At the request of the class, a section of Math. 110 will submit to a

50 min. Closed-Book Midterm Test

during the lecture-period, 1 - 2 pm., on Wed. 1 May 2002. This means that students must put away all books and papers and computing instruments before the test starts. It will be presented on one sheet of paper containing questions and blank spaces for answers. Each correct answer will earn one point; each incorrect answer will lose one point. Each answer left blank or scratched out will earn or lose nothing. Therefore, mere guesses make poor answers.

TOPICS for the second Math. 110 Midterm Test, Wed. 1 May 2002

area, volume, higher-dimensional content
associativity of addition and multiplication
basis, bases, change of basis
Choleski factorization
codomain, corange, cokernel of linear operator
column-echelon form, reduced column-echelon form
content, like area and volume, in Affine spaces
commutativity of addition but not ...
complementary projectors
cross-product of vectors in Euclidean 3-space
determinants' properties like $\det(B\cdot C) = \det(B)\cdot\det(C)$, $\det(B^T) = \det(B)$, …
dimension of a linear space
distributivity of multiplication over addition
domain of a linear operator
dual spaces of linear functionals
dyad (rank-one linear operator)
echelon forms, and canonical form under Equivalences $E^{-1}\cdot L\cdot F^{-1}$
eigenvalues and eigenvectors
elementary row- and column-operations, dilatators, shears, ...
existence and non-existence of solutions of linear equation-systems
fields of scalars
Fredholm's Alternatives for solutions of linear equation-systems
Gram-Schmidt orthogonalization
hyperplanes, equations of hyperplanes
positive inner products' connections with Euclidean spaces
inverses of linear operators and matrices: L^{-1}
intersection of two subspaces
least-squares problems and solutions
length of a vector, Euclidean length
linear spaces, affine spaces, Euclidean spaces
linear functionals
linear dependence and independence
linear operators from one space to another, or to itself, or to its dual-space
lines, equations of lines, parametric representation of a line

norm of a vector, Euclidean length
null-space or kernel of a linear operator
orientation of area, volume, higher-dimensional content
orthogonal matrices
orthonormal bases, and changing from one to another
orthogonal projections and reflections
parallel lines, parallel (hyper)planes, parallelepipeds
permutations, odd and even
planes, equations of planes, parametric representation of a plane
positive definite symmetric matrices
projector $P = P^2$
QR-factorization
range of a linear operator
rank: row-rank, column-rank, determinantal rank, tensor rank
reflection in a (hyper)plane, ... in a line, ... in a point
rotations in Euclidean 3-space
row-echelon form, reduced row-echelon form
singular (non-invertible) matrix
span of (subspace spanned by) a set of vectors
sum of two subspaces
target-space of a linear operator
transpose of a matrix
triangular matrix, triangular factorization
uniqueness and non-uniqueness of solutions of linear equation-systems
vectors, vector spaces
volume, higher-dimensional content

Relevant Readings: these notes are posted on the class web page
`http://www.cs,berkeley.edu/~wkahan/~MathH110`

2Dspaces.pdf
Cross.pdf (For this test you need not memorize triple-vector-product identities nor the formulas on pages 7 - 11.)
GEO.pdf and GEOS.pdf
RREF1.pdf
TriFact.pdf and Adjx.pdf, and scan chio.pdf lightly
pts.pdf (but for this test skip the last paragraph on p. 8 and what follows.)
the first 5 1/2 pages of Axler’s “Down with Determinants” DWD.pdf
LstSqrs.pdf, and prblms2.pdf, and scan lightly qf.pdf
the first 2 pages of NORMlite.pdf

This is a Closed-Book Midterm Test for Math. 110.

Student's SURNAME:___**MODEL SOLUTIONS**__, GIVEN NAME:__________________

Students must put away all books and papers and computing instruments before the test begins. Its one sheet of paper contains questions and blank spaces for answers. Each correct answer earns one point; each incorrect answer loses one point. Each answer left blank or scratched out earns or loses nothing. Therefore, mere guesses make poor answers. Only answer-blanks' contents will be graded, so the rest of the sheet can be used for scratch paper.

1. B is a real nonsymmetric square matrix, x and y are real nonzero column vectors, μ and ß are real nonzero scalars, $\imath = \sqrt{-1}$, and $B\cdot(x + \imath y) = (ß + \imath\mu)\cdot(x + \imath y)$. Does this imply that x and y must be linearly independent over the real field?

