At the request of the class, a section of Math. 110 will submit to a

50 min. Closed-Book Midterm Test

during one of the three lecture-periods Mon.-Wed.-Fri., 1 - 2 pm., in the week of 4 - 8 March 2002. This means that students must put away all books and papers and computing instruments before the test begins. It will be presented on one sheet of paper containing questions and blank spaces for answers. Each correct answer will earn one point; each incorrect answer will lose one point. Each answer left blank or scratched out will earn or lose nothing. Therefore, mere guesses make poor answers.

TOPICS for the Math. 110 Midterm Test, 4 - 8 Mar. 2002

area, volume, higher-dimensional content
associativity of addition and multiplication
basis, bases, change of basis
codomain, corange, cokernel of linear operator
column-echelon form, reduced column-echelon form
content, like area and volume, in Affine spaces
commutativity of addition but not ...
complementary projectors
cross-product of vectors in Euclidean 3-space
determinants' properties like $\det(B\cdot C) = \det(B)\cdot\det(C)$, $\det(B^T) = \det(B)$, …
dimension of a linear space
distributivity of multiplication over addition
domain of a linear operator
dual spaces of linear functionals
dyad (rank-one linear operator)
elementary row- and column-operations, dilatators, shears, ...
existence and non-existence of solutions of linear equation-systems
fields of scalars
hyperplanes, equations of hyperplanes
inverses of linear operators and matrices: L^{-1}
length of a vector, Euclidean length
linear spaces, affine spaces, Euclidean spaces
linear functionals
linear dependence and independence
linear operators
lines, equations of lines, parametric representation of a line
norm of a vector, Euclidean length
null-space or kernel of a linear operator
orientation of area, volume, higher-dimensional content
parallel lines, parallel (hyper)planes, parallelepipeds
permutations, odd and even
planes, equations of planes, parametric representation of a plane
projector $P = P^2$

range of a linear operator
rank, row-rank, column-rank, determinantal rank, …
reflection in a (hyper)plane, ... in a line, ... in a point
rotations in Euclidean 3-space
row-echelon form, reduced row-echelon form
singular (non-invertible) matrix
span of (subspace spanned by) a set of vectors
target-space of a linear operator
transpose of a matrix
triangular matrix, triangular factorization
uniqueness and non-uniqueness of solutions of linear equation-systems
vectors, vector spaces
volume, higher-dimensional content

Relevant Readings: these notes are posted on the class web page
`http://www.cs,berkeley.edu/~wkahan/~MathH110`

2Dspaces.pdf
Cross.pdf (For this test you need not memorize triple-vector-product identities nor the formulas on pages 7 - 11.)
GEO.pdf
GEOS.pdf (but not pages 4 - 6 for this test.)
RREF1.pdf
TriFact.pdf
pts.pdf (but for this test skip the last paragraph on p. 8 and what follows.)

This is a Closed-Book Midterm Test for Math. 110.

Student's SURNAME:___ **ANSWERS** ___, GIVEN NAME:______________________

Students must put away all books and papers and computing instruments before the test begins. Its one sheet of paper contains questions and blank spaces for answers. Each correct answer earns one point; each incorrect answer loses one point. Each answer left blank or scratched out earns or loses nothing. Therefore, mere guesses make poor answers. Only answer-blanks' contents will be graded, so the rest of the sheet can be used for scratch paper.

1. Can the columns of a 3-by-4 matrix $\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ \kappa & \lambda & \mu & \nu \end{bmatrix}$ be linearly independent?

[__] Sometimes. [__] Never. (CHOOSE ONE BY WRITING "X" IN A BOX [__].)

Answer: Never.

2. A *Tetrahedron* is a figure with four vertices, six edges and four triangular faces; each face is opposite one vertex and bounded by three edges through the other three vertices. The tetrahedron is called "nondegenerate" if no vertex lies in a plane containing the opposite face. Given any two nondegenerate tetrahedra, each with a vertex at the origin, in a 3-dimensional space, either can be mapped onto the other by an invertible *Linear Operator* ...

[__] always, in infinitely many ways.
[__] always, in finitely many ways -- more than one.
[__] always, in just one way.
[__] sometimes, not always.
[__] never.
(CHOOSE ONE BY MARKING "X" IN A BOX [__] .)

Answer: *always*, in *six* ways corresponding to the six ways to send one tetrahedron's three edges emanating from the origin to the other's. There would be infinitely many ways if the tetrahedrons were situated in a space of more than three dimensions; can you see why? Think of the edges emanating from the origin as basis vectors.

3. Knowing only that matrices B, C and D satisfy $B \cdot C \cdot D = I$ (an identity matrix), may we infer that C has an inverse and, if so, can it be expressed in terms solely of B and D ?

