At the request of the class, a section of Math. 110 will submit to a

3 hr. Closed-Book Final Exam

3 - 6 pm. Thurs. 23 May 2002 in 891 Evans Hall.

All books and papers and computing instruments must be put away before the exam starts. It will be presented on sheets of paper containing questions and blank spaces for answers. Only these will be graded; blank scratch paper will be supplied but not graded. Each correct answer will earn one point; each incorrect answer will lose one point. Each answer left blank or scratched out will earn or lose nothing; therefore mere guesses make poor answers.

TOPICS for the Math. 110 Final Exam, 3 - 6 pm. Thurs. 23 May 2002

area, volume, higher-dimensional content
associativity of addition and multiplication
basis, bases, change of basis
canonical form under Congruences $E^{T-1} \cdot A \cdot E^{-1}$
canonical form under Equivalences $E^{-1} \cdot L \cdot F^{-1}$
canonical form under Similarities $E^{-1} \cdot B \cdot E$
Cayley-Hamilton Theorem
characteristic polynomial and minimum polynomial of a square matrix
Choleski factorization
codomain, corange, cokernel of linear operator
column-echelon form, reduced column-echelon form
congruence of real symmetric matrices
content, like area and volume, in Affine spaces
commutativity of addition but not ...
complementary projectors
complex eigenvalues and eigenvectors of real square matrices
cross-product of vectors in Euclidean 3-space
determinants' properties like $\det(B \cdot C) = \det(B) \cdot \det(C)$, $\det(B^T) = \det(B)$, …
dimension of a linear space
distributivity of multiplication over addition
domain of a linear operator
dual spaces of linear functionals
dyad (rank-one linear operator)
echelon forms
eigenvalues and eigenvectors
elementary row- and column-operations, dilatators, shears, ...
existence and non-existence of solutions of linear equation-systems
fields of scalars
Fredholm's Alternatives for solutions of linear equation-systems
Gram-Schmidt orthogonalization
hyperplanes, equations of hyperplanes
positive inner products' connections with Euclidean spaces
identity I , idempotent $P = P^2$, involutory $R = R^{-1}$, and nilpotent $N^m = O =$ matrices
invariant subspaces

inverses of linear operators and matrices: L^{-1}
intersection and sum of two subspaces
Jordan’s Normal Form
least-squares problems and solutions
length of a vector, Euclidean length
linear spaces, affine spaces, Euclidean spaces
linear functionals
linear dependence and independence
linear operators from one space to another, or to itself, or to its dual-space
lines, equations of lines, parametric representation of a line
maximal irreducible invariant subspaces
norm of a vector, Euclidean length
norm of a matrix, biggest singular value
null-space or kernel of a linear operator
orientation of area, volume, higher-dimensional content
orthogonal matrices
orthonormal bases, and changing from one to another
orthogonal projections and reflections
parallel lines, parallel (hyper)planes, parallelepipeds
permutations, odd and even
planes, equations of planes, parametric representation of a plane
positive definite symmetric matrices
projector $P = P^2$
quadratic forms
QR-factorization
range of a linear operator
rank: row-rank, column-rank, determinantal rank, tensor rank
real eigenvalues and orthogonal eigenvectors of …
real symmetric matrices
reduced row- and/or column-echelon forms
reflection in a (hyper)plane, ... in a line, ... in a point
rotations in Euclidean 3-space
row-echelon form, reduced row-echelon form
signature of a real symmetric matrix
singular (non-invertible) matrix
singular value decomposition
span of (subspace spanned by) a set of vectors
Sylvester’s Inertia Theorem
target-space of a linear operator
transpose of a matrix
triangular matrix, triangular factorization
uniqueness and non-uniqueness of solutions of linear equation-systems
unitary similarity to an upper-triangular matrix --- Schur’s
vectors, vector spaces
volume, higher-dimensional content

Relevant Readings: these notes are posted on the class web page
`http://www.cs,berkeley.edu/~wkahan/MathH110`

Skim Axioms.pdf and prblms1.pdf
2dspaces.pdf
Cross.pdf (For this exam you need not memorize triple-vector-product identities nor the formulas on pages 7 - 11.)
geo.pdf and geos.pdf
RREF1.pdf
TriFact.pdf and Adjx.pdf, and skim chio.pdf lightly
pts.pdf (but for this exam skip the last paragraph on p. 8 and what follows.)
Skim all but §9 of Axler's "Down with Determinants", DownDet.pdf
lstsqrs.pdf, and prblms2.pdf, and qf.pdf but not the Theorem's proof for this exam.
Read the first 5 pages of normlite.pdf
Read jordan.pdf, but skip Sylvester's Interpolation, Cauchy's integral, Danilewski's and Frame-Souriau-Faddeev's methods, and the construction of Jordan's Normal Form for Nilpotent matrices.
Skim GIlite.pdf; Theorem 7 is important but its proof will not be on the exam.
Skim HilbMats.pdf; only the last page is important.
Read jacobi.pdf; skip the Wronskian and the Adjugate's Derivative, and skim the rest.
Read the second half of diagprom.pdf to begin a justification of Choleski factorization.
Can you follow s10Oct.pdf, prblms2.pdf, tkhms.pdf, s21nov.pdf and testexam.pdf ?
Read Model Solutions in MidTerm.pdf, Midterm2.pdf, finexms.pdf, finals.pdf ; your final exam will not be so difficult as the last two.

