Math. 55 FINAL EXAM Model Solutions

(*With corrections.*) Your section's T.A.: ____________________________

This 180 min. exam can be answered with the aid of any texts, notes and calculating instruments. Feel free to experiment; answers may be worked out as audaciously as you like on scratch paper but must then be reconsidered before being entered on this sheet in the spaces provided. Each correct proof, if required, earns two points; each almost complete proof or other correct answer earns a point; each incorrect answer loses a point; space left blank loses nothing, so **DON'T JUST GUESS.** The last problems are the hardest. Take your time; check your work. Finally hand *only* this sheet to *your* T. A. to be scored. Your score *will* determine much of your final grade.

1. Does the system of congruences

$$4x \equiv 5 \ \mathbf{mod}\ 7 \quad \text{and} \quad 2x \equiv 4 \ \mathbf{mod}\ 5$$

have any integer solutions x divisible by 3 ? (YES or NO) __ YES __ [1 pt.]

This is quick . Append congruence $x \equiv 0$ **mod** 3 to the system to see that you can solve it by the Chinese Remainder Theorem *without* having to simplify it to $x \equiv 3$ **mod** 7 and $x \equiv 2$ **mod** 5 , nor then solve it for $x \equiv 87$ **mod** 35 , nor solve the augmented system for $x \equiv 87$ **mod** 105 ,

2. A witness to a hit-and-run accident tells the police that the car's licence number had three letters followed by three decimal digits; the first letter was "A", the letter "S" appeared *just once*, and so did each of the digits "0" and "4". How many license numbers match this description? ______ $(2\cdot25)\cdot(3!\cdot8) = 2400$. __ [1 pt.]

(If "S" appeared *at least once*, and so did "0" and "4" , there are $(2\cdot26-1)\cdot(3!\cdot8 + 3 + 3) = 2754$ matches.)

3. Let $f(x) := x^2 - 121$ and $g(x) := x^2 - 11\cdot x$. There is a fault in the following attempted proof that $f(x) > g(x)$ for all $x > 0$. Circle it. [1 pt.]

$(x + 11) > x$. Therefore $(x - 11)\cdot(x + 11) > (x - 11)\cdot x$. This implies $f(x) > g(x)$.

__ " Therefore … " is invalid if $x \le 11$. __

4. The six faces on each of two identical but independent cubes are marked as follows:

one face has "1" ; two faces have "2" ; three faces have "3" on them.

When rolled as dice, the cubes' faces are each as likely to turn up as any other. Random variable X is the number that shows on one cube's uppermost face; Y is *what shows on the other cube*.

(a) What are the expected value $\overline{X}$ of X and its variance V ?

$\overline{X}$ = ____ $E(X) = (1\cdot1 + 2\cdot2 + 3\cdot3)/6 = 7/3$. ____ [1 pt.]

V = ___ $E(X^2) - \overline{X}^2 = (1\cdot1^2 + 2\cdot2^2 + 3\cdot3^2)/6 - (7/3)^2 = 6 - 49/9 = 5/9$. ____ [1 pt.]

(b) What is the expected value of X+Y ? ___ $E(X+Y) = E(X) + E(Y) = 14/3$. __ [1 pt.]

(c) What is the expected value of $(2X-Y)\cdot(X-2Y)$? ____ $-29/9$ ___ [1 pt.]

X and Y are independent, so $E(X\cdot Y) = E(X)\cdot E(Y) = 49/9$, but $2X-Y$ and $X-2Y$ are NOT independent, so $E((2X-Y)\cdot(X-2Y)) \ne E(2X-Y)\cdot E(X-2Y) = (2\overline{X}-\overline{Y})\cdot(\overline{X}-2\overline{Y}) = -49/9$. Instead we must compute $E(2X^2 - 5XY + 2Y^2) = 4E(X^2) - 5\overline{X}\cdot\overline{Y} = 4\cdot6 - 5\cdot49/9 = -29/9$.