[__] Always, and C^{-1} = ____________ .

[__] Sometimes; C^{-1} = _____________ if B and D are ________________ .

[__] No; sometimes C has no inverse.

(CHOOSE THE TRUE STATEMENT(S) BY FILLING THE BOX(ES) WITH "X", AND FILL IN ANY BLANK _____ AFTER YOUR CHOICE(S).)

Answer: If B and D are square (in which case they must be invertible too -- do you see why?) then $C^{-1} = D\cdot B$. But otherwise C may have an inverse or it may not; for example, $\begin{bmatrix} 1 & o^T \end{bmatrix} \cdot \begin{bmatrix} 1 & o^T \\ o & H \end{bmatrix} \cdot \begin{bmatrix} 1 \\ o \end{bmatrix} = 1$ but the middle matrix C has no inverse unless H has one.

4. Linear operator $\mathbf{L}$ is representable by a 4-by-3 matrix. The set of all solutions $\mathbf{z}$ of the equation $\mathbf{L}\cdot\mathbf{z} = \mathbf{o}$ sweeps out a two-dimensional subspace, a plane through the origin $\mathbf{o}$. Two different vectors $\mathbf{b} = \mathbf{L}\cdot\mathbf{u}$ and $\mathbf{c} = \mathbf{L}\cdot\mathbf{v}$ are known to be nonzero. What is the dimension of the *range* of $\mathbf{L}$? [_____] (FILL IN A NUMBER.)

Answer: [_1_] . Vectors **b** and **c** must be (anti)parallel because **L** has a two-dimensional null-space in a three-dimensional domain, and therefore the range of **L** must must be a subspace of dimension 3 – 2 = 1 within **L**'s four-dimensional target-space.

5. Complex numbers $\xi + \eta\sqrt{-1}$ form a 2-dimensional vector space over the real number field; these vectors can be multiplied to produce other vectors in the same space, and this multiplication is both commutative and associative. The cross-product $\mathbf{u}\times\mathbf{v}$ of vectors in a 3-dimensional Euclidean space is anti-commutative ($\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v}$) and non-associative; only in special cases does $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{u}\times(\mathbf{v}\times\mathbf{w})$. Real 2-by-2 matrices constitute a 4-dimensional real vector space whose vectors can be multiplied to produce other vectors in the space; its multiplication is associative but not generally commutative. Is there a 3-dimensional vector space over the real field whose vectors can be multiplied to produce other vectors in the same space, and whose multiplication is also associative but *not* generally commutative?

[__________] (WRITE "YES" OR "NO".)

Answer: YES; try 2-by-2 upper-triangular matrices.

6. Quadratic polynomials constitute a three-dimensional vector space in which every quadratic u(t) (in a conventional notation) can be represented by row-vectors with three elements. One representation $\grave{\mathbf{u}}^T := [u(0), u'(0), u''(0)]$ involves the values of u and its derivatives at t = 0 ; another $\hat{\mathbf{u}}^T := [u(-2), u(0), u(2)]$ involves values of u(t) at three arguments t . Exhibit the change-of-basis matrix $\mathbf{K}$ that figures in the equation $\hat{\mathbf{u}}^T = \grave{\mathbf{u}}^T\cdot\mathbf{K}$.

Answer: $\mathbf{K} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \\ 2 & 0 & 2 \end{bmatrix}$. (YES, IT'S WORTH NINE POINTS.)

7. Let $I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $J := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$, $K := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, and $L := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, so that

$J^2 = K^2 = L^2 = -I$, $J \cdot K = L = -K \cdot J = -L^T$, $K \cdot L = J = -L \cdot K = -J^T$, and $L \cdot J = K = -J \cdot L = -K^T$.
A *Real Quaternion* is any matrix $Q = \alpha I + \beta J + \gamma K + \delta L$ with real scalars α, β, γ and δ .

After evaluating $Q^T \cdot Q$, determine $\det(Q) =$ ______________________________ .

Answer: $\det(Q) = (\alpha^2+\beta^2+\gamma^2+\delta^2)^2$ since $Q^T \cdot Q = (\alpha^2+\beta^2+\gamma^2+\delta^2)I$ and $\det(\xi I) = \xi^4 \geq 0$, so $\det(Q) = \sqrt{(\det(Q)^2)} = \sqrt{(\det(Q^T \cdot Q))} = (\alpha^2+\beta^2+\gamma^2+\delta^2)^2$. Incidentally, most algebraists prefer to write **1**, **i**, **j**, **k** respectively for what are here written I, J, K, L , and to treat quaternion arithmetic as a non-commutative but still associative generalization of complex arithmetic.