This is a Closed-Book Final Exam for Math. 110.

Student's SURNAME:__ **MODEL SOLUTIONS** __, GIVEN NAME:__________________

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0. $\boldsymbol{S}$ is a subset of a vector space $\boldsymbol{V}$. Vectors $\mathbf{b}$, $\mathbf{c}$ and $3\mathbf{c} + 5\mathbf{d}$ belong to $\boldsymbol{S}$, but $\mathbf{d}$ and $\mathbf{e} - 2\mathbf{d}$ do not. Can $\boldsymbol{S}$ possibly be a subspace of $\boldsymbol{V}$? Answer YES or NO ..

Were $\boldsymbol{S}$ a subspace it would contain linear combination $\mathbf{d} = (1/5)\cdot(3\mathbf{c} + 5\mathbf{d}) - (3/5)\cdot\mathbf{c}$, so … ____ NO ___

1. Matrix $C := [b\cdot u^T \ \ -b\cdot u^T \ \ d\cdot v^T \ \ -d\cdot v^T]$ has as many ROWS as each column vector b and d has, and twice as many columns as row vector $[u^T \ \ v^T]$ has. Assuming both $u^T \neq o^T$ and $v^T \neq o^T$, factor C into an explicit product of three matrices none of them an identity matrix.

Answer: $$C = [b \ \ d]\cdot\begin{bmatrix} u^T & o^T \\ o^T & v^T \end{bmatrix}\cdot\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$, among others.

2. Matrix $F := [B \ \ E]$ is assembled from two matrices each with n rows: B has rank ß and E has rank ε. In the absence of any further information about B and E, determine the minimum and maximum values of rank(F) compatible with the given data n, ß and ε.

Answers:_________ $\max\{ ß, \varepsilon \} \le \mathrm{rank}(F) \le \min\{ n, ß+\varepsilon \}$ _________ … for 2 pts.

3. Solve $$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\cdot X - X\cdot\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 4 \\ 6 & -2 & -1 \end{bmatrix}$$ for $X =$ … $$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

Answer: Compute elements of X from lower right to upper left: ______________________________

4. Examine each quoted statement below about Jordan Normal Forms (JNFs) and fill in its accompanying blank with whichever of "TRUE" or "FALSE" is correct for that statement:

(i) "Every *Idempotent* matrix (Projector) $P = P^2$ other than O and I has a strictly diagonal JNF." ___ TRUE ___

… because P's eigenvalues are 0 and 1, and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \neq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and so on.

(ii) "Every *Involutory* matrix (Reflector) $R = R^{-1} \neq I$ has a strictly diagonal JNF." ___ TRUE ___

… because R's eigenvalues are -1 and 1, and $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \neq \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \neq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and so on.

(iii) "Every nonzero *Nilpotent* matrix N, with $N^m = O$ for some $m > 1$, has a non-diagonal JNF." ___ TRUE ___

… because N's eigenvalues are all 0 but $\mathrm{rank}(N) > 0$.

(iv) "Every *Companion* matrix $C = \begin{bmatrix} 0 & 0 & 0 & 0 & \varepsilon \\ 1 & 0 & 0 & 0 & \delta \\ 0 & 1 & 0 & 0 & \gamma \\ 0 & 0 & 1 & 0 & \beta \\ 0 & 0 & 0 & 1 & \alpha \end{bmatrix}$ with a nonzero last column has a strictly diagonal JNF." ___ FALSE ___

… since nullity$(C - \lambda I) = 4 - \text{rank}(C - \lambda I) = 1$ for every eigenvalue λ even if it is a multiple eigenvalue, so C has just one eigenvector per distinct eigenvalue regardless of its multiplicity.

5. Real square matrix B is invertible, and $\begin{bmatrix} O & B \\ B^T & O \end{bmatrix} = \begin{bmatrix} P & P \\ Q & -Q \end{bmatrix} \cdot \begin{bmatrix} V & O \\ O & -V \end{bmatrix} \cdot \begin{bmatrix} P & P \\ Q & -Q \end{bmatrix}^T$ in which $\begin{bmatrix} P & P \\ Q & -Q \end{bmatrix}$ is orthogonal and V is a positive diagonal matrix. Exhibit explicitly the three factors of the *Singular Value Decomposition* B = (1st Orthogonal)·(Positive Diagonal)·(2nd Orthogonal) .