5. We have unlimited supplies of three kinds of marked cards all of the same size: One kind is marked "\$", another kind "£", the third kind "¥".
(a) How many different decks of 17 cards can be made up from the supplies *if order matters*?

(See text p. 287.) ____ $3^{17} = 129140163$. ____ [1 pt.]

(b) *Disregarding order*, how many different decks of 17 cards can be made up from the supplies?
(See text p. 289.) ____ ${}^{3+17-1}C_{17} = {}^{19}C_2 = 171$. __ [1 pt.]

6. The country of Illysay has limited their legal tender to only £ 3 and £ 10 bills, and passed laws requiring all merchandise to be priced so as to be purchasable with these bills and no others. *To curb disease, vendors cannot make change; like bus drivers, they accept only exact amounts.*

For which integers $n > 0$ are prices " £ n " *illegal* ? n = 1, 2, 4, 5, 7, 8, 11, 14 or 17 . [1 pt.]
Explain why no larger price is illegal. [2 pts.]

Among legal prices £ n are those with $n = 18, 19, 20, \ldots, 27$ by inspection; and then if all larger prices n up to $N \ge 27$ are legal so is $N - 9 \ge 18$ legal, and then so is $N-9 + 10 = N+1$.

Alternatively, refer to the posted notes on *Coins and Stamps*; here $M = 10$, $N = 3$, and therefore the largest illegal n is $L = M \cdot N - M - N = 17$. And there are many other proofs.

7. If events $A_1, A_2, A_3, \ldots, A_n$ have probabilities $p_1, p_2, p_3, \ldots, p_n$ respectively then the probability of $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$ is at least $p_1 + p_2 + p_3 + \ldots + p_n + 1 - n$. Prove it. [2 pts]

The inequality is true for $n = 1$; and for $n = 2$ because the inclusion/exclusion principle says
$$1 \ge \text{Prob}(A_1 \cup A_2) = \text{Prob}(A_1) + \text{Prob}(A_2) - \text{Prob}(A_1 \cap A_2) .$$
Suppose it is true for $n \ge 2$ events. Then
$$\begin{aligned}\text{Prob}(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n \cap A_{n+1}) &\ge p_1 + p_2 + \ldots + p_{n-1} + \text{Prob}(A_n \cap A_{n+1}) + 1 - n \\ &\ge p_1 + p_2 + \ldots + p_{n-1} + (p_n + p_{n+1} - 1) + 1 - n ,\end{aligned}$$
which confirms the inequality for $n+1$ events too. See text p. 284 #8. Alternative proof:
$$1 - \text{Prob}(\cap_{1 \le j \le n} A_j) = \text{Prob}(\cup_{1 \le j \le n} \overline{A}_j) \le \Sigma_{1 \le j \le n} \text{Prob}(\overline{A}_j) = \Sigma_{1 \le j \le n} (1 - p_j) = n - \Sigma_{1 \le j \le n} p_j .$$

8. Farms in Glenroyal county raise only horses, cattle and sheep. At least one of these kinds of animal are raised on 451 farms ; horses are raised on 326 ; cattle on 114 ; sheep on 142 . All three are raised on 27 farms. How many farms raise just two kinds of animals? _ 77 _ [1 pt.]

Let H be the set of farms that raise horses among other things, C … cattle, S … sheep. By the inclusion/exclusion principle, $|H \cup C \cup S| = |H| + |C| + |S| - |H \cap C| - |C \cap S| - |S \cap H| + |H \cap C \cap S|$; so $451 = 326 + 114 + 142 - n + 27$ implies $n := |H \cap C| + |C \cap S| + |S \cap H| = 158$. Now a Venn diagram reveals that this $n = \lvert$ farms raising just two species $\rvert + 3 \cdot \lvert$ farms raising all three $\rvert$, so $n - 3 \cdot 27 = 77$ farms raise just two kinds of animals.

