Real Operator Norms in Complex Spaces

Every n-dimensional complex operator vector space $\mathbb{C}$ contains many n-dimensional real vector spaces, each obtained as the set $\mathbb{R}$ of all real linear combinations of the vectors in some complex basis chosen for $\mathbb{C}$. This process induces a definition for the complex conjugate $\bar{z}$ of any vector $z$ in $\mathbb{C}$ making $z$ the complex conjugate of $\bar{z}$, so that $\mathbb{R}$ contains all the vectors (including those basis vectors!) that are their own complex conjugates. $\mathbb{R}$ contains both $\text{Re}(z) := (\bar{z} + z)/2$ and $\text{Im}(z) := \imath (\bar{z} - z)/2$, so $z = \text{Re}(z) + \imath \cdot \text{Im}(z)$. Though $\mathbb{C}$ contains $\mathbb{R}$ it is not a subspace of $\mathbb{C}$.

A linear operator $L$ that maps $\mathbb{C}$ to itself has a complex conjugate $\overline{L}$ defined by taking $\overline{L} \cdot z$ to be the complex conjugate of $L \cdot z$. A linear operator is deemed Real if it maps $\mathbb{R}$ to itself. Both $\text{Re}(L) := (\overline{L} + L)/2$ and $\text{Im}(L) := \imath (\overline{L} - L)/2$ are instances of real linear operators; all these are their own complex conjugates. When restricted to $\mathbb{R}$, a real linear operator $R$ can have only real eigenvalues if any, though $R$ can have complex eigenvalues when it acts upon $\mathbb{C}$.

Nothing surprising has been said here yet. The surprise will come soon.

Let $\mathbb{C}$ be a normed space; its norm $\|\cdot\|$ is inherited by $\mathbb{R}$, and $\|\text{Re}(z)\| \leq (\|z\| + \|\bar{z}\|)/2$; in general $\|\bar{z}\|$ may differ from $\|z\|$. A linear operator $L$ that maps $\mathbb{C}$ to itself inherits its operator (or lub- or sup-) norm from the vector norm thus:

$$\|L\| := \max \|L \cdot z\|/\|z\|$$

over all nonzero $z$ in $\mathbb{C}$.

Evidently $\|\text{Re}(L)\| \leq (\|L\| + \|\bar{L}\|)/2$; in general $\|L\|$ may differ from $\|\bar{L}\|$. Every eigenvalue $\lambda$ of $L$ satisfies $|\lambda| \leq \|L\|$ because its associated eigenvector $z \neq 0$ has $L \cdot z = \lambda \cdot z$. Also $|\lambda| \leq \|\bar{L}\|$ because $\overline{\lambda}$ is an eigenvalue of $\overline{L}$ with eigenvector $\overline{z}$.

The same goes for a real linear operator $R = \overline{R}$; its every eigenvalue $\rho$, real or complex, must satisfy $|\rho| \leq \|R\| = \|\overline{R}\|$. However, another operator norm makes sense for real operators:

$$\|R\| := \max \|R \cdot x\|/\|x\|$$

over all nonzero $x$ in $\mathbb{R}$.

Evidently $\|R\| \leq \|R\|$, because $\mathbb{R} \subset \mathbb{C}$, and $|\rho| \leq \|R\|$ for every real eigenvalue $\rho$ of $R$.

**Surprise:** $|\rho| \leq \|R\|$ for every eigenvalue $\rho$, real or complex, of real $R$.

Prof. Ridg Scott of the University of Chicago pointed out this surprise. Hereunder is a proof:

