Real Operator Norms in Complex Spaces

Every n-dimensional complex vector space $\mathbb{C}$ contains many n-dimensional real vector spaces, each obtained as the set $\mathbb{R}$ of all real linear combinations of the vectors in some complex basis chosen for $\mathbb{C}$. This process induces a definition for the complex conjugate $\bar{\mathbf{z}}$ of any vector $\mathbf{z}$ in $\mathbb{C}$ making $\mathbf{z}$ the complex conjugate of $\bar{\mathbf{z}}$, so that $\mathbb{R}$ contains all the vectors (including those basis vectors **!**) that are their own complex conjugates. $\mathbb{R}$ contains both $Re(\mathbf{z}) := (\bar{\mathbf{z}}+\mathbf{z})/2$ and $Im(\mathbf{z}) := \imath\cdot(\bar{\mathbf{z}}-\mathbf{z})/2$, so $\mathbf{z} = Re(\mathbf{z}) + \imath\cdot Im(\mathbf{z})$. Though $\mathbb{C}$ contains $\mathbb{R}$ it is *not* a subspace of $\mathbb{C}$.

A linear operator L that maps $\mathbb{C}$ to itself has a complex conjugate $\bar{\mathrm{L}}$ defined by taking $\bar{\mathrm{L}}\cdot\mathbf{z}$ to be the complex conjugate of $\mathrm{L}\cdot\bar{\mathbf{z}}$. A linear operator is deemed *Real* if it maps $\mathbb{R}$ to itself. Both $Re(\mathrm{L}) := (\bar{\mathrm{L}}+\mathrm{L})/2$ and $Im(\mathrm{L}) := \imath\cdot(\bar{\mathrm{L}}-\mathrm{L})/2$ are instances of real linear operators; all these are their own complex conjugates. When restricted to $\mathbb{R}$, a real linear operator R can have only real eigenvalues if any, though R can have complex eigenvalues when it acts upon $\mathbb{C}$.

Nothing surprising has been said here yet. The surprise will come soon.

Let $\mathbb{C}$ be a normed space; its norm $\|\ldots\|$ is inherited by $\mathbb{R}$, and $\|\mathrm{Re}(\mathbf{z})\| \le (\|\bar{\mathbf{z}}\| + \|\mathbf{z}\|)/2$; in general $\|\bar{\mathbf{z}}\|$ may differ from $\|\mathbf{z}\|$. A linear operator L that maps $\mathbb{C}$ to itself inherits its operator (or *lub*- or *sup*-) norm from the vector norm thus:

$$\|\mathrm{L}\| := \max \|\mathrm{L}\cdot\mathbf{z}\|/\|\mathbf{z}\| \text{ over all nonzero } \mathbf{z} \text{ in } \mathbb{C}.$$

Evidently $\|Re(\mathrm{L})\| \le (\|\bar{\mathrm{L}}\| + \|\mathrm{L}\|)/2$; in general $\|\bar{\mathrm{L}}\|$ may differ from $\|\mathrm{L}\|$. Every eigenvalue λ of L satisfies $|\lambda| \le \|\mathrm{L}\|$ because its associated eigenvector $\mathbf{z} \ne \mathbf{o}$ has $\mathrm{L}\cdot\mathbf{z} = \lambda\cdot\mathbf{z}$. Also $|\lambda| \le \|\bar{\mathrm{L}}\|$ because $\bar{\lambda}$ is an eigenvalue of $\bar{\mathrm{L}}$ with eigenvector $\bar{\mathbf{z}}$.

The same goes for a real linear operator $\mathrm{R} = \bar{\mathrm{R}}$; its every eigenvalue ρ, real or complex, must satisfy $|\rho| \le \|\mathrm{R}\| = \|\bar{\mathrm{R}}\|$. However, another operator norm makes sense for real operators:

$$[\![\mathrm{R}]\!] := \max \|\mathrm{R}\cdot\mathbf{x}\|/\|\mathbf{x}\| \text{ over all nonzero } \mathbf{x} \text{ in } \mathbb{R}.$$

Evidently $[\![\mathrm{R}]\!] \le \|\mathrm{R}\|$, because $\mathbb{R} \subset \mathbb{C}$, and $|\rho| \le [\![\mathrm{R}]\!]$ for every *real* eigenvalue ρ of R.

Surprise: $|\rho| \le [\![\mathrm{R}]\!]$ for every eigenvalue ρ, real or complex, of real R.

