1$ and $x = x_2$ then " $B\cdot v = o$ " must have a nonzero solution $v := x_1 - x_2$. Thus, alternative 2) is confirmed. As for 1), observe first that if $y = B\cdot x$ then $w^T\cdot y = w^T\cdot B\cdot x$ so every solution w^T of " $w^T\cdot B = o^T$ " does make $w^T\cdot y = 0$. Conversely if every such solution w^T makes $w^T\cdot y = 0$, the existence of at least one solution x of " $B\cdot x = y$ " follows from the RREF $U := H\cdot B$ thus: (This proof is valid only for finite systems!)

Suppose the contrary, that no such solution x existed. Then $z := H·y$ would have to have a nonzero element μ in a row where U has only zeros. Let l be that column of an identity matrix with its nonzero element in the same row, so $l^T·z = \mu$ and $l^T·U = o^T$. Then for a solution $w^T := l^T·H$ of " $w^T·B = o^T$ " we would find $w^T·y = \mu \neq 0$, contradicting "Conversely if ..." above. End of proof.

Uses for the Row-Rank:

The *Row-Rank* of B is the number of nonzero rows in its RREF. It has been used to characterize the solvability of linear systems for over two centuries.

Evidently " $B·x = y$ " is *consistent* (has at least one solution) just when matrices B and (B y) have the same row-rank. This can be confirmed most easily by reducing (B y) to its RREF , which reduces B to its RREF at the same time. (Can you see why?)

Evidently a solution x of " $B·x = y$ " is unique just when no nonzero vector v satisfies " $B·v = o$ " , so x is unique just when the row-rank of B equals its number of columns. (Can you see why?)

Whenever the row-rank of B is interesting, namely when it is less than the lesser of B's dimensions, it turns out to be a discontinuous function of the elements of B . (Can you see why?) Then the computation of row-rank is very vulnerable to rounding errors, which undermine its usefulness for deciding solvability.

Column Rank = Row Rank = Rank :

By exchanging the words "row" and "column" above, we can define the *Reduced Column-Echelon Form* (RCEF) of the matrix B and its *Column Rank*. In general, the RCEF and RREF of B need not be the same unless B is *nonsingular* (invertible), as we shall see. Though not necessarily the same, the RCEF and RREF of B have something in common: their rank. This comes about because of the Corollary above which implies that row/column rank is unchanged by pre/post multiplication by invertible matrices, respectively; furthermore, the RCEF of the RREF of B can easily be seen to equal the RREF of the RCEF of B , and this twice-reduced form consists of zeros everywhere except possibly in the first few diagonal positions where the number of "1" entries is the same as the number of nonzero rows or columns. Thus, row rank equals column rank, which justifies calling them both just "rank."

A square matrix B is *nonsingular* when its finite rank equals its dimension, in which case its RREF and RCEF must both be the identity matrix I ; in other words, $E·B = I = B·F$ for products E and F of elementary invertible matrices. Then $F = E·B·F = E$ is called B^{-1} , the inverse of B . Its existence is a nontrivial theorem which cannot be deduced from either equation " $E·B = I$ " or " $B·F = I$ " separately since either equation can be satisfied and the other not if B is not square or its dimensions are infinite. (Can you see how?)

Exercise: Suppose B has finite dimensions and that $P·B·Q = I$. Must $B^{-1} = Q·P$? Justify your answer.