The Exercises after Ch. 3.2 in our textbook, *Discrete Mathematics and Its Applications* 4th. ed. by K. Rosen (1999), provide students good opportunities to prove things by Mathematical Induction. Some of these things are inequalities about which one might ask

How did someone think up that inequality?

After all, if " A > B " is true so is " A > B+ß " for all sufficiently tiny ß > 0 ; of these infinitely many inequalities how did the one actually proved get chosen? In some instances the choice seems artificial, as if the proof had been devised first and then the result presented as a puzzle:

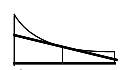
Find the proof.

In some instances the artificiality becomes obvious when an inequality I(n) > 0 proved for positive integers n turns out to be true also for positive non-integers n .

In what follows, which is not for everybody, some of the inequalities whose proofs by induction seem artificial or laborious will be proved quickly and/or improved by using the Calculus.

Harmonic Numbers

The k^{th} Harmonic Number is $H_k := 1/1 + 1/2 + 1/3 + \ldots + 1/(k-1) + 1/k$ for integers $k > 0$; see text p. 193 and p. 201 #51-52. Many estimates of H_k are best obtained from estimates of the integral $\int_x^{(x+1)} \frac{1}{t} dt = \ln(x+1) - \ln(x)$ for $x > 0$. For instance, $1/x > 1/t > 1/(x+1)$ inside the integral, so $1/x > \ln(x+1) - \ln(x) > 1/(x+1)$; and then $\ln(k+1) < H_k < 1 + \ln(k)$ follows by summing appropriate inequalities. In particular, when $k = 2^n$ for $n \ge 0$ we find that $H_k < 1+n\cdot\ln(2) < 1+n$ since $\ln(2) = 0.6931\ldots$; *cf.* p. 201 #51. Better estimates come from the observation that the graph of $y = 1/t$ is *convex* (curved like  because $d^2y/dt^2 = 2/t^3 > 0$) and thus lies below its secants but above its tangents. Consequently areas under the curve satisfy

 $= (1/x + 1/(x+1))/2 > \ln(x+1) - \ln(x) > 1/(x + 1/2) =$  .

Summing appropriate inequalities (can you see which?) now establishes for $k \ge m > 0$ that

$$\ln(k + 1/2) - \ln(m + 1/2) \ge H_k - H_m \ge \ln(k) + 1/(2k) - \ln(m) - 1/(2m) .$$

When $m = 2$ these two inequalities bracket H_k within 1% for all $k \ge 2$. In particular, when $k = 2^n \ge 2$ we find that $H_k \ge H_2 - 1/4 + (n-1)\cdot\ln(2) + 2^{-1-n} \ge 1 + n/2$, as claimed on p. 193, though the last inequality here is unobvious.

This illustrates a nasty aspect of inequalities. If you are asked to prove that A > B but not told how, you can end up proving an inequality $A > \overline{B}$ that is *stronger* (better), because $\overline{B} \ge B$, and yet remain unaware of your achievement so long as you cannot prove $\overline{B} \ge B$. For proving inequalities there are tricks but no routine procedures analogous to "simplification" procedures that so often prove the equality of algebraically equivalent expressions. This is why computerized algebra systems like *Mathematica* and *Maple* still handle inequalities ineptly.

Many people, not just students, find inequalities too troublesome, and avoid them, leaving rewarding careers open to students willing to rise to the challenge.

Bernoulli's Inequality:

If real $x > -1$, and if real $p \le 0$ or $p \ge 1$, then $(1 + px) \le (1 + x)^p$.

This ancient inequality dates from the early years of the Calculus though the text solicits a proof without Calculus and restricted to nonnegative integers p on p. 200 #11. Geometrically this inequality says that the graph of $(1 + x)^p$ is convex (U - shaped) and therefore lies above its every tangent, particularly the tangent drawn through the point on the curve where $x = 0$. Our proof will start from the derivative $d(1 + x)^p/dx = p(1 + x)^{p-1}$. Consequently the integral

$$\int_0^X p\,(\,(1+x)^{p-1} - 1)\,dx \;=\; (1+X)^p - 1 - pX\ .$$

For all x between 0 and X the integrand $p((1+x)^{p-1} - 1)$ has the same sign as X because

- if $X > 0$ and $p > 1$ then $p((1+x)^{p-1} - 1) > 0$;
- if $X > 0$ and $p < 0$ then $p((1+x)^{p-1} - 1) > 0$;
- if $-1 < X < 0$ and $p > 1$ then $p((1+x)^{p-1} - 1) < 0$;
- if $-1 < X < 0$ and $p < 0$ then $p((1+x)^{p-1} - 1) < 0$.