9. Let $f(x) := | x^3 - x^2 - 3x - 1 |$ and $g(x) := x^4/|1-x|$. Is it true that …

f(x) is $O(g(x))$ as $x \to +\infty$? (YES or NO) __ YES __ [1 pt.]

f(x) is $\Omega(g(x))$ as $x \to +\infty$? (YES or NO) __ YES __ [1 pt.]

f(x) is $O(g(x))$ as $x \to 0$? (YES or NO) __ NO __ [1 pt.]

f(x) is $\Theta(g(x))$ as $x \to 1$? (YES or NO) __ NO __ [1 pt.]

Definitions of O , Ω and Θ in the text's ch. 1.8 work for $x \to 0$ etc. as well as $x \to +\infty$.

10. The textbook uses a combinatorial argument about binomial coefficients ${}^nC_k := n!/(k!\cdot(n-k)!)$ to prove Vandermonde's Identity ${}^{m+n}C_k = \sum_{0\le j\le k} {}^mC_{k-j}\cdot{}^nC_j$ for $0 \le k \le \min\{m, n\}$. It can also be proved by a probabilistic argument involving the probability generating functions for the random variables that count the number of heads in m+n, m and n tosses of a fair coin. It can also be proved strictly algebraically by invoking a simple polynomial identity $f(x) = g(x)\cdot h(x)$. What are these three polynomials f(x) , g(x) and h(x) ?

$f(x) = (1 + x)^{m+n}$, $g(x) = (1 + x)^m$, $h(x) = (1 + x)^n$; the coefficient of x^k in f(x) is ${}^{m+n}C_k$. [1 pt.]

Actually, the identity is true for $0 \le k \le m+n$ provided we understand that ${}^nC_k = 0$ whenever $k < 0$ or $k > n$, though these possibilities have no familiar combinatorial interpretations.

11. Find *all distinct* integer solutions k **mod** 35 of the congruence " $15\cdot k \equiv 20$ **mod** 35 " .

__ The solutions k **mod** 35 are the <u>**5**</u> integers { 6, 13, 20, 27, 34 } . __ [1 pt.]

The congruence " $15\cdot k \equiv 20$ **mod** 35 " is equivalent to the equation " $15\cdot k = 20 + 35\cdot L$ " for some unknown integer L , and this is equivalent to " $k = 6 + 7\cdot(L - 2k + 2)$ ", which implies " $k \equiv 6$ **mod** 7 " . All such values k have the form $k = 6 + 7\cdot K \equiv 6$ **mod** 7 for some integer K , whence we find $15\cdot k = 90 + 105\cdot K = 20 + (2 + 3\cdot K)\cdot 35 \equiv 20 \bmod 35$; in other words, the solutions of the given congruence are all such values $k = 6 + 7\cdot K$. But some of these have the same residues **mod** 35 ; because they differ by a multiple of 35 , their values of K differ by a multiple of 5 . Therefore, all 5 of the distinct residues k **mod** 35 are obtained by letting K run through any 5 consecutive integers, say { 0, 1, 2, 3, 4 } , to generate all solutions k .

12. Prove that, if seven *distinct* integers are selected from the first ten positive integers, at least two pairs of selected integers must sum to eleven. [2 pts.]

See text p. 249 #14(a). The first ten integers can be put into five pairs {1, 10}, {2, 9}, {3, 8}, {4, 7} and {5, 6} whose members add up to 11 . Think of these pairs as pigeonholes. When seven integers are selected from the first ten, at least two of the selectees must fall into the same pigeonhole/pair, so they add up to 11 . And when this pair is removed from consideration we still have five integers to put into the four remaining pigeonholes/pairs, so it happens again.

13. If $f(x)$ is a nondecreasing function for all real arguments x, and if the sequences $\{v_0, v_1, v_2, \ldots\}$ and $\{w_0, w_1, w_2, \ldots\}$ satisfy $v_{n+1} \le f(v_n)$ and $w_{n+1} := f(w_n)$ for every $n = 0, 1, 2, 3, 4, 5, \ldots$, and if $v_0 \le w_0$, then prove that $v_n \le w_n$ for all $n > 0$. [2 pts.]