Suppose $\rho = \mu \cdot e^{i\phi}$ is a non-real eigenvalue of $R$ with $\mu > 0$ and some real $\phi \neq 0$. Then the corresponding eigenvector $z$ must be non-real; in fact $z \cdot e^{i\phi}$ must be a non-real eigenvector for every real $\phi$. Therefore some such $\phi$ achieves a positive minimum for $\|\text{Im}(z \cdot e^{i\phi})\|$. Some simplification is gained without loss of generality by assuming that a minimizing $\phi = 0$. Now the imaginary part of the equation $R \cdot z = \mu \cdot z \cdot e^{i\phi}$ yields $R \cdot \text{Im}(z) = \mu \cdot \text{Im}(z \cdot e^{i\phi})$ whence follows $\|R\| \geq \|R \cdot \text{Im}(z)\|/\|\text{Im}(z)\| = \mu \cdot \|\text{Im}(z \cdot e^{i\phi})\|/\|\text{Im}(z)\| \geq \mu$ because $\|\text{Im}(z)\|$ was minimized. []

The surprise is nontrivial; examples of norms $\|\cdot\|$ that have $\|R\| < \|R\|$ do exist, but they are not the most familiar norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. The reader is invited to find examples before turning over the page.
Examples:

Let $\mathbb{C} := \mathbb{C}^2$, the space of complex 2-columns $z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. Its usual norm is $\|z\|_2 := \sqrt{(|z_1|^2 + |z_2|^2)}$ but here we shall use $\|z\| := \|U \cdot z\|_2$ in which $U := \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$. Now $\mathbb{R} = \mathbb{R}^2$, the space of real 2-columns $x$, with the inherited norm $\|x\| = \sqrt{(x^T \cdot U^T \cdot U \cdot x)} = \sqrt{(x^T \cdot Re(U^T \cdot U) \cdot x)} = \sqrt{(x^T \cdot V^2 \cdot x)}$ in which $V := \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. The inherited operator norm for complex 2-by-2 matrices $L$ turns out to be $\|L\| = \|U \cdot L \cdot U^{-1}\|_2$ which is the larger singular value of $U \cdot L \cdot U^{-1}$. For real 2-by-2 matrices $R$ restricted to act upon only real 2-columns, the inherited operator norm is $\|[R]\| = \|V \cdot R \cdot V^{-1}\|_2$.

Complex conjugation can change $\|\ldots\|$. For instance $z := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $\|z\| = \sqrt{5} \neq \|\bar{z}\| = 1$. And $L := \begin{bmatrix} 1 & \sqrt{2} \\ 1 & 1 \end{bmatrix}$ has $\|L\| = (1 + \sqrt{3})/\sqrt{2} \neq \|L\| = (3 + \sqrt{11})/\sqrt{2}$.

Next, $\|[R]\|$ will be compared with $\|R\|$ and $\sigma(R) := \max\{|\text{eigenvalues of } R|\}$, including complex as well as real eigenvalues of real matrix $R$. (Many writers call $\sigma(R)$ the “Spectral Radius” of $R$; some other writers call $\|R\|_2$ its “Spectral Radius”. Let’s not use the term.)

- $R := \begin{bmatrix} \beta & -2\alpha \\ \alpha & \beta \end{bmatrix}$ has $\sigma(R) = \|[R]\| = \sqrt{2} \alpha^2 + \beta^2$, and complex eigenvalues if $\alpha \neq 0$. No instance $R$ has $\sigma(R) < \|[R]\| = \|R\|$.

- $\sigma(V^2) = \|[V^2]\| = 2 < \|V^2\| = (1 + \sqrt{3})/\sqrt{2}$.

- $R := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ has $\sigma(R) = 1 < \|[R]\| = \sqrt{3} < \|R\| = \sqrt{6}$, and real eigenvalues.

- $R := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ has $\sigma(R) = \sqrt{2} < \|[R]\| = (1 + \sqrt{17})/\sqrt{8} < \|R\| = 2$, and imaginary eigenvalues.

- $R := \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ has $\sigma(R) = \sqrt{3} < \|[R]\| = \sqrt{6} < \|R\| = (\sqrt{15} + \sqrt{3})/2$, and complex eigenvalues.

For more about how $\|[R]\|$ and $\|R\|$ differ, with citations of similar work going back to 1994, see “Real and Complex Operator Norms” by Olga Holtz & Michael Karow (2004) posted at arXiv:math/0512608v1.