1st Orthogonal = _ $\sqrt{2}P$ _ , Positive Diagonal = _ V _ , 2nd Orthogonal = _ $\sqrt{2}Q^T$ _ .

6. Find the eigenvalues of $F^{-1} \cdot F^T$ when $F = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Eigenvalues = _____ $3 \pm \sqrt{8}$ _____

… because $\det(\lambda I - F^{-1} \cdot F^T) = -\det(\lambda F - F^T) = -\lambda^2 + 6\lambda - 1 = -(\lambda - 3 - \sqrt{8})(\lambda - 3 + \sqrt{8})$.

Then find the following argument's flaw: " Suppose $F^{-1} \cdot F^T v = \mu v$ for some possibly complex eigenvalue μ and eigenvector $v \neq o$. Now $F^T v = \mu F v$, whence $v^* F^T v = \mu v^* F v$ where the asterisk $*$ means complex conjugate transpose. Then $|\mu| = |v^* F^T v / v^* F v| = |(v^* F v)^* / v^* F v| = 1$."

The flaw: ______ Whenever $v^* F v = 0$ the argument collapses, allowing $|\mu| \neq 1$. _______

7. A set S of points is called "convex" just when every two points in S can be joined by a straight line segment contained entirely in S . Exhibit here a short proof that the positive definite N-by-N matrices constitute a convex set.

Proof: $A(\mu) := \mu \cdot V + (1-\mu) \cdot H$ runs along a straight line segment from H to V as μ runs from 0 to 1 . Then, if H and V are any two positive definite matrices, $A(\mu)$ is another because $x^T A(\mu) x = \mu \cdot x^T V x + (1-\mu) \cdot x^T H x > 0$ for every $x \neq o$. Therefore H and V are joined by a line segment of positive definite matrices.

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8. Given two N-by-N real symmetric matrices V and H , let $A(\mu) := V + \mu H$ for all real values μ , and suppose $ß(\mu)$ is a simple (not multiple) eigenvalue of $A(\mu)$. If the eigenvalues of H are known, what do they tell us about $dß(\mu)/d\mu$?

____ $dß/d\mu$ lies between the least and largest eigenvalues of H . _______

… because, if $v(\mu)$ is an eigenvector of $A(\mu)$ belonging to eigenvalue $ß(\mu)$, then $dß/d\mu = v^T H v / v^T v$, as follows from $0 = v^T \cdot d(Av - ßv)/d\mu = v^T(Hv + A\,dv/d\mu - dß/d\mu \cdot v - ß\,dv/d\mu)$, or else from the derivative of $0 = \det(ßI - A)$.

9. What is Jordan's Normal Form for $\begin{bmatrix} 1&1&0&0&0\\ 0&2&1&0&0\\ 0&0&3&1&0\\ 0&0&0&2&1\\ 0&0&0&0&1 \end{bmatrix}$? Answer: $\begin{bmatrix} 1&1&0&0&0\\ 0&1&0&0&0\\ 0&0&2&1&0\\ 0&0&0&2&0\\ 0&0&0&0&3 \end{bmatrix}$

… since no eigenvalue of the given matrix has two independent eigenvectors.

10. An attempted Cholesky factorization of the matrix $\begin{bmatrix} 1&-1&-1&-1\\ -1&2&0&0\\ -1&0&3&1\\ -1&0&1&2 \end{bmatrix}$ fails at the last step. How many eigenvalues of this matrix are positive? ___ 3 ___

… since the last step of Cholesky's method is a square root which can fail only of it is the square root of a negative number. Then the given matrix must be congruent to diag(+1, +1, +1, –1) , whence Sylvester's Inertia Theorem implies that three eigenvalues are positive.

This problem can be solved without actually computing the factorization of the given matrix into

$$\begin{bmatrix} 1&0&0&0\\ -1&1&0&0\\ -1&-1&1&0\\ -1&-1&-1&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0\\ -1&1&0&0\\ -1&-1&1&0\\ -1&-1&-1&1 \end{bmatrix}^T ,$$

and certainly without computing its eigenvalues 4.29… , 2.36… , 1.42… , –0.07… .

11. L_N is an N-by-N unit-lower-triangular matrix whose every subdiagonal element is -1 ; for instance, $L_4 = \begin{bmatrix} 1&0&0&0\\ -1&1&0&0\\ -1&-1&1&0\\ -1&-1&-1&1 \end{bmatrix}$. What is the biggest element of L_N^{-1} ? Answer: _ $\max\{1,\ 2^{N-2}\}$ _

Write $L_N = 2I - (I-J)^{-1}$ where J is the N-by-N matrix whose only nonzero elements are 1's below and adjacent to the diagonal. Note that $J^N = O$. Then $L_N^{-1} = (I-J)(I-2J)^{-1} = I + J + 2J^2 + 4J^3 + \dots + 2^{N-2}J^{N-1}$ has its biggest element 2^{N-2} in its lower left corner unless N = 1, in which case the biggest element is 1 .