Therefore the integral is nonnegative, which confirms Bernoulli's Inequality. This inequality gets reversed if $0 \le p \le 1$; can you see why? (DRAW GRAPHS !)

Sums of Reciprocal Squares

For any integer $k > 0$ we seek close estimates for $S_k := 1/1 + 1/4 + 1/9 + 1/16 + \dots + 1/k^2$. Two good ways to find estimates both start from known formulas. One way uses the formula

$$1/(m(m+1)) + 1/((m+1)(m+2)) + \dots + 1/((k-1)k) \;=\; 1/m - 1/k$$

obtained from the text's p. 200 #6 or by *Telescoping* as in p. 79 #20. This way provides

$$S_k - 1 < \; 1/(1\cdot2) + 1/(2\cdot3) + 1/(3\cdot4) + \dots + 1/((k-1)k) \;=\; 1 - 1/k \quad \text{for } k > 1\ ,$$

which is what the text asks you to prove on p. 200 #18. A second way to estimate S_k uses the integral $\int_x^{(x+1)} t^{-2}\,dt \;=\; \frac{1}{x} - \frac{1}{x+1}$ for $x > 0$. Since the graph of $1/t^2$ is convex it lies below its secants but above its tangents (see Harmonic Numbers above); consequently

$$(\,1/x^2 + 1/(x+1)^2\,)/2 \;>\; 1/x - 1/(x+1) \;>\; 1/(x + 1/2)^2\ .$$

(These two inequalities can be proved by algebraic means alone with no appeal to Calculus; can you see how?)

Summing appropriate inequalities now establishes for $k > m > 0$ that

$$1/m - 1/(2m^2) - 1/k + 1/(2k^2) \;<\; S_k - S_m \;<\; 1/(m + 1/2) - 1/(k + 1/2)\ .$$

When $m = 2$ these two inequalities bracket S_k well within 2% for all $k \ge 2$. In particular,

$$S_k \le\; 1.65 - 1/(k + 1/2) \;<\; 2 - 1/k \qquad \text{for } k \ge 2\ .$$

Sums of Reciprocals of Square Roots

For any integer $k > 0$ we seek close estimates for $R_k := 1/\sqrt{1} + 1/\sqrt{2} + 1/\sqrt{3} + \dots + 1/\sqrt{k}$. Two good ways to find estimates both start from known formulas. One way uses a formula

$$1/(\sqrt{m} + \sqrt{(m+1)}) + 1/(\sqrt{(m+1)} + \sqrt{(m+2)}) + \dots + 1/(\sqrt{(k-1)} + \sqrt{k}) \;=\; \sqrt{k} - \sqrt{m}$$

obtained by telescoping. This way provides

$$R_k > 2/(\sqrt{1} + \sqrt{2}) + 2/(\sqrt{2} + \sqrt{3}) + 2/(\sqrt{3} + \sqrt{4}) + \dots + 2/(\sqrt{k} + \sqrt{(k+1)}) = 2\sqrt{(k+1)} - 2$$

as reqested in the text on p. 201 #53. A second way to estimate R_k uses the integral $\int_x^{(x+1)} t^{-1/2} dt = 2\sqrt{(x+1)} - 2\sqrt{x}$ for $x > 0$. Since the graph of $1/\sqrt{t}$ is convex it lies below its secants but above its tangents (see above); consequently

$$(1/\sqrt{x} + 1/\sqrt{(x+1)})/2 > 2\sqrt{(x+1)} - 2\sqrt{x} > 1/\sqrt{(x + 1/2)} .$$

(These two inequalities can be proved by algebraic means alone with no appeal to Calculus; can you see how?)

Summing appropriate inequalities now establishes for $k > m > 0$ that

$$2\sqrt{k} + 1/(2\sqrt{k}) - 2\sqrt{m} - 1/(2\sqrt{m}) < R_k - R_m < 2\sqrt{(k + 1/2)} - 2\sqrt{(m + 1/2)} .$$

When $m = 2$ these two inequalities bracket R_k well within 1% for all $k \ge 2$. In particular,

$$R_k \ge 2\sqrt{k} + 1/(2\sqrt{k}) + 1 - 7/\sqrt{8} > 2\sqrt{(k+1)} - 2 ,$$

though the last inequality is unobvious.

A pattern is emerging for these sums of series. To see how far this pattern can go look up the *Euler-Maclaurin Sum Formula* in *Advanced Calculus* texts or old *Numerical Analysis* texts. In these texts repose several centuries' lore about rapid approximate computations of functions whose exact computation would be intolerably onerous. One more example follows:

Stirling's Approximation to n!