Let's use the well-ordering of integers. The asserted inequality $v_n \le w_n$ is true when $n = 0$. If it were false for some n we could let $m+1 \ge 1$ be the least integer for which $v_{m+1} > w_{m+1}$, but $v_m \le w_m$. Then we would find $0 < v_{m+1} - w_{m+1} \le f(v_m) - f(w_m) \le f(w_m) - f(w_m) = 0$, which is impossible. End of proof. A proof by induction works too. This kind of inequality is encountered often in proofs that an algorithm cannot be arbitrarily slow. Lamé's theorem is an example.

14. Write $\bar{1}$ as an abbreviation for -1. Let string $B_n(k)$ = " $b_{n-1}b_{n-2}\ldots b_3b_2b_1b_0$ " be the n-bit Two's Complement Binary Encoding of an integer $k = -b_{n-1}\cdot 2^{n-1} + \sum_{0\le j\le n-2} b_j\cdot 2^j$, provided k lies within the range of such an encoding; here every *bit* b_j is either 1 or else o. Let string $T_n(k)$ = " $t_{n-1}t_{n-2}\ldots t_3t_2t_1t_0$ " be the n-tit *Tinary* Encoding of an integer $k = \sum_{0\le j\le n-1} t_j\cdot 2^j$, provided k lies within the range of such an encoding; here every *tit* t_j is either 1 or else $\bar{1}$. For example, $B_4(7)$ = "o111" and $T_4(7)$ = "$1\bar{1}11$" ; $B_4(-3)$ = "11o1" and $T_4(-3)$ = "$\bar{1}11\bar{1}$" .

List all integers k for which $B_4(k)$ exists but $T_4(k)$ does not:__ $0, \pm 2, \pm 4, \pm 6, -8$. __ [1 pt.]
Describe a simple algorithm to convert $T_n(k)$ into $B_{n+1}(k)$. [2 pts.]

(Though not requested, an algorithm for the conversion of $B_n(k)$ into $T_n(k)$ works whenever k is odd: First, if $k < 0$ change the sign bit b_{n-1} of $B_n(k)$ from "1" to "$\bar{1}$" . Then scan the bits of $B_n(k)$ from right to left replacing "…o1…" by "…$1\bar{1}$…" . Therefore $T_n(k)$ exists whenever $B_n(k)$ exists and k is odd. Consequently, the integers k for which $B_4(k)$ exists but $T_4(k)$ does not are the even integers k satisfying $-8 \le k < 8$, namely $0, \pm 2, \pm 4, \pm 6, -8$. An analogous scan from left to right can convert $T_n(k)$ to $B_{n+1}(k)$, but what follows is simpler.)

A simple algorithm to convert $T_n(k)$ into $B_{n+1}(k)$ goes as follows: First obtain a string P from string $T_n(k)$ by replacing its every "$\bar{1}$" by "o" and prefixing a "o" . Next obtain a string N from string $T_n(k)$ by replacing its every "1" by "o", then its every "$\bar{1}$" by "1", and prefixing a "o" . Finally, interpret P and N as nonnegative (n+1)-bit two's complement binary integers and compute $B_{n+1}(k) := B_{n+1}(P-N)$. Another algorithm simpler than this, because it requires no arithmetic, is slightly harder to justify. The algorithm replaces every "$\bar{1}$" in $T_n(k)$ by "o" , appends a "1" on the right, and *complements* (reverses) the leftmost bit to get $B_{n+1}(k)$. This works because binary integers P and N are *complementary* : $P+N = 2^n - 1$. Therefore $P-N = 2P - (P+N) = 2P - 2^n + 1$, and $B_{n+1}(P-N)$ is just what the simpler algorithm produces.

The maximum score on this exam is 28 points, but 21 points is an excellent score.