Prof. Ridg Scott of the University of Chicago pointed out this surprise. Hereunder is a proof:

Suppose $\rho = \mu\cdot e^{\imath\cdot\theta}$ is a non-real eigenvalue of R with $\mu > 0$ and some real $\theta \ne 0$. Then the corresponding eigenvector $\mathbf{z}$ must be non-real; in fact $\mathbf{z}\cdot e^{\imath\cdot\phi}$ must be a non-real eigenvector for every real ϕ. Therefore some such ϕ achieves a *positive* minimum for $\|Im(\mathbf{z}\cdot e^{\imath\cdot\phi})\|$. Some simplification is gained without loss of generality by assuming that a minimizing $\phi = 0$. Now the imaginary part of the equation $\mathrm{R}\cdot\mathbf{z} = \mu\cdot\mathbf{z}\cdot e^{\imath\cdot\theta}$ yields $\mathrm{R}\cdot Im(\mathbf{z}) = \mu\cdot Im(\mathbf{z}\cdot e^{\imath\cdot\theta})$ whence follows $[\![\mathrm{R}]\!] \ge \|\mathrm{R}\cdot Im(\mathbf{z})\|/\|Im(\mathbf{z})\| = \mu\cdot\|Im(\mathbf{z}\cdot e^{\imath\cdot\theta})\|/\|Im(\mathbf{z})\| \ge \mu$ because $\|Im(\mathbf{z})\|$ was minimized. **[]**

The surprize is nontrivial; examples of norms $\|\ldots\|$ that have $[\![\mathrm{R}]\!] < \|\mathrm{R}\|$ do exist, but they are not the most familiar norms $\|\ldots\|_1$, $\|\ldots\|_2$ and $\|\ldots\|_\infty$. The reader is invited to find examples before turning over the page.

Examples:

Let $\mathbb{C} := \mathbb{C}^2$, the space of complex 2-columns $z := \begin{bmatrix}\zeta\\ \omega\end{bmatrix}$. Its usual norm is $\|z\|_2 := \sqrt{(|\zeta|^2 + |\omega|^2)}$ but here we shall use $\|z\| := \|U\cdot z\|_2$ in which $U := \begin{bmatrix}1 & \iota\\ 0 & 1\end{bmatrix}$. Now $\mathbb{R} = \mathbb{R}^2$, the space of real 2-columns x, with the inherited norm $\|x\| = \sqrt{(x^T\cdot\overline{U}^T\cdot U\cdot x)} = \sqrt{(x^T\cdot Re(\overline{U}^T\cdot U)\cdot x)} = \sqrt{(x^T\cdot V^2\cdot x)}$ in which $V := \begin{bmatrix}1 & 0\\ 0 & \sqrt{2}\end{bmatrix}$. The inherited operator norm for complex 2-by-2 matrices L turns out to be $\|L\| = \|U\cdot L\cdot U^{-1}\|_2$ which is the larger singular value of $U\cdot L\cdot U^{-1}$. For real 2-by-2 matrices R restricted to act upon only real 2-columns, the inherited operator norm is $[\![R]\!] = \|V\cdot R\cdot V^{-1}\|_2$.

Complex conjugation can change $\|\dots\|$. For instance $z := \begin{bmatrix}\iota\\ 1\end{bmatrix}$ has $\|z\| = \sqrt{5} \neq \|\overline{z}\| = 1$. And $L := \begin{bmatrix}1 & 0\\ 1+\iota & \iota\end{bmatrix}$ has $\|L\| = (1 + \sqrt{3})/\sqrt{2} \neq \|\overline{L}\| = (3 + \sqrt{11})/\sqrt{2}$.

Next, $[\![R]\!]$ will be compared with $\|R\|$ and $\sigma(R) := \max\{\,|\text{eigenvalues of } R\,|\,\}$, including complex as well as real eigenvalues of real matrix R. (Many writers call $\sigma(R)$ the "Spectral Radius" of R; some other writers call $\|R\|_2$ its "Spectral Radius". Let's not use the term.)

- $R := \begin{bmatrix}\beta & -2\alpha\\ \alpha & \beta\end{bmatrix}$ has $\sigma(R) = [\![R]\!] = \|R\| = \sqrt{(2\alpha^2 + \beta^2)}$, and complex eigenvalues if $\alpha \neq 0$.

  No instance R has $\sigma(R) < [\![R]\!] = \|R\|$.

- $\sigma(V^2) = [\![V^2]\!] = 2 < \|V^2\| = (1 + \sqrt{5})/\sqrt{2}$.

- $R := \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$ has $\sigma(R) = 1 < [\![R]\!] = \sqrt{3} < \|R\| = \sqrt{6}$, and real eigenvalues.

- $R := \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ has $\sigma(R) = \sqrt{2} < [\![R]\!] = (1 + \sqrt{17})/\sqrt{8} < \|R\| = 2$, and imaginary eigenvalues.

- $R := \begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}$ has $\sigma(R) = \sqrt{3} < [\![R]\!] = \sqrt{6} < \|R\| = (\sqrt{15} + \sqrt{3})/2$, and complex eigenvalues.

For more about how $[\![R]\!]$ and $\|R\|$ differ, with citations of similar work going back to 1994, see "Real and Complex Operator Norms" by Olga Holtz & Michael Karow (2004) posted at arXiv:math/0512608v1 .