For any integers $n \ge k \ge 0$ let $F_{n,k}$ denote the number of permutations of n objects that leave just k unspecified objects unchanged. Then $D_n := F_{n,0}$ tells how many of those permutations leave no object unchanged; these D_n permutations are called " Derangements."

Evidently $F_{n,n} = 1$ (including $F_{0,0}$ for convenience), and $F_{n,n-1} = 0$ if $n > 0$; consequently $D_0 = 1$ and $D_1 = 0$. In general $F_{n,k} = {}^nC_k \cdot D_{n-k}$, where the combinatorial coefficient ${}^nC_k := n!/(k!\cdot(n-k)!)$, because $F_{n,k}$ is the number of ways to fix k unspecified objects and derange the rest. Since the total number of all permutations is $n!$,

$$\Sigma_{0\le k\le n} F_{n,k} = \Sigma_{0\le k\le n} {}^nC_k \cdot D_{n-k} = n! \ .$$

This turns into a recurrence

$$D_n := n! - \Sigma_{1\le k\le n} {}^nC_k \cdot D_{n-k}$$

that determines $D_0 = 1$, $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, $D_6 = 265$, … in turn.

We shall next prove the formula

$$D_n = n\cdot D_{n-1} + (-1)^n$$

for all $n > 0$ by induction. Apparently the formula holds for D_1 (among others); our induction hypothesis is that the formula holds for D_1, D_2, D_3, …, D_{N-1} and D_N . Then we find

$$\begin{aligned} D_{N+1} - (N+1)\cdot D_N &= (N+1)\cdot\Sigma_{1\le k\le N} {}^NC_k \cdot D_{N-k} - \Sigma_{1\le k\le N+1} {}^{N+1}C_k \cdot D_{N+1-k} \\ &= \Sigma_{1\le k\le N} \big(((N+1)\cdot{}^NC_k - {}^{N+1}C_k\cdot(N+1-k))\cdot D_{N-k} - {}^{N+1}C_k\cdot(-1)^{N+1-k}\big) - 1 \\ &= 0 - \Sigma_{0\le k\le N+1} {}^{N+1}C_k\cdot(-1)^{N+1-k} + (-1)^{N+1} = -(1-1)^{N+1} + (-1)^{N+1} \ , \end{aligned}$$

which vindicates the formula for D_{N+1} and consequently for all D_n with $n > 0$.

Rewriting it

$$D_n/n! = D_{n-1}/(n-1)! + (-1)^n/n! \quad \text{starting from} \quad D_0 = 1$$

turns the formula into a recurrence easily solved for

$$D_n/n! = 1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n/n! = 1/e - \Sigma_{k>n} (-1)^k/k!$$

where $e \approx 2.718281828459\ldots$ is the base of the exponential function $\exp(x) = e^x = \Sigma_{k\ge 0} x^k/k!$ explored in Calculus courses. The sum $-\Sigma_{k>n} (-1)^k/k!$ is a rapidly convergent *alternating* series that lies between any two consecutive partial sums, say the second and third:

$$(-1)^n\cdot(n+1)/(n+2)! \quad \text{and} \quad (-1)^n\cdot(n+2)^2/(n+3)! \ .$$

Therefore $D_n - n!/e$ lies between $(-1)^n/(n+2)$ and $(-1)^n\cdot(n+2)/((n+3)\cdot(n+1))$, both of which are smaller than $1/2$ in magnitude if $n > 0$. This implies, finally,

D_n = the integer nearest $n!/e$ for all integers $n > 0$.

. . . .

At Annapolis it is customary for graduating midshipmen to toss their hats in the air at the end of the graduation ceremony. If the hats are retrieved at random afterwards, what is the probability that no midshipman will retrieve his (or her) own hat? $D_n/n!$ is nearly $1/e \approx 0.367879\ldots$.

Compare pp. 365-7 & p. 368 #12-21 & #24-26 of our text *Discrete Math. & Appl'ns* 4th ed. by K. Rosen.