Problem 17 on p. 249 of our textbook, *Discrete Mathematics etc*., 4th ed., by Kenneth Rosen, is an application of the *Pigeonhole Principle* to an important problem:

How well can any given irrational number x be approximated by rational numbers m/j whose denominators j are not very big?

These rational numbers are distributed rather unevenly. For example, for a given positive integer n the rational numbers m/j with arbitrary integers m in their numerators and positive integer denominators $j \le n$ are some as close together as $1/(n-1) - 1/n = 1/((n-1)n)$, and others are as far from their neighbors as $1/n - 0/n = 1/n$. When n is big the gaps between adjacent rational numbers m/j with denominators bounded by n can be very different. All irrational numbers x that fall into a relatively narrow gap are far better approximated by some rational m/j than are the irrational numbers x near the middle of a relatively wide gap. For a given denominator j the best approximation m/j to x has as its numerator the integer m nearest jx . Given instead a bound n upon the denominator j we wish to choose it so that jx will differ as little as possible from its nearest integer m . How small can we make that difference?

This note offers a proof simpler than the text's of a stronger statement than problem 17's :

> Given irrational x and positive integer n , there exists at least one positive integer $j \le n$ for which jx and the integer m nearest jx differ in magnitude by less than $1/(n+1)$.

Here is its proof: Let $f(jx) := jx - \lfloor jx \rfloor$; this is the fractional part of jx , and it can be rational for no integer $j > 0$ lest $x = (f(jx) + \lfloor jx \rfloor)/j$ turn out to be rational. In particular $f(jx) \ne 0$; therefore $0 < f(jx) < 1$. Moreover $f(jx) \ne f(kx)$ if integers j and k are different because otherwise $x = (\lfloor kx \rfloor - \lfloor jx \rfloor)/(k-j)$ would turn out to be rational. Therefore the set

$$\{ 0,\ f(x),\ f(2x),\ f(3x),\ \dots,\ f((n-1)x),\ f(nx),\ 1 \}$$

consists of n+2 distinct numbers y all lying between 0 and 1 inclusive. We shall break this interval $0 \le y \le 1$ into n+1 disjoint subintervals to serve as pigeonholes for that set. These pigeonholes are $(i-1)/(n+1) \le y < i/(n+1)$ for $i = 1, 2, 3, \dots, n$, and $n/(n+1) \le y \le 1$. In a picture, …

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y :     ——[——)[——)[——)[  …  )[——)[——]——
i :        0    1    2    3   … n–1 n   n+1
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The last pigeonhole includes both its endpoints; other pigeonholes include only their left-hand endpoints. Each pigeonhole's width is $1/(n+1)$. Since the set has more numbers than there are pigeonholes to hold them, at least one pigeonhole contains at least two numbers from the set. Therefore four integers j_1, j_2, m_1, m_2 must exist satisfying both $0 \le j_1 < j_2 \le n$ and $|(j_2x - m_2) - (j_1x - m_1)| < 1/(n+1)$. Let $0 < j := j_2 - j_1 \le n$ so that $|jx - (m_2 - m_1)| < 1/(n+1)$. End of proof.

The bound $1/(n+1)$ cannot be reduced because when x is an irrational number extremely close to $1/(n+1)$ either 1x or nx differs from its nearest integer by barely less than $1/(n+1)$.

Thus for every positive integer n there is at least one positive integer $j \le n$ for which the integer m closest to jx satisfies $|x - m/j| < 1/((n+1)j)$, which can be rather smaller than $1/(2n)$ but maybe not if x lies near the middle of a relatively wide gap, as mentioned above. How strictly do we wish to bound the denominators j ? If we let j get slightly bigger than n can x be approximated perhaps much more closely? This question motivates what follows.

We have just deduced that among positive integers $j \le n$ at least one satisfies $|jx - m| < 1/(n+1)$ for $m = « jx » :=$ (the integer nearest jx) . Consequently $|jx - « jx »| < 1/(j+1)$ for such an integer j . How many positive integers j can satisfy this last inequality?

For every irrational number x there are infinitely many integers j that satisfy $| jx - « jx » | < 1/(j+1)$.

Proof by contradiction: Suppose there were only finitely many such integers j . Then integer N could be chosen so big that $1/(N+1) < | jx - « jx » |$ for all those integers j . But we deduced above that there exists at least one positive integer $J \le N$ satisfying $|Jx - «Jx»| < 1/(N+1)$. This is paradoxical. On the one hand $|Jx - «Jx»| < 1/(N+1) < | jx - « jx » |$ for all those finitely many integers j , so J could not be one of them; and yet $|Jx - «Jx»| < 1/(N+1) \le 1/(J+1)$, which is what characterized them. Therefore there cannot be only finitely many of them. End of proof.

Another way to state what has just been proved is …

For every irrational number x there are infinitely many pairs (m, j) of integers that satisfy $| x - m/j | < 1/((j+1)j)$.

In each such pair $m = «jx»$. These rational approximations m/j so unusually close to x can all be constructed from truncations of its non-terminating continued fraction. For example,

$\pi = 3 + 1/(7 + 1/(15 + 1/(1 + 1/(292 + 1/(1 + 1/(1 + 1/(1 + 1/(2 + 1/(1 + 1/(3 + 1/(1 + 1/(14 + \dots$
$= 3.14159\ 26535\ 89793\ 23846 \dots$.

Now $3 + 1/7 = 22/7 = 3.1428\dots$ differs from π by $0.00126\dots < 1/56$. A better approximation $3 + 1/(7 + 1/15) = 333/106 = 3.141509\dots$ differs from π by $0.000083\dots < 1/(107\cdot106)$ barely. And $3 + 1/(7 + 1/(15 + 1/1)) = 355/113 = 3.1415929\dots$ differs from π by $0.000000266\dots$ which is far tinier than $1/(114\cdot113) = 0.0000776\dots$. But π differs from the next truncated fraction $3 + 1/(7 + 1/(15 + 1/(1 + 1/292))) = 103993/33102 = 3.14159265301\dots$ by $5.8\dots/10^{10}$, which is not much smaller than $1/(33103\cdot33102) = 9.1\dots/10^{10}$. Can you see why some of these fractions m/j are much better than the bound $1/((j+1)j)$ and others not much better?

In 1891 A. Hurwitz proved that for each irrational x infinitely many pairs (m, j) of integers satisfy $| x - m/j | < 1/(j^2\sqrt{5})$; but there are some irrational numbers y for which at most finitely many pairs satisfy $| y - m/j | < 1/(j^{2+ß}\sqrt{(5+\mu)})$ no matter how small the positive increments ß and μ may be; an example of such a y is

$$y = (1 + \sqrt{5})/2 = 1 + 1/(1 + 1/(1 + 1/(1 + 1/(1 + 1/(1 + \dots .$$

In 1955 H.F. Roth proved that if an irrational z satisfies $| z - m/j | < 1/j^p$ for any fixed exponent $p > 2$ and for infinitely many integers m and $j > 0$ then z must be a *Transcendental* number; this means that z cannot be the root of a polynomial equation with all coefficients integers.

To learn more about this subject see books like I. Niven's *Diophantine Approximations* (1963, Wiley) or J.W.S. Cassel's *Intro. to Diophantine Approximation* (1957, Cambridge U.P.), or books on *Continued Fractions* like A.Ya. Khinchin's (1964, Univ. of Chicago).