Problem 14 (*l*) on page 34 :

Let variables w, x, y, z run over the students in a class.

Let propositional function C(x, y) stand for "Students x and y have chatted." It seems obvious that $C(x, y) \Leftrightarrow C(y, x)$. Not so obvious is whether C(x, x) is ever True.

The proposition to be formalized is

"There are two students in the class who have not chatted."

This proposition is ambiguous in English; both meanings will be formalized in turn. To keep expressions from overflowing the line, the abbreviation

$$G(x, y) := ((x \neq y) \wedge \neg C(x, y))$$

will be used; it is the proposition "Students x and y are distinct and have not chatted." The term "$(x \neq y)$" ensures that G(x, x) be False regardless of C(x, x) .

First meaning: "There are *at least* two students in the class who have not chatted" turns into

$$\exists x\ \exists y\ G(x, y) \qquad \text{(the answer most likely intended).}$$

Second meaning: "There are *just* two students in the class who have not chatted" turns into

$$\exists x\ \exists y\ \forall w\ \forall z\ \Big(\ G(x, y) \wedge [\ G(w, z) \rightarrow (\ ((w = x)\wedge(z = y))\vee((w = y)\wedge(z = x))\ ]\ \Big)\ .$$

It could just as well be written

$$\exists x\ \exists y\ \Big(\ G(x, y) \wedge \forall w\ \forall z\ [\ G(w, z) \rightarrow (\ ((w = x)\wedge(z = y))\vee((w = y)\wedge(z = x))\ ]\ \Big)\ .$$

(It cannot be written $\exists! x\ \exists! y\ G(x, y)$ which is always False because $G(x, y) \Leftrightarrow G(y, x)$.)

In class on Thurs. 28 Jan. a student raised the interesting question "Can this proposition be expressed with three quantifiers instead of four?" He offered an expression like

$$\exists x\ \exists y\ \Big(\ G(x, y) \wedge \forall z\ [\ \neg C(z, y) \leftrightarrow (z = x)\ ]\ \Big)\ ,$$

but this proposition is True for a class with three students X, Y, Z in which only Y and Z have chatted with each other, and Y has chatted with himself. Any correct expression of the form

$$\exists x\ \exists y\ \Big(\ G(x, y) \wedge \forall z\ P(x, y, z)\ \Big)\ ,$$

in which P is a predicate with no quantifier, would have to distinguish two classes each with four students A, B, X, Y , among others, in which X has not chatted with Y . In one class everyone except X and Y has chatted with everyone else, and the expression would have to be True; in the other class A has not chatted with B just as X has not chatted with Y but otherwise everyone has chatted with with everyone else, and the expression would have to be False. But P(X, Y, z) is the same for all choices z in both classes. Therefore no correct P without any quantifiers can be constructed.

The problem could be solved easily with two quantifiers if the variables ran not over individual students but over pairs of distinct students. Thus, our choice of *Universe of Discourse* may sometimes determine whether our problem's solution will be simple or complicated.

Math. 55 Discussion of two problems in the text.

Problem 26 on page 20 :

For definiteness we consider compound logical propositions L(p, q, r) of three logical variables. The text explains that every such proposition has a *Disjunctive Normal Form* consisting of a " disjunction of conjunctions of the variables or their negations." In other words,

$$L(p, q, r) \Leftrightarrow c_1 \vee c_2 \vee c_3 \vee \dots\ c_n$$

in which every c_j is $p_j \wedge q_j \wedge r_j$, and p_j is either p or $\neg p$, and q_j is either q or $\neg q$, and r_j is either r or $\neg r$. The number n of terms c_j depends upon L . This Disjunctive Normal Form intended by the text's author is obtained directly from the Truth Table of L as follows:

p	q	r	c	L(p, q, r)
T	T	T	$p \wedge q \wedge r$	
		F	$p \wedge q \wedge \neg r$	
	F	T	$p \wedge \neg q \wedge r$	
		F	$p \wedge \neg q \wedge \neg r$	
F	T	T	$\neg p \wedge q \wedge r$	
		F	$\neg p \wedge q \wedge \neg r$	
	F	T	$\neg p \wedge \neg q \wedge r$	
		F	$\neg p \wedge \neg q \wedge \neg r$	

For every instance of T in the column "L(p, q, r)" select the corresponding conjunction in the column "c" for inclusion in the disjunction. For instance, when L(p, q, r) is $(p \wedge q) \leftrightarrow r$,

$$(p \wedge q) \leftrightarrow r \quad \Leftrightarrow \quad (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

exhibits L(p, q, r) in its Disjunctive Normal Form. When L is the trivial proposition False it has no conjunctions; its Disjunctive Normal Form is "False". But this form is often inefficient; it can require many conjunctions for simple propositions. For example, when L is the trivial proposition True it requires the disjunction of all eight conjunctions c .

A different Disjunctive Form (DF) can be defined in a way not quite inconsistent with the wording in problem 26. It allows a conjunction c_j to omit some (or all) of the variables. This DF of True is just itself; this DF of $p \vee q \vee r$ is itself instead of the disjunction of seven conjunctions c . Unfortunately this DF is not unique; see the text's Chapter 9. For instance

$$(p \wedge q) \leftrightarrow r \quad \Leftrightarrow \quad (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg r) \quad \Leftrightarrow \quad (p \wedge q \wedge r) \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r) .$$

What is the maximum number of conjunctions c_j that any L(p, q, r) needs for this DF ?

(Answer: Four. Example: $(p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r)$.)