[_YES_] … because if $x = \lambda\cdot y$ then $B\cdot y = (ß + \imath\mu)\cdot y$ would be real, so $\mu = 0$. **!**

2. B is a real nonsymmetric square matrix; what is the maximum value taken by $v^T\cdot B\cdot v/v^T\cdot v$ as v runs over all nonzero real column vectors of the same dimension as B ?

[_ the largest eigenvalue of $(B + B^T)/2$ _] … since $2\cdot v^T\cdot B\cdot v/v^T\cdot v = v^T\cdot(B + B^T)\cdot v/v^T\cdot v$.

3. **K** is a given linear operator that maps one finite-dimensional vector space into another in which **k** is a given vector. State *Fredholm's Alternatives* for the solution **x** of $\mathbf{K}\cdot\mathbf{x} = \mathbf{k}$:

(i): At least one solution **x** must exist if and only if …

[_ $\mathbf{w}^T\mathbf{k} = 0$ for every linear functional $\mathbf{w}^T$ that satisfies $\mathbf{w}^T\mathbf{K} = \mathbf{o}^T$. _]

(ii): At most one solution **x** can exist if and only if …

[_ $\mathbf{z} = \mathbf{o}$ is the only vector **z** that satisfies $\mathbf{K}\cdot\mathbf{z} = \mathbf{o}$. _]

4. Given a collection of more than 5 distinct pairs $[\tau_k, \eta_k]$ of real numbers, we seek a cubic polynomial $q(\tau) = \xi_0 + \xi_1\tau + \xi_2\tau^2 + \xi_3\tau^3$ that minimizes $\sum_k (q(\tau_k) - \eta_k)^2$. Any column vector $x := [\xi_0, \xi_1, \xi_2, \xi_3]^T$ that achieves the minimum must satisfy the *Normal Equations* $Ax = b$; obtain explicit closed-form expressions for the elements α_{ij} of A and β_i of b in terms of the pairs $[\tau_k, \eta_k]$ using $\sum$-notation:

[$\alpha_{ij} = \sum_k \tau_k^{i+j}$, and $\beta_i = \sum_k \tau_k^{i}.\eta_k$ for $0 \le i \le 3$ and $0 \le j \le 3$] .

5. On the back of this page, exhibit a proof that any norm $||\ldots||$ that satisfies the conditions

$||\mathbf{x}|| > 0$ except $||\mathbf{o}|| = 0$, and $||\mu\mathbf{x}|| = |\mu|\cdot||\mathbf{x}||$, and $||\mathbf{x}+\mathbf{y}|| \le ||\mathbf{x}|| + ||\mathbf{y}||$

must satisfy $|\,||\mathbf{w}-\mathbf{x}|| - ||\mathbf{y}-\mathbf{z}||\,| \le ||\mathbf{w}-\mathbf{y}|| + ||\mathbf{x}-\mathbf{z}||$ for any **w**, **x**, **y**, **z** in a normed space.

[Proof: Observe that $||\mathbf{w}-\mathbf{x}|| = ||\mathbf{w}-\mathbf{y} + \mathbf{y}-\mathbf{z} + \mathbf{z}-\mathbf{x}|| \le ||\mathbf{w}-\mathbf{y}|| + ||\mathbf{y}-\mathbf{z}|| + ||\mathbf{x}-\mathbf{z}||$, and similarly $||\mathbf{y}-\mathbf{z}|| = ||\mathbf{y}-\mathbf{w} + \mathbf{w}-\mathbf{x} + \mathbf{x}-\mathbf{z}|| \le ||\mathbf{w}-\mathbf{y}|| + ||\mathbf{w}-\mathbf{x}|| + ||\mathbf{x}-\mathbf{z}||$. From these inequalities follow respectively $||\mathbf{w}-\mathbf{x}|| - ||\mathbf{y}-\mathbf{z}|| \le ||\mathbf{w}-\mathbf{y}|| + ||\mathbf{x}-\mathbf{z}||$ and $||\mathbf{y}-\mathbf{z}|| - ||\mathbf{w}-\mathbf{x}|| \le ||\mathbf{w}-\mathbf{y}|| + ||\mathbf{x}-\mathbf{z}||$, which together yield the desired conclusion $|\,||\mathbf{w}-\mathbf{x}|| - ||\mathbf{y}-\mathbf{z}||\,| \le ||\mathbf{w}-\mathbf{y}|| + ||\mathbf{x}-\mathbf{z}||$.]