Example 10 on p. 195 of our textbook proves an estimate $n! > 2^n$ for $n \ge 4$ by induction. This is too crude an estimate for the needs of this class. What follows proves an old (published in 1730) formula,

$$\text{James Stirling's Approximation} \quad n! \approx \sqrt{2\pi \cdot n} \cdot (n/e)^n ,$$

whose *relative* (not *absolute*) error approaches zero as n approaches $+\infty$. For example …

Table 1: Stirling's Approximation

n	n!	$\sqrt{2\pi \cdot n} \cdot (n/e)^n$	Rel. error
10	3,628,800	$3.60 \cdot 10^6$	0.8 %
20	$2.433 \cdot 10^{18}$	$2.423 \cdot 10^{18}$	0.4 %
40	$8.159 \cdot 10^{47}$	$8.142 \cdot 10^{47}$	0.2 %
80	$7.157 \cdot 10^{118}$	$7.149 \cdot 10^{118}$	0.1 %
160	$4.715 \cdot 10^{284}$	$4.712 \cdot 10^{284}$	0.05 %

A much better approximation can be obtained from the (nonconvergent!) *Asymptotic Series*

$$n! \approx \sqrt{2\pi \cdot n} \cdot (n/e)^n \cdot \exp\left(1/(12 \cdot n) - 1/(360 \cdot n^3) + 1/(1260 \cdot n^5) - 1/(1680 \cdot n^7) + \dots \right) \quad \text{for large } n ,$$

but it lies far beyond the scope of this course. Instead the integral $\int \ln(x) \cdot dx = x \cdot \ln(x) - x$ will be exploited to estimate upper and lower bounds for the finite series

$$\ln(n!) = \sum_{k>1} \ln(k) = \ln(2) + \ln(3) + \ln(4) + \dots + \ln(n-1) + \ln(n) \quad \text{for } n > 1$$

as was done before except that the graph of $\ln(x)$ is *concave* (curved like ⌒) now because $\ln(x)'' = -1/x^2 < 0$, so the graph lies *above* its secants but *below* its tangents. Consequently

$$(\ln(x) + \ln(x+1))/2 \ < \ \int_x^{(x+1)} \ln(t)\,dt \ = \ (x+1)\cdot\ln(x+1) - 1 - x\cdot\ln(x) \ < \ \ln(x + 1/2)\ .$$

As before, summing appropriate inequalities implies

$$(n+1/2)\cdot\ln(n+1/2) - n - (3/2)\cdot\ln(3/2) + 1 \ < \ln(n!) < \ (n+1/2)\cdot\ln(n) - n + 2 - (3/2)\cdot\ln(2)\ .$$

The upper bound exceeds the lower by

$$1 - (3/2)\cdot\ln(4/3) - (n+1/2)\cdot\ln((n+1/2)/n) \ = \ (1/2)\cdot\ln(1-z)/z + 0.568477\ldots \ < \ 0.0685\ ,$$

where $z := 1/(2n+1)$ and $\ln(1-z)/z = -1 - z/2 - z^2/3 - z^3/4 - z^4/5 - \ldots$. Consequently

$$0.89 < 0.962\ldots - 0.0685 < \ Ç(n) := \ln(n!) - (n+1/2)\cdot\ln(n) + n \ < \ 2 - (3/2)\cdot\ln(2) = 0.962\ldots\ .$$

This Ç(n) is a decreasing function of n because, after some algebra,

$$Ç(n+1) - Ç(n) = \ 1 + (1/2)\cdot\ln\big((1-z)/(1+z)\big)/z \ = \ -z^2/3 - z^4/5 - z^6/7 - \ldots \ < 0\ .$$

Therefore, as n increases towards infinity, Ç(n) decreases towards a limit Ç > 0.89 . Although Stirling did not know it at first, this constant Ç turns out to be $\ln(\sqrt{2\pi}) = 0.919\ldots$, as shall be proved in a moment. For now we conclude, for some constant Ç between 0.962 and 0.89 , that $\ln(n!) - (n+1/2)\cdot\ln(n) + n - Ç$ approaches zero or, equivalently, that $n!/\big(e^{Ç}\cdot\sqrt{n}\cdot(n/e)^n\big)$ approaches 1 , descending as n increases towards infinity.

To determine Ç we obtain an estimate for π found first by John Wallis (who died in 1730) but derived nowadays more rigorously by using *Integration by Parts* as follows. For $m \ge 2$ set

$$J_m := \int_0^{\pi/2} (\sin x)^m dx \ = \ -\int_0^{\pi/2} (\sin x)^{m-1} d\cos x \ = \ \int_0^{\pi/2} (\cos x)\, d(\sin x)^{m-1} \quad (\text{ using } I\text{-}by\text{-}P\ )$$

$$= \ (m-1)\int_0^{\pi/2} (\cos x)^2 (\sin x)^{m-2} dx \ = \ (m-1)\int_0^{\pi/2} (1 - (\sin x)^2)\,(\sin x)^{m-2} dx\ ,$$

from which follows that $J_m = (m-1)\cdot(J_{m-2} - J_m) = (1 - 1/m)\cdot J_{m-2}$ provided we also set

$$J_1 := \int_0^{\pi/2} (\sin x)^1 dx \ = \ 1 \quad \text{and} \quad J_0 := \int_0^{\pi/2} (\sin x)^0 dx \ = \ \pi/2\ .$$

Now induction on k = 0, 1, 2, 3, … in turn provides confirmation for the formulas

$$J_{2k+1} = (2^k\cdot k!)^2/(2k+1)! \quad \text{and} \quad J_{2k} = \ (2k)!\cdot(\pi/2)/(2^k\cdot k!)^2\ .$$

Moreover, because $0 < \sin x < 1$ inside the range of integration, $0 < J_m < J_{m-1}$. Consequently $1 > J_m/J_{m-1} = (1 - 1/m)\cdot J_{m-2}/J_{m-1} > (1 - 1/m) \to 1$ and therefore $J_m/J_{m-1} \to 1$ as $m \to +\infty$, and so does $(\pi/2)\cdot J_{2k+1}/J_{2k} \ = \ (2^k\cdot k!)^4/((2k+1)!\cdot(2k)!) \ \to \ \pi/2$ as $k \to +\infty$. That quotient of factorials *etc*. is Wallis' estimate for π/2 .

Replace each factorial in that quotient by its Stirling approximation $n! \approx e^{Ç}\cdot\sqrt{n}\cdot(n/e)^n$ and let $k \to +\infty$. We find that Stirling's approximation to $(2^k\cdot k!)^4/((2k+1)!\cdot(2k)!)$ simplifies, after a lot of algebra, to $e^{2Ç+1}\cdot 2^{-2}\cdot(1 + 1/(2k))^{-2k-3/2} \to e^{2Ç}/4$ as $k \to +\infty$ since $(1 + 1/(2k))^{2k} \to e$. This implies that $e^{2Ç}/4 = \pi/2$, whence $e^{Ç} = \sqrt{2\pi}$, completing the vindication of

$$\text{Stirling's Approximation} \quad n! \ \approx \ \sqrt{2\pi\cdot n}\cdot(n/e)^n\ .$$

Arithmetic *vs*. Geometric Means

Given collections of positive variables x_j and positive *weights* w_j, where we restrict subscript j to some finite set solely to avoid the technicalities associated with convergence if j were allowed to range over an infinite set, let

$$w := \sum_j w_j\ , \qquad A := (\textstyle\sum_j w_j \cdot x_j)/w \quad \text{and} \qquad G := \left(\prod_j x_j^{\,w_j}\right)^{1/w} .$$

Here A is the *Weighted Arithmean Mean* (Average) of the numbers x_j, and G is their *Weighted Geometric Mean*. To some extent these definitions are redundant because $\ln(G)$ is the weighted arithmetic mean of the numbers $\ln(x_j)$, but this is no time to quibble about terms that have been in use for millennia. Our objective is to prove that

$$A \ge G$$

with equality just when all the x_j 's have the same positive value. This inequality is the same as on the text's p. 201 #55 except that the text considers only the special case when all weights $w_j = 1$, and supplies a long unobvious proof by induction of which only Gauss could be proud (and was). Our proof will be very short. First we simplify the notation by defining *fractional weights* $f_j := w_j/w > 0$ so that

$$\sum_j f_j = 1\ , \qquad A := \textstyle\sum_j f_j \cdot x_j \qquad \text{and} \qquad G := \prod_j x_j^{\,f_j} .$$

Next observe for any $x > 0$ that $0 \le \int_x^G \left(\frac{1}{t} - \frac{1}{G}\right) dt \;=\; \ln(G) - \ln(x) - 1 + x/G$ because, so long as the integrand's t lies strictly between x and G, the signs of $G–x$ and of $1/t - 1/G$ must be the same. Of course " $0 \le \dots$ " becomes " $0 = \dots$ " just when $x = G$. Now replace x by x_j, multiply by f_j, and sum over j to deduce that $0 \le \ 0 - 1 + A/G$ as was claimed.